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Inverse Problem Having Special Singularity Type From Two Spectra *

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Abstract

It is well known that the two spectra $\{\lambda_n\}$ and $\{\mu_n\}$ uniquely determine the potential function $q(x)$ in a Sturm-Liouville equation defined on the unit interval and having of the special singularity type $q(x) = \frac{\delta}{x^p} + q_0(x)$ (where δ is an constant, 1<p<2) at the point zero. In this work, we give the solution of the inverse problem on two partially non-coinciding spectra for the Sturm-Liouville equation with the peculiarity at zero. In particular in this case we obtain Hochstadt's theorem concerning the structure of the difference $q(x) - \tilde{q}(x)$.

Keywords and Phrases: *Inverse problem, Singular Sturm-Liouville operator, Spectra, Hochstadt theorem.*

1. Introduction

Sturm-Liouville Problems have been an important research issue in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Particularly in relation to our article, Hochstadt showed that the potential is an even function, the potential is uniquely determined [10]. Many scientists have been studing on the problems who use different spectral data. Some of them are [1-19].

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We consider the singular problem

$$Ly = -y'' + \left[\frac{\delta}{x^p} + q_0(x)\right] y = \mu y \qquad \left(\mu = \lambda^2,\, 0 \leqslant x \leqslant \pi\right) \tag{1.1}$$

$$y(0) = 0, \tag{1.2}$$

$$y(\pi,\lambda)\cos\alpha + y'(\pi,\lambda)\sin\alpha = 0. \tag{1.3}$$

Let $\lambda_0 < \lambda_1 < ... < \lambda_n < ...$ be the spectrum of the problem (1.1)-(1.3), and $\mu_0 < \mu_1 < ... < \mu_n < ...$ be the spectrum of the problem

$$Ly = -y'' + \left[\frac{\delta}{x^p} + q_0(x)\right] y = \mu y \qquad \left(\mu = \lambda^2,\, 0 \leqslant x \leqslant \pi\right) \tag{1.4}$$

$$y(0) = 0, \tag{1.5}$$

$$y(\pi,\lambda)\cos\gamma + y'(\pi,\lambda)\sin\gamma = 0, \tag{1.6}$$

where the real potential $q(x)$ satisfy the condition

$$\int_0^\pi x\,|q(x)|\,dx < \infty, \tag{1.7}$$

and δ =constant, $q_0(x) \in L_2[0,\pi]$, $1 < p < 2$, $q(x) = \frac{\delta}{x^p} + q_0(x)$.

It is well known [7] that a knowledge of the two spectra $\{\lambda_n\}$ and $\{\mu_n\}$ uniquely determines the operator L. Note that this type of inverse problems for singular Sturm-Liouville operator was investigated in [6] and [8]. It is also well known that the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are alternating and conform to the asymptotics:

$$\lambda_n = n - \frac{1}{2} - \frac{h}{\pi\left(n - \frac{1}{2}\right)} + O\left(\frac{1}{n^{4-2p}}\right) \tag{1.8}$$

$$\mu_n = n - \frac{1}{2} - \frac{h_1}{\pi\left(n - \frac{1}{2}\right)} + O\left(\frac{1}{n^{4-2p}}\right). \tag{1.9}$$

The converse is also true; if the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are alternating and satisfy the asymptotics, there exists a function $q(x)$ and α, γ $(\alpha \neq \gamma)$ such that $\{\lambda_n\}$ is the spectrum of the problem (1.1)-(1.3) and the $\{\mu_n\}$ is the spectrum of the problem (1.4)-(1.6).

From the latter result it follows that if we fix one of the sequences, say $\{\mu_n\}$ and change an arbitrary number of the first terms of the second sequence $\{\lambda_n\}$ so that the new sequence $\left\{\tilde{\lambda}_n\right\}$ $\left(\lambda_n = \tilde{\lambda}_n \quad for \quad n > N\right)$ alternates with the fixed sequence $\{\mu_n\}$. We can find a function $\tilde{q}(x)$ such that the sequence $\left\{\tilde{\lambda}_n\right\}$ is the spectrum of the problem

$$\tilde{L}y = -y'' + \left[\frac{\delta}{x^p} + \tilde{q}_0(x)\right] y = \mu y \qquad \left(\mu = \lambda^2,\, 0 \leqslant x \leqslant \pi\right) \tag{1.1'}$$

$$y(0) = 0, \tag{1.2'}$$

$$y(\pi,\lambda)\cos\alpha + y'(\pi,\lambda)\sin\alpha = 0 \tag{1.3'}$$

and $\{\tilde{\mu}_n\}$ is the spectrum of the problem

$$\tilde{L}y = -y'' + \left[\frac{\delta}{x^p} + \tilde{q}_0(x)\right] y = \mu y, \qquad \left(\mu = \lambda^2, 0 \leqslant x \leqslant \pi\right) \tag{1.4.'}$$

$$y(0) = 0, \tag{1.5'}$$

$$y(\pi,\lambda)\cos\gamma + y'(\pi,\lambda)\sin\gamma = 0, \tag{1.6'}$$

where

$$\int_0^\pi x\,|\tilde{q}(x)|\,dx < \infty, \tag{1.7'}$$

$\delta =$ constant and $\tilde{q}_0(x) \in L_2[0,\pi]$, $\qquad 1 < p < 2, \qquad \tilde{q}(x) = \frac{\delta}{x^p} + \tilde{q}_0(x)$.

