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Real Paley-Wiener Theorems for the Modified q-Mellin Transform*

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Abstract

In this paper, a modified q-Mellin transform is introduced and studied, and a Plancherel formula as well as a Hausdorff-Young inequality are shown. Next, new type Paley-Wiener theorems for this transform are established, using real variable methods.

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1. Introduction

In the q-theory, (see [2] and [5]), for a real parameter $q \in]0,1[$, we write

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C},$$

$$(a;q)_0 = 1, \text{ and } (a;q)_n = \prod_{k=0}^{n}(1-aq^k), \quad n \in \mathbb{N} \cup \{\infty\}, \quad a \in \mathbb{C},$$

and

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}.$$

The q-Jackson's integrals from 0 to a, from 0 to ∞ and in a generic interval $[a,b]$ are defined by (see [4])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{1}$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{2}$$

provided the sums converge absolutely, and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{3}$$

Using the q-Jackson's integral, the q-Mellin transform of a function f on $\mathbb{R}_{q,+}$ is defined in [1], by

$$M_q(f)(s) = M_q[f(t)](s) = \int_0^\infty t^{s-1}f(t)d_qt.$$

It is analytic on a strip $\langle \alpha_{q,f}; \beta_{q,f} \rangle$, called the fundamental strip, and it is a periodic function, with period $\dfrac{2i\pi}{\text{Log}q}$. Furthermore, we have

$$\forall n \in \mathbb{N}, \forall s \in \langle \alpha_{q,f}; \beta_{q,f} \rangle, \qquad \frac{d^n}{ds^n}M_q(f)(s) = M_q\left[(\text{Log}(t))^n f(t)\right](s). \tag{4}$$

The inversion formula for this transform is given by

$$f(x) = \frac{\mathrm{Log}(q)}{2i\pi(1-q)} \int_{c-\frac{i\pi}{\mathrm{Log}(q)}}^{c+\frac{i\pi}{\mathrm{Log}(q)}} M_q(f)(s) x^{-s} ds, \quad x \in \mathbb{R}_{q,+},$$

where $\alpha_{q,f} < c < \beta_{q,f}$.
It was shown in [1], that for $c \in \langle \alpha_{q,f}; \beta_{q,f} \rangle \cap \langle 1-\beta_{q,g}; 1-\alpha_{q,g} \rangle$, we have

$$\frac{\mathrm{Log}(q)}{2i\pi(1-q)} \int_{c-i\frac{\pi}{\mathrm{Log}(q)}}^{c+i\frac{\pi}{\mathrm{Log}(q)}} M_q(f)(s) M_q(g)(1-s) ds = \int_0^\infty f(x) g(x) d_q x. \quad (5)$$

The aim of the present paper is to introduce and study a q-analogue of the modified Mellin transform, that will be called modified q-Mellin transform. In particular, we prove for this new transform a Plancherel formula and a Hausdorff-Young inequality. Next, inspired by the ideas developed in [7], we establish for the modified q-Mellin transform some real Paley-Wiener theorems.

This paper is organized as follows: in Section 2, we introduce the modified q-Mellin transform and we prove a Plancherel formula and a Hausdorff-Young inequality. In Section 3, we establish a relation between the support of a function f on $\mathbb{R}_{q,+}$ and differentiability properties of its modified q-Mellin transform. We investigate then the support of a function only in terms of its q-Mellin transform, using real variable techniques.

2. Modified q-Mellin Transform

Notation. The notation $L_q^p(\mathbb{R}_{q,+})$ will stand for the Banach space induced by the norm $\|f\|_{L_q^p(\mathbb{R}_{q,+})} = \left(\int_0^\infty |f(t)|^p d_q t \right)^{\frac{1}{p}}$ and in the presence of a weight, we will write $\|f\|_{L_q^p(\mathbb{R}_{q,+}, w(t) d_q t)} = \left(\int_0^\infty |f(t)|^p w(t) d_q t \right)^{\frac{1}{p}}$.

Definition 1. Let f be a function defined on $\mathbb{R}_{q,+}$. We define the modified q-Mellin transform $M_{q,\gamma}(f)$, $\gamma \in \mathbb{R}$, of f as

$$M_{q,\gamma}(f)(x) = \int_0^\infty t^{\gamma + ix - 1} f(t) d_q t, \quad x \in \mathbb{R}, \quad (6)$$

provided the q-integral converges.

