Tamsui Oxford Journal of Information and Mathematical Sciences **28(4)** (2012) 437-469
Aletheia University

Some New Modular Equations and their Applications *

M. S. Mahadeva Naika† and S. Chandankumar‡

Department of Mathematics, Bangalore University,
Central College Campus, Bangalore-560 001, India

Received January 3, 2012, Accepted May 9, 2012.

Abstract

In this paper, we establish several new modular equations of degree two by using Ramanujan's modular equations. We obtain several general formulas for the explicit evaluations of ratios of Ramanujan's theta–functions φ. We also establish some new explicit evaluations for Ramanujan–Göllnitz–Gordon continued fraction, Ramanujan–Selberg continued fraction and a continued fraction of Eisenstein.

Keywords and Phrases: *Modular equation, Theta-function, Continued fraction.*

1. Introduction, Definitions and Notations

In Chapter 16 of his second notebook [15], Ramanujan develops the theory of theta-function and his theta-function is defined by

$$\begin{aligned} f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \\ &= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \end{aligned} \tag{1.1}$$

*2000 *Mathematics Subject Classification.* Primary 33D10, 11F27, 11A55.
Research Supported by UGC grant No.F.No.34–140\2008 (SR)

†Corresponding author. E-mail: msmnaika@rediffmail.com
‡E-mail: chandan.s17@gmail.com

The three special cases of Ramanujan's theta-function are as follows:

$$\begin{aligned}
\varphi(q) &:= f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \\
\psi(q) &:= f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \\
f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,
\end{aligned}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \ \mid x \mid < 1,$$

where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), \text{ for } n \geq 1.$$

Let

$$Z := Z(x) := \ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right),$$

where $0 < x < 1$.

Let n denote a fixed natural number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}. \tag{1.2}$$

Then a modular equation of degree n in the classical theory is a relation between α and β induced by (1.2).

In [19], J. Yi introduced two parameters $h_{k,n}$ as

$$h_{k,n} := \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})} \tag{1.3}$$

and established some properties and several explicit evaluations of $h_{k,n}$ for different positive rational values of k and n. In [7], M. S. Mahadeva Naika, S. Chandankumar and M. Manjunatha have established several new modular equations of degree 2. In [13], Mahadeva Naika, K. Sushan Bairy and Manjunatha have established several new modular equations of degree 4. They have also established general formulas for explicit evaluations of $h_{4,n}$. In [8], [9] Mahadeva Naika, Chandankumar and Bairy have established several new modular equations of degree 9 and degree 3, and also established several general formulas for the explicit evaluations of ratios of Ramanujan's theta functions φ and ψ.

In Section 2, we collect some results which are useful to prove our main results. In Section 3, we establish several new modular equations of degree 2 for the ratios of Ramanujan's theta–function. In Section 4, we establish several general formulas for the explicit evaluations of $h_{2,n}$, for positive rational values of n. In section 5, we establish some explicit evaluations for the Ramanujan–Göllnitz–Gordon continued fraction, Ramanujan–Selberg continued fraction and a continued fraction of Eisenstein using the values of $h_{2,n}$.

2. Preliminary Results

In this section, we collect some results which are useful to prove our main results.

Lemma 2.1. *For* $0 < x < 1$,

1. [4, Entry 10(i), p. 122]
$$\varphi(e^{-y}) = \sqrt{z}, \tag{2.1}$$

2. [4, Entry 10(iv), p. 122]
$$\varphi(e^{-2y}) = \sqrt{z}\left(\frac{1}{2}(1+\sqrt{1-x})\right)^{1/2}, \tag{2.2}$$

3. [4, Entry 10(vi), p. 122] *We have*
$$\varphi(e^{-y/2}) = \sqrt{z}(1+\sqrt{x})^{1/2}, \tag{2.3}$$