Since the problems (1.1$'$)-(1.3$'$) and (1.4$'$)-(1.6$'$) are obtained from the problems (1.1)-(1.3) and (1.4)-(1.6) by changing a finite number of the parameters, we can regard the first of these problems as a finite-dimensional perturbation of the second set of problems.

Now let's examine asymptotic behaviour of function $G(x,s,\lambda)$ for $|\lambda| \longrightarrow \infty$ in the region

$$\Gamma_n(2) = \begin{cases} |\mathrm{Re}\,\lambda| \le n \\ |\mathrm{Im}\,\lambda| \le n \end{cases}$$

where $\lambda = \tau^2$, $\tau = \sigma + it$.

Before proving the main theorem of the present paper,we prove the following lemma.

Lemma 1.1. *For function $G(x,s,\lambda)$ on the region $\Gamma_n(2)$ one has the following inequality,*

$$|G(x,s,\lambda)| \leq M\frac{e^{-|\operatorname{Im}\tau||x-s|}}{|\tau|}, \tag{1.10}$$

where M is a positive constant and independent from x, s and n.

Proof. We denote by $\varphi(x,\lambda)$ the soluion of (1.1) satisfying the initial condition (1.2). Then for large $|\lambda|$

$$|\varphi(x,\lambda)| \leq M_1\frac{e^{|\operatorname{Im}\tau|x}}{|\tau|}, \tag{1.11}$$

$$|\varphi(x,\lambda)| \leq \tilde{M}_1.x.e^{|\operatorname{Im}\tau|x}. \tag{1.12}$$

Let $\psi(x,\lambda)$ the solution of the equation (1.1) satisfying the conditions.

Thus

$$\psi(\pi,\lambda)=1, \qquad \psi'(\pi,\lambda)=h. \tag{1.13}$$

$$-\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda\psi(x,\lambda) \qquad 0\leq x\leq\pi \tag{1.14}$$

now let us obtain the necessary inequality for $\psi(x,\lambda)$. In (1.14), if we replace x for $\pi-x$, then we obtain the following equation

$$-\psi''(\pi-x,\lambda)+q(\pi-x)\psi(\pi-x,\lambda)=\lambda\psi(\pi-x,\lambda) \qquad 0\leq x\leq\pi. \tag{1.15}$$

Let

$$\theta(x,\lambda)=\psi(\pi-x,\lambda).$$

Hence

$$\theta'(x,\lambda)=-\psi'(\pi-x,\lambda) \tag{1.16}$$

$$\theta''(x,\lambda)=\psi''(\pi-x,\lambda). \tag{1.17}$$

Therefore using

$$q_1(x)=q(\pi-x),$$

then we find

$$-\theta''(x,\lambda)+q_1(x)\theta(x,\lambda)=\lambda\theta(x,\lambda). \tag{1.18}$$

Now we obtain the potential $q_1(x)$ having a singular point at $x = \pi$ and $(x-\pi)\, q_1(x) \in L_1(0,\pi)$. Furthermore from (1.13) we obtain

$$\theta(0,\lambda) = 1 \qquad \theta'(0,\lambda) = -h, \tag{1.19}$$

and

$$\theta(x,\lambda) = \cos\tau x + O\left(\frac{e^{|\operatorname{Im}\tau| x}}{|\tau|}\right) \tag{1.20}$$

is the solution of (1.1). From (1.15), we obtain the following equality

$$\psi(\pi - x,\lambda) = \cos\tau x + O\left(\frac{e^{|\operatorname{Im}\tau| x}}{|\tau|}\right). \tag{1.21}$$

Replacing x for $\pi - x$, we get

$$\psi(x,\lambda) = \cos\tau(\pi - x) + O\left(\frac{e^{|\operatorname{Im}\tau|(\pi - x)}}{|\tau|}\right). \tag{1.22}$$

Therefore from the last equality, it follows for large $|\lambda|$

$$|\psi(x,\lambda)| \le M_2 e^{|\operatorname{Im}\tau|(\pi - x)}, \tag{1.23}$$

$$w(\lambda) = \cos\tau\pi + O\left(e^{|\operatorname{Im}\tau|\pi}\right). \tag{1.24}$$

Thus on the counters $\Gamma_n(2)$, we can get the following inequality

$$|\cos\lambda\pi| \ge c e^{|\operatorname{Im}\tau|\pi}. \tag{1.25}$$

Then we can write $w(\lambda)$ as follows

$$w(\lambda) = \cos\tau\pi[1 + o(1)]. \tag{1.26}$$

Thus for large $|\tau|$, $[1 + o(1)] > \frac{1}{2}$, by virtue of the inequality (1.26) for large n, we obtain from (1.24)

$$|w(\lambda)| \ge M_3 e^{|\operatorname{Im}\tau|\pi}, \qquad \forall\tau \in \Gamma_n \tag{1.27}$$

where $M_3-$is the constant independent from n.

