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Strengthening of Nevanlinna's Five-Value Theorem *

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Abstract

In this paper, we generalise Nevanlinna's five-value theorem for derivatives of meromorphic functions by considering weaker assumptions of sharing five values to partially sharing $k(\geq 5)$ values. As a particular cases of our results, we deduce earlier results of C.-C. Yang [8, Theorem 3.2] and T.-G. Chen, K.-Y. Chen and Y.-L. Tsai [1].

Keywords and Phrases: *Value distribution theory, Meromorphic functions, Sharing values and uniqueness.*

1. Introduction, Definitions and Main Results

Nevanlinna's five-value theorem [8] is a very important result of Nevanlinna on the uniqueness of meromorphic functions, which says that if two meromorphic functions share five values ignoring multiplicity, then these two functions must be identical.
C.-C. Yang [8, Theorem 3.2] observed that one can weaken the assumption sharing five values to " partially $''$ sharing five values in Nevanlinna's five-value theorem.

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We say that a meromorphic function $f(z)$ partially shares a value a with a meromorphic function $g(z)$ if

$$E(a,f) \subseteq E(a,g).$$

Under this terminology, Yang [8, Theorem 3.2] proved that if a meromorphic function $f(z)$ partially share five values $a_1, a_2, \cdots, a_5$ with a meromorphic function $g(z)$ and

$$\varliminf_{r\to\infty}\left[\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{f-a_j}\right)\Big|\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{g-a_j}\right)\right] > \frac{1}{2}$$

then, $f(z)$ and $g(z)$ must be identical. In Nevanlinna's five-value theorem, we have $E\left(a_j,f\right) = E\left(a_j,g\right)$ for all $1 \le j \le 5$. In this case,

$$\varliminf_{r\to\infty}\left[\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{f-a_j}\right)\Big|\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{g-a_j}\right)=1\right] > \frac{1}{2}$$

So, $f(z) \equiv g(z)$. Hence Yang's result is a generalization of Nevanlinna's five-value theorem.
We assume that the reader is familiar with the basic notations and fundamental results of Nevanlinna's theory of meromorphic functions, as found in [8, 9]. In particular, we use E to denote a subset of $(0,\infty)$ such that E is of finite linear measure, which may be varied in different places.

Definition 1.1. Let $h(z)$ be a non-constant meromorphic function and a be a value in the extended complex plane. We define

$$E(a,h) = \{z|h(z)-a=0\}$$

denotes zero set of $h(z)-a$ in which each zero is counted according to its multiplicity and

$$\overline{E}(a,h) = \{z|h(z)-a=0\}$$

denotes zero set of $h(z)-a$, in which each zero is counted only once.
In 2003, Yang[8, Theorem 3.2] proved the following theorem.

Theorem 1.A. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_1, a_2, \cdots, a_5$ be five distinct values. If $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$ for all $1 \leq j \leq 5$ and*

$$\varliminf_{r\to\infty} \left[\sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) \right] > \frac{1}{2} \tag{1.1}$$

then $f(z) \equiv g(z)$.

In 2007, Chen, Chen and Tsai [1, Theorem A] extended Theorem 1.A by considering $f(z)$ and $g(z)$ partially sharing more than five values and proved the following theorems.

Theorem 1.B. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_1, a_2, \cdots, a_k$ be k distinct values, where $k \geq 5$ and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$ for all $1 \leq j \leq k$. If*

$$\varliminf_{r\to\infty} \left[\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right) \right] > \frac{1}{k-3} \tag{1.2}$$

then $f(z) \equiv g(z)$.

It is a natural question to ask if $f^{(n)}(z)$ and $g^{(n)}(z)$ partially share more than five values for a positive integer n, what the corresponding inequality becomes?
In this paper, we answer this question by proving the following theorem.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_1, a_2, \cdots, a_k$ be k distinct complex numbers, where $k \geq 5$ and for a non-negative integer n, if*

$$E\left(a_j, f^{(n)}\right) \subseteq E\left(a_j, g^{(n)}\right) \qquad \textit{for all } 1 \leq j \leq k \tag{1.3}$$

$$E(0, f) \subseteq E\left(0, f^{(n)}\right) \quad \textit{and} \quad E(0, g) \subseteq E\left(0, g^{(n)}\right) \tag{1.4}$$

and

$$\varliminf_{r\to\infty} \left[\sum_{j=1}^{k} N\left(r, \frac{1}{f^{(n)}-a_j}\right) \Big/ \varliminf_{r\to\infty} \sum_{j=1}^{k} N\left(r, \frac{1}{g^{(n)}-a_j}\right) \right] > \frac{n+1}{k-(n+3)} \tag{1.5}$$

then $f^{(n)}(z) \equiv g^{(n)}(z)$.

Remark 1.1. If $n = 0$ in Theorem 1.1, then the conditions $E(0, f) \subseteq E\left(0, f^{(n)}\right)$ and $E(0, g) \subseteq E\left(0, g^{(n)}\right)$ are obvious and hence in this case, Theorem 1.1 reduces to Theorem 1.B.

Remark 1.2. If $n = 0$ and $k = 5$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem 1.A, a result of Yang.

Remark 1.3. If $f^{(n)}$ and $g^{(n)}$ share partially $2n + 5$ complex numbers IM, then $f^{(n)} \equiv g^{(n)}$.

2. Lemmas

In this section, we state the following lemma which will be needed in the proof of Theorem 1.1.

