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Some Inclusion Properties for Certain Subclass of Meromorphically Multivalent Functions Involving the Srivastava-Attiya Operator *

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Abstract

The Srivastava-Attiya operator is used here to define a new subclass of meromorphically multivalent functions in the punctured open unit disk $\mathbb{U}_0 = \{z : 0 < |z| < 1\}$. For this new function class, several inclusion relationships are established. Some interesting corollaries and consequences of the main inclusion relationship are also considered.

Keywords and Phrases: *Analytic function, Meromorphically multivalent function, Hadamard product (or convolution), Srivastava-Attiya operator, Subordinate.*

1. Introduction

Let $\Sigma(p)$ denote the class of meromorphically multivalent functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary
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which are analytic in the punctured open unit disk $\mathbb{U}_0 = \{z : 0 < |z| < 1\}$ with a pole at $z = 0$. The class $\Sigma(p)$ is closed under the Hadamard product (or convolution)

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^{n-p} = (f_2 * f_1)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,j}z^{n-p} \in \Sigma(p) \quad (j = 1, 2).$$

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U} = \mathbb{U}_0 \cup \{0\}$, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g \prec f$ or (more precisely) $g(z) \prec f(z) \quad (z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathbb{U})$ such that $g(z) = f(w(z)) \quad (z \in \mathbb{U})$. In particular, if $f(z)$ is univalent in $\mathbb{U}$, we have the following equivalence

$$g(z) \prec f(z) \quad (z \in \mathbb{U}) \Leftrightarrow g(0) = f(0) \quad \text{and} \quad g(\mathbb{U}) \subset f(\mathbb{U}).$$

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

which are analytic in $\mathbb{U}$. A function $f(z) \in A$ is said to be in the class $S^*(\alpha)$ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha (\alpha < 1)$. When $0 \leq \alpha < 1$, $S^*(\alpha)$ is the class of starlike functions of order α in $\mathbb{U}$. A function $f(z) \in A$ is said to be prestarlike of order α in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha) \quad (\alpha < 1), \tag{1.4}$$

where the symbol $*$ means the familiar Hadamard product (or convolution) of two analytic functions in $\mathbb{U}$. We denote this class by $R(\alpha)$ (see [16]). Clearly a function $f(z) \in A$ is in the class $R(0)$ if and only if $f(z)$ is convex univalent in $\mathbb{U}$ and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

With a view to introducing the Srivastava-Attiya convolution operator $L_{s,b}$, we begin by recalling a general Hurwitz-Lerch Zeta function $\Phi(z,s,b)$ defined by (cf., e.g., [20, p.121 et seq.])

$$\Phi(z,s,b) := \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^s} = \frac{1}{b^s} + \frac{z}{(1+b)^s} + \frac{z^2}{(2+b)^s} + \cdots$$

$(b \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0,-1,-2,\cdots\}; s \in \mathbb{C}$ when $|z|<1; \mathrm{R}e(s)>1$ when $|z|=1)$.

Some interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,b)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and López [5], Garg et al. [6], Lin and Srivastava [9], Lin et al. [10], Luo and Srivastava [12], and others.

Very recently, Srivastava and Attiya [19] introduced the linear operator

$$L_{s,b} : A \to A$$

defined, in terms of the Hadamard product (or convolution), by

$$L_{s,b}f(z) := G_{s,b}(z) * f(z) \quad (z \in \mathbb{U}; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}),$$

where, for convenience,

$$G_{s,b}(z) := (1+b)^s[\Phi(z,s,b) - b^{-s}] \quad (z \in \mathbb{U}).$$

The operator $L_{s,b}$ is called the Srivastava-Attiya operator (see [15][23]). It is well known that the Srivastava-Attiya operator $L_{s,b}$ contains, among its special cases, the integral operators introduced and investigated earlier by (for example) Alexander [1], Libera [8], Bernardi [2], and Jung et al. [7].