Therefore for the function $G(x,s,\lambda)$ for $\tau \in \Gamma_n$ and $x \leq s$ we have the following estimate

$$\begin{aligned} |G(x,s,\lambda)| &\leq \frac{1}{|w(\lambda)|}|\varphi(x,\lambda)||\psi(s,\lambda)| \\ &= M\frac{e^{-|\operatorname{Im}\tau|(x-s)}}{|\tau|}. \end{aligned} \tag{1.28}$$

Furthermore, for $x \geq s$ we obtain

$$\begin{aligned} |G(x,s,\lambda)| &\leq \frac{1}{|w(\lambda)|}|\varphi(s,\lambda)||\psi(x,\lambda)| \\ &= M\frac{e^{-|\operatorname{Im}\tau||x-s|}}{|\tau|}. \end{aligned}$$

Where $M = \frac{M_1 M_2}{M_3}$ is constant and independent from x, s and n.

Consequently we have the following asymptotic formula

$$G(x,s,\lambda) = O\left(\frac{e^{-|\operatorname{Im}\tau||x-s|}}{|\tau|}\right). \tag{1.29}$$

In [10] Hochstadt showed inverse problem for Regular Sturm Liouville equation according to eigenvalues. The purpose of our paper is to give the sctructure concerning the difference $q(x) - \tilde{q}(x)$ for the differential operators having the singularity type $\frac{\delta}{x^p} + q_0(x)$, by using

Hochstadt method. Now, let's give the main theorem and its proof.

Main Results

2. The Theorem of Hochstadt for Singular Sturm-Liouville Operator

Consider the operator

$$Ly = -y'' + \left[\frac{\delta}{x^p} + q_0(x)\right]y = \mu y \qquad \left(\mu = \lambda^2, 0 \leqslant x \leqslant \pi\right), \tag{2.1}$$

subject to the boundary conditions

$$y\,(0) = 0, \tag{2.2}$$

$$y(\pi,\lambda)\cos\alpha + y'(\pi,\lambda)\sin\alpha = 0, \tag{2.3}$$

where

$$\int_0^\pi x\,|q\,(x)|\,dx < \infty,$$

and $q\,(x)$ is square integrable real function on $[0,\pi]$. Let $\{\lambda_n\}$ be the spectrum of L with (2.2) and (2.3) boundary conditions. The condition (2.3) is replaced by a new boundary condition

$$y(\pi,\lambda)\cos\gamma + y'(\pi,\lambda)\sin\gamma = 0, \tag{2.4}$$

where $\sin(\alpha-\gamma) \neq 0$. Let $\{\mu_n\}$ be the spectrum of L with (2.2) and (2.4) conditions.

Consider now a second operator

$$\tilde{L}y = -y'' + \left[\frac{\delta}{x^p} + \tilde{q}_0\,(x)\right] y = \mu y \qquad \left(\mu = \lambda^2, 0 \leqslant x \leqslant \pi\right) \tag{2.5}$$

where

$$\int_0^\pi x\,|\tilde{q}\,(x)|\,dx < \infty$$

and $\tilde{q}\,(x)$ is a square integrable real function on $[0,\pi]$. Suppose that, under the boundary conditions (2.2) and (2.4), $\tilde{L}$ has the spectrum $\left\{\tilde{\lambda}_n\right\}$, with $\lambda_n = \tilde{\lambda}_n$ for all except a finite number of values of n. $\tilde{L}$ with the boundary conditions (2.2) and (2.4) is assumed to have the spectrum $\{\tilde{\mu}_n\}$.

Thus, we assume that the second-named spectra, i.e $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$, coincide, and that $\{\lambda_n\}$ and $\left\{\tilde{\lambda}_n\right\}$ differ in a finite number of their terms. We shall denote by Λ_0 the finite index set for which $\lambda_n \neq \tilde{\lambda}_n$ and by Λ the infinite index set for which $\lambda_n = \tilde{\lambda}_n$. Under the assumptions, we can give the following theorem.

Main Theorem. *If the spectra $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ coincide and the spectra $\{\lambda_n\}$ and $\left\{\tilde{\lambda}_n\right\}$ differ in a finite number of their terms, then*

$$q-\tilde{q}=\sum_{\Lambda_0}\left(\tilde{\psi}_n\varphi_n\right)' \tag{2.6}$$

where $\widetilde{\psi}_n$ *and* φ_n *are suitable solutions of the following equations, respectively*

$$\tilde{\psi}_n'' + \left[\lambda^2 - \frac{\delta}{x^p} - \tilde{q}_0(x)\right]\tilde{\psi}_n = 0,$$

$$\varphi_n'' + \left[\lambda^2 - \frac{\delta}{x^p} - q_0(x)\right]\varphi_n = 0.$$