It is clear that $M_{q,\gamma}(f)$ is the restriction of the q-Mellin transform of f on $\gamma+i\mathbb{R}$. So, $M_{q,\gamma}(f)$ is defined on $\mathbb{R}$ if and only if γ is a real in the fundamental strip of $M_q(f)$. In the sequel, we assume that this condition holds. Moreover, it is a periodic function, with period $\frac{2\pi}{\mathrm{Log}q}$.

Theorem 1. *Let f be a function defined on $\mathbb{R}_{q,+}$ such that $t^{\gamma-1}f(t)\in L_q^1(\mathbb{R}_{q,+})$. Then $M_{q,\gamma}(f)\in L^\infty\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)$ and*

$$\|M_{q,\gamma}(f)\|_{L^\infty\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}\leq\|t^{\gamma-1}f(t)\|_{L_q^1(\mathbb{R}_{q,+})}. \tag{7}$$

Proof. For all $x\in\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right]$, we have

$$|M_{q,\gamma}(f)(x)|=\left|\int_0^\infty t^{\gamma+ix-1}f(t)d_qt\right|\leq\int_0^\infty t^{\gamma-1}|f(t)|d_qt=\|t^{\gamma-1}f(t)\|_{L_q^1(\mathbb{R}_{q,+})}.$$

Theorem 2. *(Plancherel formula)*
Let f be a function defined on $\mathbb{R}_{q,+}$ such that $t^{\gamma-1/2}f(t)\in L_q^2(\mathbb{R}_{q,+})$. Then $M_{q,\gamma}(f)$ is in $L^2\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)$ and

$$\left(\frac{\mathrm{Log}q}{2\pi(q-1)}\right)^{\frac{1}{2}}\|M_{q,\gamma}(f)\|_{L^2\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}=\|t^{\gamma-1/2}f(t)\|_{L_q^2(\mathbb{R}_{q,+})}. \tag{8}$$

Proof. Using (5) and the fact

$$\forall\lambda\in\mathbb{C},\quad M_q[t^\lambda f(t)](s)=M_q(f)(\lambda+s),$$

we obtain

$$\begin{aligned}\int_0^{+\infty}|f(x)|^2x^{2\gamma-1}d_qx &= \int_0^{+\infty}f(x)\overline{f}(x)x^{2\gamma-1}d_qx\\ &= \frac{\mathrm{Log}q}{2i\pi(1-q)}\int_{\gamma-i\frac{\pi}{Logq}}^{\gamma+i\frac{\pi}{Logq}}M_q(f)(s)M_q(\overline{f})(2\gamma-s)ds\\ &= \frac{\mathrm{Log}q}{2\pi(1-q)}\int_{-\frac{\pi}{\mathrm{Log}q}}^{\frac{\pi}{\mathrm{Log}q}}M_q(f)(\gamma+it)M_q(\overline{f})(\gamma-it)dt\\ &= \frac{\mathrm{Log}q}{2\pi(1-q)}\int_{-\frac{\pi}{\mathrm{Log}q}}^{\frac{\pi}{\mathrm{Log}q}}|M_{q,\gamma}(f)(t)|^2dt.\end{aligned}$$

Thus,

$$\|x^{\gamma-1/2}f\|_{L_q^2(\mathbb{R}_{q,+})} = \left(\frac{\text{Log}q}{2\pi(q-1)}\right)^{1/2} \|M_{q,\gamma}(f)\|_{L^2([\frac{\pi}{\text{Log}q},-\frac{\pi}{\text{Log}q}],dx)}.$$

We are now in a situation to state a Hausdorff-Young inequality for the modified q-Mellin transform.

Theorem 3. *(Hausdorff-Young inequality)*
Let f be a function defined on $\mathbb{R}_{q,+}$ and $1 < n \leq 2$ (resp. $n = 1$) such that $t^{\gamma-\frac{1}{n}}f(t) \in L_q^n(\mathbb{R}_{q,+})$. Then for $m = \frac{n}{n-1}$ (resp. $m = \infty$), we have $M_{q,\gamma}(f) \in L^m\left(\left[\frac{\pi}{\text{Log}q}, -\frac{\pi}{\text{Log}q}\right], dx\right)$ and

$$\|M_{q,\gamma}(f)\|_{L^m([\frac{\pi}{\text{Log}q},-\frac{\pi}{\text{Log}q}],dx)} \leq C(q,n)\|t^{\gamma-\frac{1}{n}}f(t)\|_{L_q^n(\mathbb{R}_{q,+})}. \tag{9}$$

Proof. Consider the linear operator T defined by, $T(f) = M_{q,\gamma}(t^{-\gamma}f)$.