Analogous to $L_{s,b}$, we now introduce the following generalized Srivastava-Attiya operator $J_{s,b}$

$$J_{s,b} : \Sigma(p) \to \Sigma(p)$$

$$J_{s,b}f(z) := G_{p,s,b}(z) * f(z) \quad (z \in \mathbb{U}; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}), \tag{1.5}$$

where

$$G_{p,s,b}(z) := (1+b)^s[\Phi_p(z,s,b) - b^{-s}] \tag{1.6}$$

and

$$\Phi_p(z,s,b) = \frac{1}{b^s} + \frac{z^{-p}}{(1+b)^s} + \frac{z^{-p+1}}{(2+b)^s} + \cdots . \tag{1.7}$$

It is not difficult to see from (1.5) to (1.7) that

$$J_{s,b}f(z) = z^{-p} + \sum_{n=1}^{\infty} \left(\frac{1+b}{n+1+b} \right)^s a_n z^{n-p}. \tag{1.8}$$

Let P be the class of functions $h(z)$ with $h(0) = 1$, which are analytic and convex univalent in $\mathbb{U}$.

Definition. A function $f(z) \in \Sigma(p)$ is said to be in the class $M_{s,b}(\lambda; h)$ if it satisfies the subordination condition

$$\frac{(\lambda-1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda}{p(p+1)} z^{p+2}(J_{s,b}f(z))'' \prec h(z), \tag{1.9}$$

where λ is a complex number and $h(z) \in P$.

The main object of this paper is to present a systematic investigation of the class $M_{s,b}(\lambda; h)$ defined above by means of the generalized Srivastava-Attiya operator $J_{s,b}$. Several inclusion relationships of the class $M_{s,b}(\lambda; h)$ are established. Some interesting corollaries and consequences of the main inclusion relationship are also considered.

For our purpose, we shall need the following lemmas to derive our main results for the class $M_{s,b}(\lambda; h)$.

Lemma 1. (see [14]). *Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.10}$$

where $Re\mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of (1.10).

Lemma 2. (see [16]). *Let $\alpha < 1$, $f(z) \in S^*(\alpha)$ and $g(z) \in R(\alpha)$. Then, for any analytic function $F(z)$ in $\mathbb{U}$,*

$$\frac{g * (fF)}{g * f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ denotes the closed convex hull of $F(\mathbb{U})$.

2. Properties of the Class $M_{s,b}(\lambda; h)$

Theorem 1. *Let $\lambda_1 < \lambda_2 \leq 0$. Then $M_{s,b}(\lambda_1; h) \subset M_{s,b}(\lambda_2; h)$.*
Proof. Let $\lambda_1 < \lambda_2 \leq 0$ and suppose that

$$g(z) = -\frac{z^{p+1}(J_{s,b}f(z))'}{p} \tag{2.1}$$

for $f(z) \in M_{s,b}(\lambda_1; h)$. Then the function $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$. Differentiating both sides of (2.1) with respect to z and using (1.9), we have

$$\begin{aligned} &\frac{(\lambda_1 - 1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda_1}{p(p+1)} z^{p+2}(J_{s,b}f(z))'' \\ &= g(z) - \frac{\lambda_1}{p+1} z g'(z) \prec h(z). \end{aligned} \tag{2.2}$$

Hence an application of Lemma 1 yields

$$g(z) \prec h(z). \tag{2.3}$$

Noting that $0 < \frac{\lambda_2}{\lambda_1} < 1$ and that $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.1) to (2.3) that

$$\begin{aligned} &\frac{(\lambda_2 - 1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda_2}{p(p+1)} z^{p+2}(J_{s,b}f(z))'' \\ &= \frac{\lambda_2}{\lambda_1}\left(\frac{(\lambda_1 - 1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda_1}{p(p+1)} z^{p+2}(J_{s,b}f(z))''\right) + \left(1 - \frac{\lambda_2}{\lambda_1}\right) g(z) \\ &\prec h(z). \end{aligned}$$

Thus $f(z) \in M_{s,b}(\lambda_2; h)$ and the proof of Theorem 1 is completed.

Theorem 2. *Let $f(z) \in M_{s,b}(\lambda; h), g(z) \in \Sigma(p)$ and*

$$Re\{z^p g(z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{2.4}$$

Then

$$(f * g)(z) \in M_{s,b}(\lambda; h).$$

Proof. For $f(z) \in M_{s,b}(\lambda;h)$ and $g(z) \in \Sigma(p)$, we have

$$\begin{aligned}
&\frac{(\lambda-1)}{p} z^{p+1}(J_{s,b}(f*g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2}(J_{s,b}(f*g)(z))'' \\
&= \frac{(\lambda-1)}{p}(z^p g(z)) * (z^{p+1}(J_{s,b}f(z))') + \frac{\lambda}{p(p+1)}(z^p g(z)) * (z^{p+2}(J_{s,b}f(z))'') \\
&= (z^p g(z)) * \psi(z), \qquad (2.5)
\end{aligned}$$

where

$$\psi(z) = \frac{(\lambda-1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda}{p(p+1)} z^{p+2}(J_{s,b}f(z))''. \qquad (2.6)$$