In particularly, if Λ_0 *is empty and* $q=\tilde{q}$ *almost everywhere.*

Proof. Let $\{\varphi_n\}\,(\{\tilde{\varphi}_n\})$ denote the set of eigenfunctions of the operator defined by (2.1) ((2.5)) with the (2.2) and (2.3) boundary conditions. We define two Hilbert spaces that are subspaces of $L_2[0,\pi]$ as follows:

$$H=\{\, f\in L_2[0,\pi] \mid \langle f,\varphi_n\rangle = 0, \quad n\in\Lambda_0\},$$

$$\tilde{H}=\{\, f\in L_2[0,\pi] \mid \langle f,\tilde{\varphi}_n\rangle = 0, \quad n\in\Lambda_0\}.$$

By hypothesis, the spectrum of L restricted to H and the spectrum of $\tilde{L}$ restricted to $\tilde{H}$ coincide.

We define an operator T that maps H into $\tilde{H}$ by

$$T\varphi_n=\tilde{\varphi}_n, \qquad n\in\Lambda. \tag{2.7}$$

Let $f\in H$ and expand f in terms of the set $\{\varphi_n\}$ so that

$$f=\sum_{\Lambda} f_n\varphi_n. \tag{2.8}$$

If λ is an arbitrary element in the complex plane, different from all the eigenvalues, the operator $(\lambda-L)^{-1}$ exists and it is bounded. It can be written

$$(\lambda-L)^{-1}f=\sum_{\Lambda}\frac{f_n\varphi_n}{(\lambda-\lambda_n)}. \tag{2.9}$$

if we apply T , we obtain

$$T(\lambda-L)^{-1}f=\sum_{\Lambda}\frac{f_n\tilde{\varphi}_n}{(\lambda-\lambda_n)}. \tag{2.10}$$

Since

$$\tilde{L}\tilde{\varphi}_n = \lambda_n \widetilde{\varphi}_n$$
$$\left(\lambda - \tilde{L}\right) T \left(\lambda - L\right)^{-1} f = Tf,$$

it follows that L, $\tilde{L}$ and T are related by

$$\left(\lambda - \tilde{L}\right) T \left(\lambda - L\right)^{-1} = T. \tag{2.11}$$

By means of the second spectrum, we can determine the structure of T. When T is established, conclusion of theorem can be deduced. One has the following asymptotic formula [7,8] as $\lambda \to \infty$

$$\varphi\left(x, \lambda\right) = \frac{\sin \lambda x}{\lambda} + O\left(\frac{e^{|\operatorname{Im}\lambda| x}}{|\lambda|^{5-2p}}\right) \tag{2.12}$$

$$\varphi'\left(x, \lambda\right) = \cos \lambda x + O\left(\frac{e^{|\operatorname{Im}\lambda| x}}{|\lambda|^{4-2p}}\right). \tag{2.13}$$

We define two functions of λ by using the boundary conditions (2.3) and (2.4)

$$w\left(\lambda\right) = \sin \alpha \varphi'\left(\pi, \lambda\right) + \cos \alpha \varphi\left(\pi, \lambda\right), \tag{2.14}$$

$$v\left(\lambda\right) = \sin \gamma \varphi'\left(\pi, \lambda\right) + \cos \gamma \varphi\left(\pi, \lambda\right). \tag{2.15}$$

Using asymptotic results, we conclude that φ and φ' are entire functions of λ. It is well known [12]. That $w\left(\lambda\right)$ and $v\left(\lambda\right)$ determined by (2.14) and (2.15) are two entire analytic functions and therefore they can also be determined by their zeros. As follows

$$w\left(\lambda\right) = a\prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right), \tag{2.16}$$

$$v\left(\lambda\right) = b\prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\mu_n}\right), \tag{2.17}$$

where a and b are constants. Similarly, we can write

$$\tilde{w}\left(\lambda\right) = \tilde{a}\prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\tilde{\lambda}_n}\right), \tag{2.18}$$

$$\tilde{v}(\lambda) = \tilde{b}\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\tilde{\mu}_n}\right), \quad (\tilde{\mu}_n = \mu_n,\ n = 1,2,...). \tag{2.19}$$

By hypothesis, $\frac{v(\lambda)}{\tilde{v}(\lambda)}$ is constant and $\frac{w(\lambda)}{\tilde{w}(\lambda)}$ is a rational function.

If β is different from zero, we shall show that T is a bounded and invertible operator. Then using the asymptotic formulas

$$\lambda_n = n - \frac{1}{2} + O(\tau_n)\ , n \to \infty, \quad O(\tau_n) \to 0 \tag{2.20}$$

$$\alpha_n = \frac{\pi}{2\left(n-\frac{1}{2}\right)^2} + O\left(\frac{\tilde{\tau}_n}{n^2}\right), \tag{2.21}$$

where $\tau_n = \int_0^{\frac{2}{n}} x\,|q(x)|\,dx + \frac{1}{n}\int_{\frac{1}{2n}}^{\pi} |q(x)|\,dx$, (see [7] and [8]) and using the asymptotic results for large λ,

$$w(\lambda) = \sin\alpha\cos\lambda\pi + \frac{\sin\lambda\pi}{\lambda}\cos\alpha + O\left(\frac{e^{|\mathrm{Im}\,\lambda|x}}{|\lambda|^{5-2p}}\right) \tag{2.22}$$

we obtain

$$\|\varphi_n\|^2 = \int_0^{\pi} \varphi_n^2(x)\,dx = \frac{\pi^2}{2\left(n-\frac{1}{2}\right)^2} + O\left(\frac{\tau_n}{n^2}\right). \tag{2.23}$$