4. [4, Entry 12(ii), p. 124] *We have*

$$f(e^{-y}) = \sqrt{z}2^{-1/6}\{x(1-x)e^{y}\}^{1/24}, \tag{2.4}$$

5. [4, Entry 12(iv), p. 124] *We have*

$$f(-e^{-4y}) = \sqrt{z}2^{-2/3}(1-x)^{1/24}\{xe^{y}\}^{1/6}, \tag{2.5}$$

6. [4, Entry 12(v), Ch.17, p. 124] *We have*

$$\chi(e^{-y}) = 2^{1/6}\{x(1-x)e^{y}\}^{-1/24}, \tag{2.6}$$

7. [4, Entry 12(vi), Ch.17, p. 124] *We have*

$$\chi(-e^{-y}) = 2^{1/6}(1-x)^{1/12}(xe^{y})^{-1/24}, \tag{2.7}$$

where

$$y = \pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)}.$$

Lemma 2.2. [19, Theorem 2.2(ii)] *We have*

$$h_{k,n}h_{k,1/n} = 1. \tag{2.8}$$

Lemma 2.3. [19, Theorem 4.6] *We have*

$$\sqrt{2}\left(h_{2,n}h_{2,4n} + \frac{1}{h_{2,n}h_{2,4n}}\right) = 2 + \frac{h_{2,4n}}{h_{2,n}}. \tag{2.9}$$

Lemma 2.4. *We have*

1. [4, Eq.(24.22), p.215] *If β is of degree 4 over α, then*

$$\beta = \left\{\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right\}^{4}. \tag{2.10}$$

2. [4, Entry 13(i), p.280] *If β is of degree 5 over α, then*

$$\{\alpha\beta\}^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\left\{16\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/6} = 1. \tag{2.11}$$

3. [4, Entry 19(i), p.314] *If β is of degree 7 over α, then*

$$\{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}=1. \tag{2.12}$$

4. [4, Entry 24(v), p.217] *If β is of degree 8 over α, then*

$$(1-(1-\alpha)^{1/4})(1-\beta^{1/4})=2\sqrt{2}(\beta(1-\alpha))^{1/8}. \tag{2.13}$$

5. [4, Entry 3(x),(xi), p.352] *If β is of degree 9 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}=\sqrt{m}, \tag{2.14}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}=\frac{3}{\sqrt{m}}. \tag{2.15}$$

6. [4, Entry 7, p.363] *If β is of degree 11 over α, then*

$$\{\alpha\beta\}^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\left\{16\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/12}=1. \tag{2.16}$$

7. [4, Entry 21, p.435] *If β is of degree 15 over α, then*

$$\begin{aligned}&(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\&=\left\{\frac{1}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)\right\}^{1/2}.\end{aligned} \tag{2.17}$$

8. [5, Entry 17.3.26, pp.391-392] *If β has degree 17 over α, then*

$$\begin{aligned}m=&\left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}\\&-2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left\{1+\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\},\end{aligned} \tag{2.18}$$

$$\begin{aligned}\frac{17}{m}=&\left(\frac{\alpha}{\beta}\right)^{1/4}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\\&-2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}\left\{1+\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\right\}.\end{aligned} \tag{2.19}$$

Throughout this article we use the following notations.

$$a=\sqrt[8]{\alpha\beta},\ \ a_2=\sqrt[4]{\alpha\beta},\ \ a_4=\sqrt{\alpha\beta},$$
$$b=\sqrt[8]{(1-\alpha)(1-\beta)},\ \ b_2=\sqrt[4]{(1-\alpha)(1-\beta)},$$
$$c=\sqrt[4]{1-\alpha},\ \ \ d=\sqrt[4]{\beta},\ \ \ d_2=\sqrt{\beta}.$$

3. Some New Modular Equations of Degree Two

In this section, we establish several modular equations of degree two for the ratios of Ramanujan's–theta function φ.