In view of (2.4), the function $z^p g(z)$ has the Herglotz representation

$$z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \qquad (2.7)$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.5) to (2.7) that

$$\begin{aligned}
&\frac{(\lambda-1)}{p} z^{p+1}(J_{s,b}(f*g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2}(J_{s,b}(f*g)(z))'' \\
&= \int_{|x|=1} \psi(xz)d\mu(x) \prec h(z).
\end{aligned}$$

This shows that $(f*g)(z) \in M_{s,b}(\lambda;h)$ and the theorem is proved.

Corollary 1. *Let $f(z) \in M_{s,b}(\lambda;h)$ be given by (1.1) and let*

$$s_m(z) = z^{-p} + \sum_{n=1}^{m-1} a_n z^{n-p} \quad (m \in \mathbb{N} \setminus \{1\}).$$

Then the function

$$\sigma_m(z) = \int_0^1 t^p s_m(tz)dt$$

is also in the class $M_{s,b}(\lambda; h)$.

Proof. We have

$$\sigma_m(z) = z^{-p} + \sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n-p} = (f * g_m)(z) \quad (m \in \mathbb{N} \setminus \{1\}),$$

where

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \in M_{s,b}(\lambda; h)$$

and

$$g_m(z) = z^{-p} + \sum_{n=1}^{m-1} \frac{z^{n-p}}{n+1} \in \Sigma(p).$$

Also, for $m \in \mathbb{N} \setminus \{1\}$, it is known from [18] that

$$\mathrm{Re}\,\{z^p g_m(z)\} = \mathrm{Re}\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}).$$

An application of Theorem 2 leads to $\sigma_m(z) \in M_{s,b}(\lambda; h)$.

Theorem 3. *Let* $f(z) \in M_{s,b}(\lambda; h), g(z) \in \Sigma(p)$ *and*

$$z^{p+1} g(z) \in R(\alpha) \quad (\alpha < 1).$$

Then

$$(f * g)(z) \in M_{s,b}(\lambda; h).$$

Proof. For $f(z) \in M_{s,b}(\lambda; h)$ and $g(z) \in \Sigma(p)$, from (2.5) we have

$$\begin{aligned} &\frac{(\lambda - 1)}{p} z^{p+1} (J_{s,b}(f * g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (J_{s,b}(f * g)(z))'' \\ &= \frac{(z^{p+1} g(z)) * (z\psi(z))}{(z^{p+1} g(z)) * z} \quad (z \in \mathbb{U}), \end{aligned} \tag{2.8}$$

where $\psi(z)$ is defined as in (2.6).

Since $h(z)$ is convex univalent in $\mathbb{U}$,

$$\psi(z) \prec h(z), \quad z^{p+1} g(z) \in R(\alpha) \quad \text{and} \quad z \in S^*(\alpha) \quad (\alpha < 1),$$

it follows from (2.8) and Lemma 2 the desired result.

Taking $\alpha = 0$ and $\alpha = \frac{1}{2}$, Theorem 3 reduces to the following.

Corollary 2. *Let $f(z) \in M_{s,b}(\lambda; h)$ and let $g(z) \in \Sigma(p)$ satisfy either of the following conditions*

(i) $z^{p+1}g(z)$ is convex univalent in $\mathbb{U}$

or

(ii) $z^{p+1}g(z) \in S^*\left(\frac{1}{2}\right)$.

Then

$$(f * g)(z) \in M_{s,b}(\lambda; h).$$

Theorem 4. *Let $G(z) \in M_{s,b}(\lambda; h)$. If the function $f(z)$ is defined by*

$$G(z) = \frac{\mu - p}{z^{\mu}} \int_0^z t^{\mu-1} f(t)dt \quad (\mu > p), \tag{2.9}$$

then

$$\sigma^p f(\sigma z) \in M_{s,b}(\lambda; h),$$

where

$$\sigma = \frac{\sqrt{1 + (\mu - p)^2} - 1}{\mu - p} \in (0, 1). \tag{2.10}$$

Proof. For $G(z) \in \Sigma(p)$, it is easy to verify that

$$G(z) = G(z) * \frac{z^{-p}}{1 - z} \quad \text{and} \quad zG'(z) = G(z) * \left(\frac{z^{-p}}{(1 - z)^2} - (p + 1)\frac{z^{-p}}{1 - z} \right).$$