Similarly, it can be written as

$$\|\tilde{\varphi}_n\|^2 = \int_0^{\pi} \tilde{\varphi}_n^2(x)\,dx = \frac{\pi^2}{2\left(n-\frac{1}{2}\right)^2} + O\left(\frac{\tilde{\tau}_n}{n^2}\right). \tag{2.24}$$

From the equality (2.23) and (2.24) we conclude that

$$\lim_{n\to\infty}\frac{\|\varphi_n\|}{\|\tilde{\varphi}_n\|} = 1.$$

It follows that $\frac{\|T\varphi_n\|}{\|\varphi_n\|}$ is uniformly bounded so that T is bounded and has an inverse.

Let f be an arbitrary function in H. Then,

$$f_n = \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2}, \tag{2.25}$$

and

$$(\lambda - L)^{-1} f = \sum_{\Lambda} \frac{f_n \varphi_n}{(\lambda - \lambda_n)}, \tag{2.26}$$

provided $\lambda \neq \lambda_n$ for all n. To verify the above equation we observe that $(\lambda - L)^{-1}$ is a compact operator and $(\lambda - L)^{-1} \varphi_n = (\lambda - \lambda_n)^{-1} \varphi_n$. Now we can write

$$T (\lambda - L)^{-1} f = \sum_{\Lambda} \frac{f_n \tilde{\varphi}_n}{(\lambda - \lambda_n)}. \tag{2.27}$$

The right side of (2.27) is in the range of the unbounded operator $\tilde{L}$. To show this we recall a necessary and sufficient condition for

$$g = \sum_{n=0}^{\infty} g_n \tilde{\varphi}_n$$

to be in the range of $\tilde{L}$ is that

$$\sum_{n=0}^{\infty} \left| \tilde{\lambda}_n g_n \right|^2 < \infty.$$

we see from (2.20) that

$$\sum_{\Lambda} \left| \frac{\lambda_n f_n}{\lambda - \lambda_n} \right|^2 \leq \frac{1}{\min\limits_n |\lambda/\lambda_n - 1|^2} \sum_{\Lambda} |f_n|^2 < \infty.$$

Clearly, $\left(\lambda - \tilde{L}\right) \tilde{\varphi}_n = (\lambda - \lambda_n) \tilde{\varphi}_n$ and

$$\left(\lambda - \tilde{L}\right) T (\lambda - L)^{-1} f = \sum_{\Lambda} f_n \tilde{\varphi}_n = Tf. \tag{2.28}$$

Since (2.28) holds for an arbitrary $f \in H$.

Now, we shall seek a different representation for T. To do this we shall employ the Green's function of the operator L with the boundary conditions (2.2) and (2.3). Let $\psi (x)$ be the solution of the following problem

$$\psi'' + \left[\lambda^2 - \frac{\delta}{x^p} - q_0 (x)\right] \psi = 0 \tag{2.29}$$

$$\psi(\pi, \lambda) = -\sin\alpha, \quad \psi'(\pi, \lambda) = \cos\alpha$$

and $y(x)$ be solution of the equation

$$y'' + \left[\lambda^2 - \frac{\delta}{x^p} - q_0(x)\right] y = f(x).$$

The Green's function $G(x, s)$ is given by

$$(\lambda - L)\int_0^1 G(x, s) f(s)\, ds = f(x). \tag{2.30}$$

By a standard computation, we can easily show that

$$G(x, s) = \begin{cases} \frac{\varphi(x)\psi(s)}{w(\lambda)}, & x < s \\ \frac{\varphi(s)\psi(x)}{w(\lambda)}, & s < x \end{cases}, \tag{2.31}$$

where $\varphi(x)$ is the solution of (1.1), $\psi(x)$ is the solution of (2.29) and $w(\lambda)$ satisfies (2.14). A more convenient notation for $G(x, s)$ is written as

$$G(x, s) = \frac{\varphi(x_<)\,\phi(x_>)}{w(\lambda)}, \tag{2.32}$$

where

$$x_< = \min(x, s) \quad , \quad x_> = \max(x, s). \tag{2.33}$$

By the asymptotic results, both the numerator and the denominator of $G(x, s)$ are entire functions of λ of order $\frac{1}{2}$. Then for large λ, bounded away from the poles of $G(x, s)$, we have (1.27) from Lemma 1.1.