Theorem 3.1. *If* $P=\dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^4)}{\varphi(q^8)}$ *and* $Q=\dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^8)}{\varphi(q^4)}$, *then*

$$\begin{aligned}&4Q^2+\frac{1}{Q^2}-4\left(2Q+\frac{3}{Q}\right)+4\left(P+\frac{2}{P}\right)+\left(P^2+\frac{4}{P^2}\right)+16\\&=4\left(PQ+\frac{1}{PQ}\right)+2\left(\frac{P}{Q}+\frac{4Q}{P}\right).\end{aligned}\tag{3.1}$$

Poorf. Employing the equations (2.1) and (2.2) in (2.10), we find that

$$\begin{aligned}&-8Q^3P-2P^3Q-P^2-4Q^2+4QP+8Q^2P^2+2P^3cQ\\&-8Q^3cP+8Q^2P^2c+4Q^4P^2-4Q^3P^3+P^4Q^2=0.\end{aligned}\tag{3.2}$$

Isolating the terms having c on one side of the equation and squaring both sides, we deduce that

$$\begin{aligned}&(P^4Q^2+P^2-2P^3Q-12QP^2-4QP+4Q^2P^3+16Q^2P^2+8Q^2P+4Q^2\\&-4Q^3P^3-8Q^3P^2-8Q^3P+4Q^4P^2)(P^4Q^2+P^2-2P^3Q+12QP^2-4QP\\&-4Q^2P^3+16Q^2P^2-8Q^2P+4Q^2-4Q^3P^3+8Q^3P^2-8Q^3P+4Q^4P^2)=0.\end{aligned}\tag{3.3}$$

By examining the behaviour of the factors near $q=0$, it can be seen that there is a neighbourhood about the origin, where the first factor is zero, whereas the second factor is not zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. Hence, we obtain the equation (3.1). □

Theorem 3.2. *If* $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^5)}{\varphi(q^{10})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{10})}{\varphi(q^5)}$, *then*

$$\begin{aligned} & Q^3 - \frac{1}{Q^3} + 20\left(Q^2 + \frac{1}{Q^2}\right) + 5\left(Q - \frac{1}{Q}\right) + 16\left(P^2 + \frac{4}{P^2}\right) \\ & + 120 = 40\left(Q + \frac{1}{Q}\right)\left(P + \frac{2}{P}\right). \end{aligned} \tag{3.4}$$

Poorf. Using the equations (2.1) and (2.2) in (2.11), we find that

$$\begin{aligned} & -384PQ^2 - 384PQ^4 + 480P^2Q^3 + 144P^2Q^5 - 120P^3Q^2 - 120P^3Q^4 - 8P^3 \\ & + 144P^2Q + 64a_4P^2Q^3 - 64a_4P^3Q^2 - 64a_4P^3Q^4 + 12a_4P^4Q + 168a_4P^4Q^3 \\ & + 12a_4P^4Q^5 - 64a_4P^5Q^2 - 64a_4P^5Q^4 + 32a_4P^6Q^3 + 256Q^3 - 8P^3Q^6 = 0. \end{aligned} \tag{3.5}$$

Isolating the terms having a_4 on one side of the equation (3.5) and squaring both sides, we find that

$$\begin{aligned} & (-80PQ^2 - P^2 - 5P^2Q^2 + 5P^2Q^4 - 80PQ^4 + 120P^2Q^3 - 40P^3Q^2 \\ & - 40P^3Q^4 + 20P^2Q + 64Q^3 + 16P^4Q^3 + 20P^2Q^5 + P^2Q^6)(-80PQ^2 \\ & + P^2 + 5P^2Q^2 - 5P^2Q^4 - 80PQ^4 + 120P^2Q^3 - 40P^3Q^2 - 40P^3Q^4 \\ & + 20P^2Q + 64Q^3 + 16P^4Q^3 + 20P^2Q^5 - P^2Q^6) = 0. \end{aligned} \tag{3.6}$$

By examining the behaviour of the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is zero, whereas the first factor is not zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. Hence, we obtain (3.4). □