Hence, by (2.9), we have

$$f(z) = \frac{\mu G(z) + zG'(z)}{\mu - p} = (G * g)(z) \quad (z \in \mathbb{U}_0; \mu > p), \tag{2.11}$$

where

$$g(z) = \frac{1}{\mu - p} \left((\mu - p - 1)\frac{z^{-p}}{1 - z} + \frac{z^{-p}}{(1 - z)^2} \right) \in \Sigma(p). \tag{2.12}$$

Next we show that

$$\mathrm{Re}\left\{z^p g(z)\right\} > \frac{1}{2} \quad (|z| < \sigma), \tag{2.13}$$

where σ is given by (2.10). Setting

$$\frac{1}{1-z} = Re^{i\theta} \quad (R>0) \quad \text{and} \quad |z| = r < 1,$$

we see that

$$\cos\theta = \frac{1+R^2(1-r^2)}{2R} \quad \text{and} \quad R \geq \frac{1}{1+r}. \tag{2.14}$$

For $\mu > p$ it follows from (2.12) and (2.14) that

$$\begin{aligned}
2\mathrm{Re}\{z^p g(z)\} &= \frac{2}{\mu-p}\left[(\mu-p-1)R\cos\theta + R^2(2\cos^2\theta-1)\right] \\
&= \frac{1}{\mu-p}\left[(\mu-p-1)(1+R^2(1-r^2)) + (1+R^2(1-r^2))^2 - 2R^2\right] \\
&= \frac{R^2}{\mu-p}\left[R^2(1-r^2)^2 + (\mu-p+1)(1-r^2) - 2\right] + 1 \\
&\geq \frac{R^2}{\mu-p}\left[(1-r)^2 + (\mu-p+1)(1-r^2) - 2\right] + 1 \\
&= \frac{R^2}{\mu-p}(\mu-p-2r-(\mu-p)r^2) + 1.
\end{aligned}$$

This evidently gives (2.13), which is equivalent to

$$\mathrm{Re}\{z^p\sigma^p g(\sigma z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{2.15}$$

Let $G(z) \in M_{s,b}(\lambda; h)$. Then by using (2.11) and (2.15), an application of Theorem 2 yields

$$\sigma^p f(\sigma z) = G(z) * (\sigma^p g(\sigma z)) \in M_{s,b}(\lambda; h).$$

Theorem 5. *Let $\lambda < 0, \beta > 0$ and $f(z) \in M_{s,b}(\lambda; \beta h + 1 - \beta)$. If $\beta \leq \beta_0$, where*

$$\beta_0 = \frac{1}{2}\left(1 + \frac{p+1}{\lambda}\int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u}du\right)^{-1}, \tag{2.16}$$

then $f(z) \in M_{s,b}(0; h)$. The bound β_0 is sharp when $h(z) = \frac{1}{1-z}$.

Proof. Let us define

$$g(z) = -\frac{z^{p+1}(J_{s,b}f(z))'}{p} \tag{2.17}$$

for $f(z) \in M_{s,b}(\lambda; \beta h + 1 - \beta)$ with $\lambda < 0$ and $\beta > 0$. Then we have

$$\begin{aligned} g(z) - \frac{\lambda}{p+1} z g'(z) &= \frac{(\lambda - 1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda}{p(p+1)} z^{p+2}(J_{s,b}f(z))'' \\ &\prec \beta h(z) + 1 - \beta. \end{aligned}$$

Hence an application of Lemma 1 yields

$$\begin{aligned} g(z) &\prec -\frac{\beta(p+1)}{\lambda} z^{\frac{p+1}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda}-1} h(t)dt + 1 - \beta \\ &= (h * \psi)(z), \end{aligned} \tag{2.18}$$

where

$$\psi(z) = -\frac{\beta(p+1)}{\lambda} z^{\frac{p+1}{\lambda}} \int_0^z \frac{t^{-\frac{p+1}{\lambda}-1}}{1-t} dt + 1 - \beta. \tag{2.19}$$

If $0 < \beta \leq \beta_0$, where $\beta_0 (> 1)$ is given by (2.16), then it follows from (2.19) that

$$\begin{aligned} \mathrm{Re}\psi(z) &= -\frac{\beta(p+1)}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} \mathrm{Re}\left(\frac{1}{1-uz}\right) du + 1 - \beta \\ &> -\frac{\beta(p+1)}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du + 1 - \beta \\ &\geq \frac{1}{2} \quad (\lambda < 0; z \in \mathbb{U}). \end{aligned}$$

Now, by using the Herglotz representation for $\psi(z)$, from (2.17) and (2.18) we arrive at

$$-\frac{z^{p+1}(J_{s,b}f(z))'}{p} \prec (h * \psi)(z) \prec h(z)$$

because $h(z)$ is convex univalent in $\mathbb{U}$. This shows that $f(z) \in M_{s,b}(0; h)$.