Let $\{\Gamma_n\}$ be a sequence of square which intersects the positive λ-axis between λ_n and λ_{n+1}. Then,

$$\lim_{n\to\infty}\int_{\Gamma_n} \frac{G(x, s)}{\lambda - \mu} d\mu = 0. \tag{2.34}$$

By using a residue integration, it follows that

$$\frac{1}{2\pi i}\int_{\Gamma_n} \frac{G(x, s)}{\lambda - \mu} d\mu = -G(x, s) + \sum_{k=0}^{n} \frac{\varphi_n(x_<)\,\psi_n(x_>)}{w'(\lambda_n)(\lambda - \lambda_n)}. \tag{2.35}$$

From (2.34) and (2.35) we see that the Mittag-Leffler expansion [12]. For $G(x,s)$ as a function of λ is

$$G(x,s)=\sum_{n=0}^{\infty}\frac{\varphi_n(x_<)\,\psi_n(x_>)}{w'(\lambda_n)(\lambda-\lambda_n)}, \tag{2.36}$$

where φ_n and ψ_n are both eigenfunctions corresponding to the simple eigenvalues λ_n. Therefore, these functions are linearly dependent. Then

$$k_n\varphi_n(x)=\psi_n(x), \tag{2.37}$$

for $x=\pi$,

$$k_n=\begin{cases}-\frac{\sin\alpha}{\varphi_n(\pi)} & , \quad \alpha\neq 0\\ \frac{1}{\varphi_n'(\pi)} & , \quad \alpha=0\end{cases} \tag{2.38}$$

From (2.30) we obtain

$$(\lambda-L)^{-1}f=\int_0^{\pi}G(x,s)\,f(s)\,ds. \tag{2.39}$$

Hence,

$$(\lambda-L)^{-1}f=\sum_{\Lambda}\frac{k_n\varphi_n(x)\int_0^{\pi}\varphi_n(s)\,f(s)\,ds}{w'(\lambda_n)(\lambda-\lambda_n)}. \tag{2.40}$$

Now, using (2.25), (2.40) reduces to

$$(\lambda-L)^{-1}f=\sum_{\Lambda}\frac{k_n\varphi_n(x)\,f_n\,\|\varphi_n\|^2}{w'(\lambda_n)(\lambda-\lambda_n)}. \tag{2.41}$$

Comparing (2.26) with (2.41), we see that

$$\|\varphi_n\|^2=\frac{w'(\lambda_n)}{k_n}. \tag{2.42}$$

So, from (2.32) and (2.39) we obtain

$$(\lambda-L)^{-1}f=\frac{\psi(x)\int_0^{x}\varphi(s)\,f(s)\,ds+\varphi(x)\int_x^{\pi}\psi(s)\,f(s)\,ds}{w(\lambda)}. \tag{2.43}$$

Our next claim is that

$$T\left(\lambda-L\right)^{-1}f=\sum_{\Lambda}\frac{\tilde{\psi}_{n}\left(x\right)\int\limits_{0}^{x}\varphi_{n}\left(s\right)f\left(s\right)ds+\tilde{\varphi}_{n}\left(x\right)\int\limits_{x}^{\pi}\psi_{n}\left(s\right)f\left(s\right)ds}{w'\left(\lambda_{n}\right)\left(\lambda-\lambda_{n}\right)},\quad(2.44)$$

where $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ are solutions of (1.1) and (2.29), respectively, with $q\left(x\right)$ replaced by $\widetilde{q}\left(x\right)$. To prove this we have to show that the right side of (2.44) and (2.27) coincide. Using the equalities,

$$\begin{aligned}\psi_{n}\left(x\right)&=k_{n}\varphi_{n}\left(x\right),\\ \tilde{\psi}_{n}\left(x\right)&=\tilde{k}_{n}\tilde{\varphi}_{n}\left(x\right).\end{aligned}\quad(2.45)$$

We transform, the right side of (2.44) to the form,

$$\sum_{\Lambda}\frac{\tilde{k}_{n}\tilde{\varphi}_{n}\left(x\right)\int\limits_{0}^{x}\varphi_{n}\left(s\right)f\left(s\right)ds+k_{n}\tilde{\varphi}_{n}\left(x\right)\int\limits_{x}^{\pi}\varphi_{n}\left(s\right)f\left(s\right)ds}{w'\left(\lambda_{n}\right)\left(\lambda-\lambda_{n}\right)},\quad(2.46)$$

and if $k_n=\tilde{k}_n$, (2.42) reduces to $\sum\limits_{\Lambda}\frac{k_n\tilde{\varphi}_n(x)\int\limits_0^{\pi}\varphi_n(s)f(s)ds}{w'(\lambda_n)(\lambda-\lambda_n)}=\sum\limits_{\Lambda}\frac{\tilde{\varphi}_n(x)\int\limits_0^{\pi}\varphi_n(s)f(s)ds}{\|\varphi_n\|^2(\lambda-\lambda_n)}$. Now, we have to show that $k_n=\tilde{k}_n$. So far, the second spectrum has not been used. It must be used only in this step. We return to (2.14) and (2.15) and let $\lambda=\lambda_n$. Since $w\left(\lambda_n\right)=0$, these reduce to

$$\varphi_n(\pi,\lambda)\cos\alpha+\varphi_n'(\pi,\lambda)\sin\alpha=0,\quad(2.48)$$

$$\varphi_n(\pi,\lambda)\cos\gamma+\varphi_n'(\pi,\lambda)\sin\gamma=v\left(\lambda_n\right),\quad(2.49)$$

and solving for $\varphi_n\left(\pi,\lambda\right)$, we find that

$$\varphi_n(\pi)=-\frac{\sin\alpha v\left(\lambda_n\right)}{\sin\left(\alpha-\gamma\right)}\quad,\quad(2.50)$$

and by using (2.38), we can obtain

$$k_n=\frac{\sin\left(\gamma-\alpha\right)}{v\left(\lambda_n\right)}.\quad(2.51)$$