Theorem 3.3. *If* $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^7)}{\varphi(q^{14})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{14})}{\varphi(q^7)}$, *then*

$$\begin{aligned} & Q^4 + \frac{1}{Q^4} + 56\left(Q^3 + \frac{1}{Q^3}\right) + 252\left(Q^2 + \frac{1}{Q^2}\right) + 1064\left(Q + \frac{1}{Q}\right) \\ & - 64\left(P^3 + \frac{8}{P^3}\right) + 1078 = 112\left\{\left(P + \frac{2}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right)\right.\right. \\ & \left.\left. +3\left(Q + \frac{1}{Q}\right) + 8\right] - \left(P^2 + \frac{4}{P^2}\right)\left[2\left(Q + \frac{1}{Q}\right) + 1\right]\right\}. \end{aligned} \tag{3.7}$$

Poorf. Using the equations (2.1) and (2.2) in (2.12), we deduce that

$$-4Q+2PQ^2+2P-4P^2Qa+6P^2Qa_2-4P^2Qa_2a+P^2Qa_4=0. \tag{3.8}$$

Isolating the terms having a on one side of the equation (3.8) and squaring both sides, we deduce that

$$\begin{aligned}&16P^4Q^2a_2-4P^4Q^2a_4+4P^4Q^2a_2a_4-32Q^2-4P^2+32PQ-24P^3Qa_2\\&-4P^3Q^3a_4+32PQ^3-24P^2Q^2-24P^3Q^3a_2-4P^3Qa_4+8P^2Q^2a_4\\&+48P^2Q^2a_2-4P^2Q^4=0.\end{aligned} \tag{3.9}$$

Collecting the terms containing a_2 on one side of the equation (3.9) and squaring, we find that

$$\begin{aligned}&-768P^3Q^3a_4-2048Q^4+3136P^4Q^2+4096PQ^3-2560P^2Q^2+320P^4Q^6a_4\\&+15008Q^4P^4-5376P^5Q^3-5376P^5Q^5+1792P^6Q^4-16P^4Q^8-2560P^2Q^6\\&+512P^3Q^7-7168P^3Q^5+3136P^4Q^6+4096Q^5P+256P^8Q^4a_4-768P^7Q^5a_4\\&+2752P^6Q^4a_4-768P^7Q^3a_4+320P^4Q^2a_4+512P^3Q-7168P^3Q^3-16P^4\\&-32P^5Q^7a_4+544a_4P^6Q^2-32P^5Qa_4+3200a_4Q^4P^4-2528a_4P^5Q^5\\&-2528a_4P^5Q^3+544a_4P^6Q^6-768Q^5P^3a_4+512Q^4P^2a_4=0.\end{aligned} \tag{3.10}$$

Isolating the terms having a_4 on one side of the equation (3.10) and squaring both sides, we deduce that

$$\begin{aligned}&(-P^3+512Q^4+224P^4Q^2-896PQ^3+1792P^2Q^4+448P^2Q^2-448PQ^4\\&+896Q^4P^4-224P^5Q^3-224P^5Q^5+64P^6Q^4+448P^2Q^6-56P^3Q^7\\&-1064P^3Q^5+224P^4Q^6-896Q^5P-56P^3Q-1064P^3Q^3-252P^3Q^2\\&-1078Q^4P^3-P^3Q^8+672P^2Q^5-252P^3Q^6+672P^2Q^3-112P^5Q^4\\&+336P^4Q^5+336P^4Q^3)(P^3+512Q^4+224P^4Q^2-896PQ^3+1792P^2Q^4\\&+448P^2Q^2+448PQ^4+896Q^4P^4-224P^5Q^3-224P^5Q^5+64P^6Q^4\\&+448P^2Q^6-56P^3Q^7-1064P^3Q^5+224P^4Q^6-896Q^5P-56P^3Q\\&-1064P^3Q^3+252P^3Q^2+1078Q^4P^3+P^3Q^8-672P^2Q^5+252P^3Q^6\\&-672P^2Q^3+112P^5Q^4-336P^4Q^5-336P^4Q^3)=0.\end{aligned} \tag{3.11}$$

By examining the behaviour of the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is zero, whereas the first factor is not zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. Hence, we obtain (3.7). $\square$