From Theorem 1, we have for all $f \in L_q^1(\mathbb{R}_{q,+}, \frac{d_qx}{x})$,

$$\|T(f)\|_{L^\infty([\frac{\pi}{\text{Log}q},-\frac{\pi}{\text{Log}q}],dx)} \leq \|f\|_{L_q^1(\mathbb{R}_{q,+},\frac{d_qx}{x})}$$

and from Theorem 2, we have for all $f \in L_q^2(\mathbb{R}_{q,+}, \frac{d_qx}{x})$,

$$\|T(f)\|_{L^2([\frac{\pi}{\text{Log}q},-\frac{\pi}{\text{Log}q}],dx)} = \left(\frac{\text{Log}q}{2\pi(q-1)}\right)^{-\frac{1}{2}} \|f\|_{L_q^2(\mathbb{R}_{q,+},\frac{d_qx}{x})}.$$

So, by the Riesz-Thorin interpolation theorem (see [3] and [6]), we obtain the result with $C(q,n) = \left(\frac{\text{Log}q}{2\pi(q-1)}\right)^{\frac{1-n}{n}}$.

3. Paley-Wiener Theorems for the Modified q-Mellin Transform

From (4), one can see that $M_{q,\gamma}(f)$ is a C^∞-function on $\mathbb{R}$ and for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, we have

$$\frac{d^n}{dx^n}M_{q,\gamma}(f)(x) = M_{q,\gamma}[(i\text{Log}t)^n f(t)](x).$$

Then, for a polynomial function $P(x)$, we have

$$P\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)(x) = M_{q,\gamma}\left[P(\text{Log}t)f(t)\right](x). \tag{10}$$

For a real positive number r, we put

$$\Omega_{P,r} = \{t > 0 : |P(\text{Log}(t)| \leq r\}.$$

We begin by the following useful lemma.

Lemma 1. *Let $p > 0$, F and Q be two functions defined on $\mathbb{R}_{q,+}$, such that $Q^nF \in L^p_q(\mathbb{R}_{q,+})$ for all $n = 0, 1, 2, ...$, then*

$$\lim_{n\to+\infty} \|Q^nF\|^{\frac{1}{n}}_{L^p_q(\mathbb{R}_{q,+})} = \sup_{t\in supp(F)\cap\mathbb{R}_{q,+}} |\ Q(t)\ |\ .$$

Proof. The case $F = 0$ is trivial, since in this case $supp(F) = \emptyset$. Suppose now that $F \neq 0$ and define a measure μ on $\mathbb{R}_{q,+}$ by

$$d\mu = \|F\|^{-p}_{L^p_q(\mathbb{R}_{q,+})}|F(x)|^p d_qx.$$

We have : $\mu(\mathbb{R}_{q,+}) = 1$ and

$$\|Q^nF\|^{\frac{1}{n}}_{L^p_q(\mathbb{R}_{q,+})} = \|F\|^{\frac{1}{n}}_{L^p_q(\mathbb{R}_{q,+})}\|Q\|_{L^{pn}_q(\mathbb{R}_{q,+},d\mu)}.$$

On the other hand, we have

$$\lim_{n\to+\infty} \|Q\|_{L^{pn}_q(\mathbb{R}_{q,+},d\mu)} = \|Q\|_{L^\infty_q(\mathbb{R}_{q,+},d\mu)}$$

and

$$\|Q\|_{L^\infty_q(\mathbb{R}_{q,+},d\mu)} = \sup_{t\in supp(\mu)} |Q(t)| = \sup_{t\in supp(F)\cap\mathbb{R}_{q,+}} |Q(t)|.$$

Then, the result follows from the fact that $\lim\limits_{n\to+\infty} \|F\|^{\frac{1}{n}}_{L^p_q(\mathbb{R}_{q,+})} = 1.$

Theorem 4. *Let f be a function defined on $\mathbb{R}_{q,+}$ such that $(1 + |Logt|)^k t^{\gamma-1/2} f(t) \in L^2_q(\mathbb{R}_{q,+})$ for all $k = 0, 1, 2....$ Then*

$$\lim_{k\to+\infty} \left\|P^k\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|^{1/k}_{L^2\left(\left[\frac{\pi}{\text{Log}q},-\frac{\pi}{\text{Log}q}\right],dx\right)} = \sup_{t\in supp(f)\cap\mathbb{R}_{q,+}} |P(\text{Log}t)|. \tag{11}$$