Similarly,

$$\tilde{k}_n = \frac{\sin(\gamma - \alpha)}{\tilde{v}(\lambda_n)}. \tag{2.52}$$

Using of asymptotic formulas shows that $v(\lambda)$ and $\tilde{v}(\lambda)$ have the same asymptotic forms. Then we have $k_n = \tilde{k}_n$ for $n \in \Lambda$. So that (2.44) holds. From the formulas (2.11) and (2.44), we obtain

$$Tf = \left(\lambda - \tilde{L}\right) \sum_{\Lambda} \frac{\tilde{\psi}_n(x) \int_0^x \varphi_n(s) f(s)\, ds + \tilde{\varphi}_n(x) \int_x^\pi \psi_n(s) f(s)\, ds}{w'(\lambda_n)(\lambda - \lambda_n)}. \tag{2.53}$$

We calculate

$$g(x) = \frac{\tilde{\psi}(x) \int_0^x \varphi(s) f(s)\, ds + \tilde{\varphi}(x) \int_x^\pi \psi(s) f(s)\, ds}{w(\lambda)}. \tag{2.54}$$

By virtue of Mittag-Leffler expansion of $g(x)$, we get the following equation:

$$\begin{aligned} g(x) &= \sum_{\Lambda_0} \frac{\tilde{u}_n(x) \int_0^x \varphi_n(s) f(s)\, ds + \tilde{z}_n(x) \int_x^\pi \psi_n(s) f(s)\, ds}{w'(\lambda_n)(\lambda - \lambda_n)} \\ &\quad + \sum_{\Lambda} \frac{\psi_n(x) \int_0^x \varphi_n(s) f(s)\, ds + \tilde{\varphi}_n(x) \int_x^\pi \psi_n(s) f(s)\, ds}{w'(\lambda_n)(\lambda - \lambda_n)}. \end{aligned} \tag{2.55}$$

The second summation is $T(\lambda - L)^{-1} f$ as in (2.44). In the first term $\tilde{u}_n(x)$ and $\tilde{z}_n(x)$ represent $\tilde{\psi}(x)$ and $\tilde{\varphi}(x)$ evaluated at λ_n, respectively.

Hence,

$$(\lambda - \tilde{L})^{-1} T f = g(x) - \sum_{\Lambda_0} \frac{\tilde{u}_n(x) \int_0^x \varphi_n(s) f(s)\, ds + \tilde{z}_n(x) \int_x^\pi \psi_n(s) f(s)\, ds}{w'(\lambda_n)(\lambda - \lambda_n)} \tag{2.56}$$

$$
\begin{aligned}
(\lambda-\tilde{L})^{-1}Tf \;&=\; \frac{\tilde{\psi}\left(x\right)\int\limits_{0}^{x}\varphi\left(s\right)f\left(s\right)ds+\tilde{\varphi}\left(x\right)\int\limits_{x}^{\pi}\phi\left(s\right)f\left(s\right)ds}{w\left(\lambda\right)} \\
&\quad -\sum_{\Lambda_0}\frac{\tilde{u}_n\left(x\right)\int\limits_{0}^{x}\varphi_n\left(s\right)f\left(s\right)ds+\tilde{z}_n\left(x\right)\int\limits_{x}^{\pi}\phi_n\left(s\right)f\left(s\right)ds}{w'\left(\lambda_n\right)\left(\lambda-\lambda_n\right)}. \qquad (2.57)
\end{aligned}
$$

Applying $(\lambda-\tilde{L})$ to the both sides we observe that it is continuous and it is differentiable. Using the formulas (2.54) and differentiation of the right side of (2.56), we obtain

$$
\begin{aligned}
&\frac{\tilde{\psi}'\left(x\right)\int\limits_{0}^{x}\varphi\left(s\right)f\left(s\right)ds+\tilde{\varphi}'\left(x\right)\int\limits_{x}^{\pi}\psi\left(s\right)f\left(s\right)ds}{w\left(\lambda\right)} \\
&-\sum_{\Lambda_0}\frac{\tilde{u}_n'\left(x\right)\int\limits_{0}^{x}\varphi_n\left(s\right)f\left(s\right)ds+\tilde{z}_n'\left(x\right)\int\limits_{x}^{\pi}\psi_n\left(s\right)f\left(s\right)ds}{w'\left(\lambda_n\right)\left(\lambda-\lambda_n\right)} \\
&+\left\{\frac{\tilde{\psi}\left(x\right)\varphi\left(x\right)-\tilde{\varphi}\left(x\right)\psi\left(x\right)}{w\left(\lambda\right)}-\sum_{\Lambda_0}\frac{\tilde{u}_n\left(x\right)\varphi_n\left(x\right)-\tilde{z}_n\left(x\right)\psi_n\left(x\right)}{w'\left(\lambda_n\right)\left(\lambda-\lambda_n\right)}\right\}f\left(x\right). \qquad (2.58)
\end{aligned}
$$