Lemma 2.1. [8; Theorem 3.35] *Let $f(z)$ be a non-constant meromorphic function and b_j $(j = 1, 2, \cdots, q)$ be distinct finite non-zero complex numbers. Then for any positive integer n, we have*

$$\begin{aligned} qT(r, f) \quad < \quad & \overline{N}(r, f) + qN\left(r, \frac{1}{f}\right) + \sum_{j=1}^{q} N\left(r, \frac{1}{f^{(n)} - b_j}\right) \\ & - \left[(q-1)N\left(r, \frac{1}{f^{(n)}}\right) + N\left(r, \frac{1}{f^{(n+1)}}\right)\right] + S(r, f) \end{aligned} \tag{2.1}$$

3. Proof of Theorem 1.1

By the Lemma 2.1, we have

$$\begin{aligned} (k-2)T(r, f) \quad < \quad & \overline{N}(r, f) + (k-2)N\left(r, \frac{1}{f}\right) + \sum_{j=1}^{k-2} N\left(r, \frac{1}{f^{(n)} - a_j}\right) \\ & -(k-3)N\left(r, \frac{1}{f^{(n)}}\right) + S(r, f) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(k-2)T(r,g) &< \overline{N}(r,g)+(k-2)N\left(r,\frac{1}{g}\right)+\sum_{j=1}^{k-2}N\left(r,\frac{1}{g^{(n)}-a_j}\right)\\ &\quad -(k-3)N\left(r,\frac{1}{g^{(n)}}\right)+S(r,g) \qquad (3.2)\end{aligned}$$

Using (1.4), (3.1) and (3.2) reduces to

$$(k-2)T(r,f) < \overline{N}(r,f)+N\left(r,\frac{1}{f^{(n)}}\right)+\sum_{j=1}^{k-2}N\left(r,\frac{1}{f^{(n)}-a_j}\right)+S(r,f) \quad (3.3)$$

and

$$(k-2)T(r,g) < \overline{N}(r,g)+N\left(r,\frac{1}{g^{(n)}}\right)+\sum_{j=1}^{k-2}N\left(r,\frac{1}{g^{(n)}-a_j}\right)+S(r,g). \quad (3.4)$$

Without loss of generality, we may assume $a_k=\infty$ and $a_{k-1}=0$.
First we may assume that all a_j $(1\le j\le k)$ in (1.3) are finite. Then by (3.3) and (3.4), we have

$$(k-3)T(r,f) < \sum_{j=1}^{k-1}N\left(r,\frac{1}{f^{(n)}-a_j}\right)+S(r,f) \qquad (3.5)$$

and

$$(k-3)T(r,f) < \sum_{j=1}^{k-1}N\left(r,\frac{1}{f^{(n)}-a_j}\right)+S(r,g). \qquad (3.6)$$

Assume $f^{(n)}(z)\not\equiv g^{(n)}(z)$. Then from (1.3), we have

$$\begin{aligned}\sum_{j=1}^{k-1}N\left(r,\frac{1}{f^{(n)}-a_j}\right) &\le N\left(r,\frac{1}{f^{(n)}-g^{(n)}}\right)\\ &\le T\left(r,f^{(n)}\right)+T\left(r,g^{(n)}\right)+O(1)\\ &\le (n+1)\left[T(r,f)+T(r,g)\right]+O(1)\end{aligned}$$

Again from (3.5) and (3.6), we have

$$\sum_{j=1}^{k-1} N\left(r, \frac{1}{f^{(n)}-a_j}\right)$$
$$\leq \left(\frac{n+1}{k-3}+o(1)\right)\left[\sum_{j=1}^{k-1} N\left(r, \frac{1}{f^{(n)}-a_j}\right)+\sum_{j=1}^{k-1} N\left(r, \frac{1}{g^{(n)}-a_j}\right)\right] (r \notin E)$$

Therefore,

$$\left(\frac{k-(n+4)}{k-3}+o(1)\right)\sum_{j=1}^{k-1} N\left(r, \frac{1}{f^{(n)}-a_j}\right)$$

$$\leq \left(\frac{n+1}{k-3}+o(1)\right)\sum N\left(r, \frac{1}{g^{(n)}-a_j}\right) \quad (r \notin E)$$

It follows that,

$$\frac{\underline{\lim}_{r\to\infty}\sum_{j=1}^{k-1} N\left(r, \frac{1}{f^{(n)}-a_j}\right)}{\underline{\lim}_{r\to\infty}\sum_{j=1}^{k-1} N\left(r, \frac{1}{g^{(n)}-a_j}\right)} \leq \frac{n+1}{k-(n+4)} \tag{3.7}$$

Relace $k-1$ by k in (3.7), we get

$$\frac{\underline{\lim}_{r\to\infty}\sum_{j=1}^{k} N\left(r, \frac{1}{f^{(n)}-a_j}\right)}{\underline{\lim}_{r\to\infty}\sum_{j=1}^{k} N\left(r, \frac{1}{g^{(n)}-a_j}\right)} \leq \frac{n+1}{k-(n+3)} \tag{3.8}$$

where a_k is finite (since all a_j $(1 \leq j \leq k)$ are finite).
(3.8) contradicts to (1.5) and hence $f^{(n)}(z) \equiv g^{(n)}(z)$. Now assume that one of the a_j $(1 \leq j \leq k)$ in (1.3) is infinity say $a_k = \infty$. Taking any finite value a such that $a \neq a_j$ $(1 \leq j \leq k-1)$. Set

$$F^{(n)}(z) = \frac{1}{f^{(n)}-a}, \qquad G^{(n)}(z) = \frac{1}{g^{(n)}-a}$$

Put $b_j = \frac{1}{a_j - a} \quad (1 \leq j \leq k-1) \quad \text{and} \quad b_k = 0.$

So, $F^{(n)}(z)$ and $G^{(n)}(z)$ partially share finite values b_j $(1 \leq j \leq k)$ IM. Then by the above case $F^{(n)}(z) \equiv G^{(n)}(z)$.

This completes proof of Theorem 1.1.

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