Theorem 3.4. *If* $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^8)}{\varphi(q^{16})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{16})}{\varphi(q^8)}$, *then*

$$\begin{aligned}
&16Q^4 + \frac{1}{Q^4} + 8\left(8Q^3 - \frac{11}{Q^3}\right) + 64\left(4Q^2 + \frac{13}{Q^2}\right) + 16\left(74Q + \frac{113}{Q}\right) \\
&+ \left(P^4 + \frac{16}{P^4}\right) - 4\left(P^3 + \frac{8}{P^3}\right)\left[\left(2Q + \frac{1}{Q}\right) + 6\right] + 2\left(P^2 + \frac{4}{P^2}\right) \\
&\times\left[4\left(14Q + \frac{19}{Q}\right) + 3\left(4Q^2 + \frac{1}{Q^2}\right) + 160\right] + 1936 - 4\left(P + \frac{2}{P}\right) \\
&\times\left[12\left(10Q + \frac{21}{Q}\right) + 2\left(20Q^2 + \frac{21}{Q^2}\right) + \left(8Q^3 + \frac{1}{Q^3}\right) + 168\right] = 0.
\end{aligned} \tag{3.12}$$

Theorem 3.5. *If* $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^9)}{\varphi(q^{18})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{18})}{\varphi(q^9)}$, *then*

$$\begin{aligned}
&Q^6 + \frac{1}{Q^6} - 136\left(Q^5 + \frac{1}{Q^5}\right) + 2458\left(Q^4 + \frac{1}{Q^4}\right) - 12264\left(Q^3 + \frac{1}{Q^3}\right) \\
&+ 40911\left(Q^2 + \frac{1}{Q^2}\right) - 73104\left(Q + \frac{1}{Q}\right) + 256\left(P^4 + \frac{16}{P^4}\right) \\
&+ 3840\left(P^3 + \frac{8}{P^3}\right) + 17280\left(P^2 + \frac{4}{P^2}\right) + 51072\left(P + \frac{2}{P}\right) + 95532 \\
&= 256\left(P^4 + \frac{16}{P^4}\right)\left(Q + \frac{1}{Q}\right) \\
&+ 1152\left(P^3 + \frac{8}{P^3}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) - \left(Q + \frac{1}{Q}\right)\right] + 576\left(P^2 + \frac{4}{P^2}\right) \\
&\times\left[25\left(Q + \frac{1}{Q}\right) - 11\left(Q^2 + \frac{1}{Q^2}\right) + 3\left(Q^3 + \frac{1}{Q^3}\right)\right] + 192\left(P + \frac{2}{P}\right) \\
&\times\left[221\left(Q + \frac{1}{Q}\right) - 110\left(Q^2 + \frac{1}{Q^2}\right) + 35\left(Q^3 + \frac{1}{Q^3}\right) - 5\left(Q^4 + \frac{1}{Q^4}\right)\right].
\end{aligned} \tag{3.13}$$

Theorem 3.6. *If* $P = \frac{\varphi(q)}{\varphi(q^2)}\frac{\varphi(q^{11})}{\varphi(q^{22})}$ *and* $Q = \frac{\varphi(q)}{\varphi(q^2)}\frac{\varphi(q^{22})}{\varphi(q^{11})}$, *then*