An inspection of the term in the second braces shows that it vanishes identically. To do this one merely computes the residue at each λ_n and observes that it becomes zero. One can differentiate the expression in the first braces in last expression and then obtain from (2.58)

$$
\begin{aligned}
Tf \;&=\; \left[\frac{\tilde{\psi}'\left(x\right)\varphi\left(x\right)-\tilde{\varphi}'\left(x\right)\psi\left(x\right)}{w\left(\lambda\right)}-\sum_{\Lambda_0}\frac{\tilde{u}_n'\left(x\right)\varphi_n\left(x\right)-\tilde{z}_n'\left(x\right)\psi_n\left(x\right)}{w'\left(\lambda_n\right)\left(\lambda-\lambda_n\right)}\right]f\left(x\right) \\
&\quad -\sum_{\Lambda_0}\frac{\tilde{u}_n\left(x\right)\int\limits_{0}^{x}\varphi_n\left(s\right)f\left(s\right)ds+\tilde{z}_n\left(x\right)\int\limits_{x}^{\pi}\psi_n\left(s\right)f\left(s\right)ds}{w'\left(\lambda_n\right)}. \qquad (2.59)
\end{aligned}
$$

The operator T must be independent of λ. To deduce the value of the expression in the braces in (2.59) we let $\lambda\to\infty$. Using the asymptotic formulas we see that the term in the braces must, in fact, reduce to unity. To simplify the second term in (2.59) we recall that

$$\phi_n = k_n \varphi_n,$$

$$\int_0^\pi \varphi_n(s) f(s)\, ds = 0, \quad n \in \Lambda_0$$

hence,

$$Tf = f - \sum_{\Lambda_0} \frac{\tilde{u}_n(x) - k_n \tilde{z}_n(x)}{w'(\lambda_n)} \int_0^x \varphi_n(s) f(s)\, ds,$$

$$Tf = f - \frac{1}{2} \sum_{\Lambda_0} \tilde{\psi}_n(x) \int_0^x \varphi_n(s) f(s)\, ds, \tag{2.60}$$

where

$$\frac{\tilde{u}_n(x) - k_n \tilde{z}_n(x)}{w'(\lambda_n)} = \frac{1}{2} \tilde{\psi}_n(x).$$

Now, from the (2.11) we conclude that

$$\begin{aligned} \left(\lambda - \tilde{L}\right) T (\lambda - L)^{-1} &= T, \\ \tilde{L}T &= TL, \end{aligned}$$

and this equality holds for each $f \in H$

$$\tilde{L}Tf = TLf. \tag{2.61}$$

We substitute (2.60) into (2.61) and then by straightforward computations, we obtain that

$$\begin{aligned} \tilde{L}Tf &= -(Tf)'' + \left[\frac{\delta}{x^p} + \tilde{q}_0(x)\right] Tf \\ &= -\left[f - \frac{1}{2} \sum_{\Lambda_0} \tilde{\psi}_n(x) \int_0^x \varphi_n(s) f(s)\, ds\right]'' \\ &\quad + \tilde{q}(x) \left[f - \frac{1}{2} \sum_{\Lambda_0} \tilde{\psi}_n(x) \int_0^x \varphi_n(s) f(s)\, ds\right], \end{aligned}$$

and

$$\begin{aligned} TLf &= T\left\{-f'' + \left[\frac{\delta}{x^p} + q_0(x)\right] f\right\} \\ &= -f'' + \left[\frac{\delta}{x^p} + q_0(x)\right] f \\ &\quad -\frac{1}{2}\sum_{\Lambda_0} \tilde{\psi}_n(x) \int_0^x \varphi_n(s) \left\{-f'' + \left[\frac{\delta}{x^p} + q_0(x)\right] f(s)\right\} ds. \end{aligned}$$

Consequently,

$$(q - \tilde{q}) = \sum_{\Lambda_0} \left[\tilde{\psi}_n(x)\, \varphi_n(x)\right]'.$$

If Λ_0 is empty, then T is the identity operator and $L = \tilde{L}$. Hence, $q(x) = \tilde{q}(x)$. This completes the proof of theorem.

Finally, we should note that the similar results were obtained for regular and singular Sturm-Liouville operators with a different method in [12].

Conclusion. In this paper, we give the solution of the inverse Sturm-Liouville problem having special singularity type at zero on two partially coinciding spectra. In this case, we proved Hocshtadt's theorem concerning the structure of the difference $q(x) - \tilde{q}(x)$.

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