In particular, $supp(f) \cap \mathbb{R}_{q,+} \subset \Omega_{P,r}$ if and only if

$$\lim_{k\to+\infty} \left\| P^k \left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|^{1/k}_{L^2\left(\left[\frac{\pi}{Logq}, -\frac{\pi}{Logq}\right], dx\right)} \leq r. \tag{12}$$

Proof. On the one hand, from the relation (10) and the Plancherel formula, we have

$$\left(\frac{\mathrm{Log}q}{2\pi(q-1)}\right)^{\frac{1}{2}} \left\| P^k \left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|_{L^2\left(\left[\frac{\pi}{Logq}, -\frac{\pi}{Logq}\right], dx\right)}$$
$$= \|t^{\gamma-1/2} P^k(\mathrm{Log}t) f(t)\|_{L^2_q(\mathbb{R}_{q,+})}.$$

On the other hand, Lemma 1 gives

$$\begin{aligned} & \lim_{k\to+\infty} \left\| P^k \left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|^{1/k}_{L^2\left(\left[\frac{\pi}{\mathrm{Log}q}, -\frac{\pi}{\mathrm{Log}q}\right], dx\right)} \\ = & \lim_{k\to+\infty} \|t^{\gamma-1/2} P^k(\mathrm{Log}t) f(t)\|^{1/k}_{L^2_q(\mathbb{R}_{q,+})} \\ = & \sup_{t\in supp(t^{\gamma-1/2}f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)| \\ = & \sup_{t\in supp(f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)|. \end{aligned}$$

Finally, the fact that $supp(f) \cap \mathbb{R}_{q,+} \subset \Omega_{P,r}$ if and only if

$$\sup_{t\in supp(f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)| \leq r$$

completes the proof.

In the particular case $P(t) = t$, we have the following result.

Corollary 1. *A function F is the modified q-Mellin transform $M_{q,\gamma}(f)$ of a function $f \in L^2_q(\mathbb{R}_{q,+})$ with support in the interval $[e^{-r}, e^r]$ if and only if $\frac{d^k}{dx^k} F \in L^2\left(\left[\frac{\pi}{Logq}, -\frac{\pi}{Logq}\right], dx\right)$ for all $k = 0, 1, \ldots$ and*

$$\lim_{k\to+\infty} \left\| \frac{d^k}{dx^k} F(x) \right\|^{1/k}_{L^2\left(\left[\frac{\pi}{Logq}, -\frac{\pi}{Logq}\right], dx\right)} \leq r.$$

Owing to the Hausdorff-Young inequality, the previous theorem can be generalized by the substitution of the L^2 norm by an L^p norm, $2 \leq p \leq \infty$. This is the aim of the following result.

Theorem 5. *For $2 \leq p \leq \infty$, we have for all polynomial function P with real coefficients*

$$\lim_{k\to+\infty} \left\| P^k\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|^{1/k}_{L^p\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} = \sup_{t\in supp(f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)|. \quad (13)$$

Proof. For $2 \leq p \leq \infty$, we note p' its conjugate number (**i.e.** $\frac{1}{p}+\frac{1}{p'}=1$).
If $2 \leq p < \infty$, then from the Hausdorff-Young inequality and the relation (10), we have

$$\left\| P^k\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|_{L^p\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \leq C(q,p)\|P^k(Logt)t^{\gamma-1/p'}f(t)\|_{L^{p'}_q(\mathbb{R}_{q,+})}.$$

So, by Lemma 1, we get

$$\begin{aligned}
&\limsup_{k\to+\infty} \left\| P^k\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|^{1/k}_{L^p\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \\
&\qquad \leq \limsup_{k\to+\infty} C(q,p)^{1/k}\|P^k(\mathrm{Log}t)t^{\gamma-1/p'}f(t)\|^{1/k}_{L^{p'}_q(\mathbb{R}_{q,+})} \qquad (14)\\
&\qquad = \sup_{t\in supp\ (t^{\gamma-1/p'}f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)| = \sup_{t\in supp(f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)|.
\end{aligned}$$

If $p=\infty$, then we have from Theorem 1 and the Hölder's inequality

$$\begin{aligned}
\|M_{q,\gamma}(f)\|_{L^\infty\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} &\leq \|t^{\gamma-1}f\|_{L^1_q(\mathbb{R}_{q,+})} \\
&= \int_0^\infty (1+t^2)^{-1}|(1+t^2)t^{\gamma-1}f(t)|d_qt \\
&\leq \|(1+t^2)^{-1}\|_{L^2_q(\mathbb{R}_{q,+})}\|(1+t^2)t^{\gamma-1}f(t)\|_{L^2_q(\mathbb{R}_{q,+})} \\
&\leq C\|(1+t^2)t^{\gamma-1}f(t)\|_{L^2_q(\mathbb{R}_{q,+})}.
\end{aligned}$$