$$\begin{aligned}
&Q^6 - \frac{1}{Q^6} - 286\left(Q^5 + \frac{1}{Q^5}\right) + 11660\left(Q^4 - \frac{1}{Q^4}\right) - 29766\left(Q^3 + \frac{1}{Q^3}\right)\\
&+ 120021\left(Q^2 - \frac{1}{Q^2}\right) - 179388\left(Q + \frac{1}{Q}\right) + 1024\left(P^5 + \frac{32}{P^5}\right)\\
&- 5632\left(P^4 + \frac{16}{P^4}\right)\left(Q + \frac{1}{Q}\right) + 704\left(P^3 + \frac{8}{P^3}\right)\left[16\left(Q^2 + \frac{1}{Q^2}\right) + 45\right.\\
&\left.-8\left(Q - \frac{1}{Q}\right)\right] + 44\left(P + \frac{2}{P}\right)\left[79\left(Q^4 + \frac{1}{Q^4}\right) - 652\left(Q^3 - \frac{1}{Q^3}\right)\right.\\
&\left.+3386 + 1404\left(Q^2 + \frac{1}{Q^2}\right) - 1548\left(Q - \frac{1}{Q}\right)\right] - 176\left(P^2 + \frac{4}{P^2}\right)\\
&\times\left[56\left(Q^3 + \frac{1}{Q^3}\right) - 129\left(Q^2 - \frac{1}{Q^2}\right) + 370\left(Q + \frac{1}{Q}\right)\right] = 0.
\end{aligned} \tag{3.14}$$

Theorem 3.7. *If* $P = \frac{\varphi(q)}{\varphi(q^2)}\frac{\varphi(q^{15})}{\varphi(q^{30})}$ *and* $Q = \frac{\varphi(q)}{\varphi(q^2)}\frac{\varphi(q^{30})}{\varphi(q^{15})}$, *then*

$$\begin{aligned}
&Q^{12} + \frac{1}{Q^{12}} + 1080\left(Q^{11} + \frac{1}{Q^{11}}\right) + 207540\left(Q^{10} + \frac{1}{Q^{10}}\right) + 2771560\left(Q^9 + \frac{1}{Q^9}\right)\\
&+ 387029\left(Q^8 + \frac{1}{Q^8}\right) + 57329160\left(Q^7 + \frac{1}{Q^7}\right) - 27833756\left(Q^6 + \frac{1}{Q^6}\right)\\
&+ 297924120\left(Q^5 + \frac{1}{Q^5}\right) - 866201937\left(Q^4 + \frac{1}{Q^4}\right) + 317655600\left(Q^3 + \frac{1}{Q^3}\right)\\
&+ 4062669080\left(Q^2 + \frac{1}{Q^2}\right) + 4943360\left(Q + \frac{1}{Q}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 16 \left\{ 4096 \left(P^8 + \frac{256}{P^8} \right) \right. \\
&+ 1024 \left(P^7 + \frac{128}{P^7} \right) \left[15 - 40 \left(Q + \frac{1}{Q} \right) + \left(Q^4 + \frac{1}{Q^4} \right) \right] + 256 \left(P^6 + \frac{64}{P^6} \right) \\
&\times \left[1691 - 30 \left(Q^5 + \frac{1}{Q^5} \right) + 15 \left(Q^4 + \frac{1}{Q^4} \right) - 46 \left(Q^3 + \frac{1}{Q^3} \right) + 680 \left(Q^2 + \frac{1}{Q^2} \right) \right. \\