For $h(z) = \frac{1}{1-z}$ and $f(z) \in \Sigma(p)$ defined by

$$-\frac{z^{p+1}(J_{s,b}f(z))'}{p} = -\frac{\beta(p+1)}{\lambda} z^{\frac{p+1}{\lambda}} \int_0^z \frac{t^{-\frac{p+1}{\lambda}-1}}{1-t} dt + 1 - \beta,$$

it is easy to verify that

$$\frac{(\lambda-1)}{p} z^{p+1}(J_{s,b}f(z))' + \frac{\lambda}{p(p+1)} z^{p+2}(J_{s,b}f(z))'' = \beta h(z) + 1 - \beta.$$

Thus $f(z) \in M_{s,b}(\lambda; \beta h + 1 - \beta)$. Also, for $\beta > \beta_0$, we have

$$\begin{aligned} \mathrm{Re}\{-\frac{z^{p+1}(J_{s,b}f(z))'}{p}\} &\to -\frac{\beta(p+1)}{\lambda}\int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u}du + 1 - \beta \\ &< \frac{1}{2} \quad (z \to -1), \end{aligned}$$

which implies that $f(z) \notin M_{s,b}(0; h)$. Hence the bound β_0 cannot be increased when $h(z) = \frac{1}{1-z}$.

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math. Ser.2*, **17** (1915), 12-22.

[2] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135** (1969), 429-446.

[3] N. E. Cho, O. S. Kwon, and H. M. Srivastava, Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu-Srivastava operator, *Integral Transforms Spec. Funct.*, **21** (2010), 589-601.

[4] J. Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, *Appl. Math. Comput.*, **170** (2005), 399-409.

[5] C. Ferreira and J. L. López, Asymptotic expansions of the Hurwitz-Lerch Zeta function, *J. Math. Anal. Appl.*, **298** (2004), 210-224.

[6] M. Garg, K. Jain, and H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.*, **17** (2006), 803-815.

[7] I. B. Jung, Y. C. Kim, and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.*, **176** (1993), 138-147.

[8] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, **16** (1969), 755-758.

[9] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.*, **154** (2004), 725-733.

[10] S.-D. Lin, H. M. Srivastava, and P.-Y. Wang, Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.*, **17** (2006), 817-827.

[11] J.-L. Liu and H. M. Srivastava, A class of multivalently analytic functions associated with the Dziok-Srivastava operator, *Integral Transforms Spec. Funct.*, **20** (2009), 401-417.

[12] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, *J. Math. Anal. Appl.*, **308** (2005), 290-302.

[13] T. H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.*, **104** (1962), 532-537.

[14] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** (1981), 157-171.

[15] D. Răducanu and H. M. Srivastava, A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.*, **18** (2007), 933-943.

[16] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Les Presses de 1'Université de Montréal, Montréal, 1982.

[17] T. N. Shanmugam, S. Sivasubramanian, and S. Owa, On sandwich results for some subclasses of analytic functions involving certain linear operator, *Integral Transforms Spec. Funct.*, **21** (2010), 1-11.

[18] R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.*, **106** (1989), 145-152.

[19] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transforms Spec. Funct.*, **18** (2007), 207-216.

[20] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[21] H. M. Srivastava, S. S. Eker, and B. Seker, Inclusion and neighborhood properties for certain classes of multivalently analytic functions of complex order associated with the convolution structure, *Appl. Math. Comput.*, **212** (2009), 66-71.

[22] Z.-G. Wang, Y.-P. Jiang, and H. M. Srivastava, Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function, *Comput. Math. Appl.*, **57** (2009), 571-586.

[23] Z.-G. Wang, Q.-G. Li, and Y.-P. Jiang, Certain subclasses of multivalent analytic functions involving the generalized Srivastava-Attiya operator, *Integral Transforms Spec. Funct.*, **21** (2010), 221-234.