Therefore,

$$\left\| P^k\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f) \right\|_{L^\infty\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \leq C\|P^k(\mathrm{Log}t)(1+t^2)t^{\gamma-1}f(t)\|_{L^2_q(\mathbb{R}_{q,+})}.$$

Also, the use of Lemma 1 gives

$$\limsup_{k\to+\infty}\left\|P^k\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|^{1/k}_{L^\infty\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \leq \sup_{t\in supp\quad(1+t^2)t^{\gamma-1}f\cap\mathbb{R}_{q,+}}|P(\mathrm{Log}t)| = \sup_{t\in supp(f)\cap\mathbb{R}_{q,+}}|P(\mathrm{Log}t)|. \tag{15}$$

On the other hand, since $M_{q,\gamma}(f)$ is a $\dfrac{2\pi}{\mathrm{Log}q}$ periodic function, then some integrations by parts give

$$\int_{\frac{\pi}{\mathrm{Log}q}}^{-\frac{\pi}{\mathrm{Log}q}} P^k\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)(t)P^k\left(i\frac{d}{dx}\right)\overline{M_{q,\gamma}(f)(t)}dt = \int_{\frac{\pi}{\mathrm{Log}q}}^{-\frac{\pi}{\mathrm{Log}q}} \overline{M_{q,\gamma}(f)(t)}P^{2k}\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)(t)dt.$$

So, by the Hölder's inequality, we obtain

$$\left\|P^k\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|^2_{L^2\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \leq \|M_{q,\gamma}(f)\|_{L^{p'}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}\left\|P^{2k}\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|_{L^p\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}. \tag{16}$$

But, from Theorem 4, we have

$$\begin{aligned}
\sup_{t\in suppf\cap\mathbb{R}_{q,+}}|P(\mathrm{Log}t)| &= \lim_{k\to+\infty}\left\|P^k\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|^{1/k}_{L^2\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}\\
&\leq \lim_{k\to+\infty}\|M_{q,\gamma}(f)\|^{1/2k}_{L^{p'}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}\\
&\quad \liminf_{k\to+\infty}\left\|P^{2k}\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|^{1/2k}_{L^p\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}\\
&= \liminf_{k\to+\infty}\left\|P^{2k}\left(-i\frac{d}{dx}\right)M_{q,\gamma}(f)\right\|^{1/2k}_{L^p\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}.
\end{aligned}$$

Now, replacing $M_{q,\gamma}(f)$ by $P(-i\frac{d}{dx})M_{q,\gamma}(f)$ in formula (16), we obtain

$$\begin{aligned} & \left\| P^{k+1}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|^2_{L^2\left(\left[\frac{\pi}{Logq},-\frac{\pi}{Logq}\right],dx\right)} \\ \leq \quad & \left\| P\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|_{L^{p'}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \\ & \left\| P^{2k+1}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|_{L^{p}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sup_{t\in suppf\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)| = \lim_{k\to+\infty} \left\| P^{k+1}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|^{\frac{1}{k+1}}_{L^2\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \\ = \quad & \lim_{k\to+\infty} \left\| P^{k+1}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|^{\frac{2}{2k+1}}_{L^2\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \\ \leq \quad & \lim_{k\to+\infty} \|M_{q,\gamma}(f)\|^{1/2k+1}_{L^{p'}([\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}])} \\ & \liminf_{k\to+\infty} \left\| P^{2k+1}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|^{1/2k+1}_{L^{p}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \\ = \quad & \liminf_{k\to+\infty} \left\| P^{2k+1}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|^{1/2k+1}_{L^{p}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)}. \end{aligned}$$

Thus,

$$\liminf_{k\to+\infty} \left\| P^{k}\left(-i\frac{d}{dx}\right) M_{q,\gamma}(f)\right\|^{1/k}_{L^{p}\left(\left[\frac{\pi}{\mathrm{Log}q},-\frac{\pi}{\mathrm{Log}q}\right],dx\right)} \geq \sup_{t\in supp(f)\cap\mathbb{R}_{q,+}} |P(\mathrm{Log}t)|.$$

Finally, the result follows from this relation and formulas (14) and (15).

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