&\left. -570 \left(Q + \frac{1}{Q} \right) \right] + 128 \left(P^5 + \frac{32}{P^5} \right) \left[1097 + 180 \left(Q^6 + \frac{1}{Q^6} \right) - 210 \left(Q^5 + \frac{1}{Q^5} \right) \right. \\
&\left. +735 \left(Q^4 + \frac{1}{Q^4} \right) - 3530 \left(Q^3 + \frac{1}{Q^3} \right) + 5020 \left(Q^2 + \frac{1}{Q^2} \right) - 15238 \left(Q + \frac{1}{Q} \right) \right] \\
&+ 16 \left(P^4 + \frac{16}{P^4} \right) \left[645646 - 2200 \left(Q^7 + \frac{1}{Q^7} \right) + 4620 \left(Q^6 + \frac{1}{Q^6} \right) \right. \\
&-18900 \left(Q^5 + \frac{1}{Q^5} \right) + 55125 \left(Q^4 + \frac{1}{Q^4} \right) - 109260 \left(Q^3 + \frac{1}{Q^3} \right) \\
&\left. +333360 \left(Q^2 + \frac{1}{Q^2} \right) - 320360 \left(Q + \frac{1}{Q} \right) \right] + 20 \left(P^3 + \frac{8}{P^3} \right) \left[1437 \left(Q^8 + \frac{1}{Q^8} \right) \right. \\
&+769902 - 4656 \left(Q^7 + \frac{1}{Q^7} \right) + 28056 \left(Q^6 + \frac{1}{Q^6} \right) - 58216 \left(Q^5 + \frac{1}{Q^5} \right) \\
&+171884 \left(Q^4 + \frac{1}{Q^4} \right) - 515688 \left(Q^3 + \frac{1}{Q^3} \right) + 468008 \left(Q^2 + \frac{1}{Q^2} \right) \\
&\left. -1643008 \left(Q + \frac{1}{Q} \right) \right] + \left(P^2 + \frac{4}{P^2} \right) \left[114518210 - 11886 \left(Q^9 + \frac{1}{Q^9} \right) \right. \\
&+39595 \left(Q^8 + \frac{1}{Q^8} \right) - 696270 \left(Q^7 + \frac{1}{Q^7} \right) + 662840 \left(Q^6 + \frac{1}{Q^6} \right) \\
&-5205000 \left(Q^5 + \frac{1}{Q^5} \right) + 13248180 \left(Q^4 + \frac{1}{Q^4} \right) - 11892680 \left(Q^3 + \frac{1}{Q^3} \right) \\
&\left. +65473160 \left(Q^2 + \frac{1}{Q^2} \right) - 19525620 \left(Q + \frac{1}{Q} \right) \right] + \left(P + \frac{2}{P} \right) \\
&\times \left[13968720 + 2030 \left(Q^{10} + \frac{1}{Q^{10}} \right) + 15615 \left(Q^9 + \frac{1}{Q^9} \right) \right. \\
&+524408 \left(Q^8 + \frac{1}{Q^8} \right) + 144775 \left(Q^7 + \frac{1}{Q^7} \right) + 5585270 \left(Q^6 + \frac{1}{Q^6} \right) \\
&-8897204 \left(Q^5 + \frac{1}{Q^5} \right) + 16404960 \left(Q^4 + \frac{1}{Q^4} \right) - 79416580 \left(Q^3 + \frac{1}{Q^3} \right) \\
&\left.\left. +12559260 \left(Q^2 + \frac{1}{Q^2} \right) - 226488270 \left(Q + \frac{1}{Q} \right) \right] \right\} = 6893052420.
\end{aligned} \tag{3.15}
$$

Theorem 3.8. *If* $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{17})}{\varphi(q^{34})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{34})}{\varphi(q^{17})}$, *then*

$$\begin{aligned}
&Q^9+\frac{1}{Q^9}-1938\left(Q^8+\frac{1}{Q^8}\right)+698649\left(Q^7+\frac{1}{Q^7}\right)-13577968\left(Q^6+\frac{1}{Q^6}\right)\\
&-138056524\left(Q^5+\frac{1}{Q^5}\right)-945602424\left(Q^4+\frac{1}{Q^4}\right)-3499343036\left(Q^3+\frac{1}{Q^3}\right)\\
&-8483099280\left(Q^2+\frac{1}{Q^2}\right)-14485100818\left(Q+\frac{1}{Q}\right)-65536\left(P^8+\frac{256}{P^8}\right)\\
&+278528\left\{2\left(P^7+\frac{128}{P^7}\right)\left[2+\left(Q+\frac{1}{Q}\right)\right]-\left(P^6+\frac{64}{P^6}\right)\left[42+32\left(Q+\frac{1}{Q}\right)\right.\right.\\
&\left.+7\left(Q^2+\frac{1}{Q^2}\right)\right]+\left(P^5+\frac{32}{P^5}\right)\left[328+247\left(Q+\frac{1}{Q}\right)+104\left(Q^2+\frac{1}{Q^2}\right)\right.\\
&\left.\left.+13\left(Q^3+\frac{1}{Q^3}\right)\right]\right\}-4352\left(P^4+\frac{16}{P^4}\right)\left[85519\left(Q+\frac{1}{Q}\right)+43066\left(Q^2+\frac{1}{Q^2}\right)\right.\\
&\left.+11265\left(Q^3+\frac{1}{Q^3}\right)+879\left(Q^4+\frac{1}{Q^4}\right)+110030\right]+2176\left(P^3+\frac{8}{P^3}\right)\\
&\left[810176+677050\left(Q+\frac{1}{Q}\right)+357624\left(Q^2+\frac{1}{Q^2}\right)+124911\left(Q^3+\frac{1}{Q^3}\right)\right.\\
&\left.+20840\left(Q^4+\frac{1}{Q^4}\right)+1047\left(Q^5+\frac{1}{Q^5}\right)\right]-1088\left\{\left(P^2+\frac{4}{P^2}\right)\left[3807186\left(Q+\frac{1}{Q}\right)\right.\right.\\
&+2192811\left(Q^2+\frac{1}{Q^2}\right)+816475\left(Q^3+\frac{1}{Q^3}\right)+197160\left(Q^4+\frac{1}{Q^4}\right)+4582352\\
&\left.+19219\left(Q^5+\frac{1}{Q^5}\right)+645\left(Q^6+\frac{1}{Q^6}\right)\right]+\left(P+\frac{2}{P}\right)\left[8103555\left(Q+\frac{1}{Q}\right)\right.\\
&+4816651\left(Q^2+\frac{1}{Q^2}\right)+1931229\left(Q^3+\frac{1}{Q^3}\right)+78927\left(Q^5+\frac{1}{Q^5}\right)+9749708\\
&\left.\left.+2501\left(Q^6+\frac{1}{Q^6}\right)+81\left(Q^7+\frac{1}{Q^7}\right)+489418\left(Q^4+\frac{1}{Q^4}\right)\right]\right\}=169023853880.
\end{aligned} \tag{3.16}$$

Proofs of the identities (3.12)–(3.16) are similar to the proof of the identity (3.7) given above except that in place of result (2.12), result (2.13) is used for proving (3.12); result (2.14) and (2.15) is used for proving (3.13); result (2.16) is used for proving (3.14); result (2.17) is used for proving (3.15) and results (2.18) and (2.19) are used for proving (3.16).

4. General Formulas for Explicit Evaluations of $h_{2,n}$

We shall employ modular equations established in Section 3 to establish several general formulas for explicit evaluations of $h_{2,n}$ for any positive rational number n.

Theorem 4.1. *If* $X = h_{2,n}h_{2,16n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,16n}}$, *then*

$$\begin{aligned} &4Y^2 + \frac{1}{Y^2} - 4\left(2Y + \frac{3}{Y}\right) + 4\sqrt{2}\left(X + \frac{1}{X}\right) + 2\left(X^2 + \frac{1}{X^2}\right) + 16 \\ &= 4\left(\sqrt{2}XY + \frac{1}{\sqrt{2}XY}\right) + 2\sqrt{2}\left(\frac{X}{Y} + \frac{2Y}{X}\right). \end{aligned} \tag{4.1}$$

Proof. Employing the equation (1.3) with $k = 2$ in the equation (3.1), we obtain (4.1). □

Corollary 4.1. *We have*

$$h_{2,4} = 1 + \sqrt{2} - \sqrt{1 + \sqrt{2}}, \tag{4.2}$$

$$h_{2,1/4} = \frac{1 + \sqrt{\sqrt{2} - 1}}{\sqrt{2}}, \tag{4.3}$$

$$h_{2,16} = 5 + 3\sqrt{2} + (\sqrt{2} + 1)^{5/2} - \sqrt{76 + 54\sqrt{2} + (488\sqrt{2} + 690)(\sqrt{2} - 1)^{5/2}}, \tag{4.4}$$

$$h_{2,1/16} = \left(\frac{\sqrt{\sqrt{2} + 1}}{\sqrt{2}}\right)\left(\sqrt{\sqrt{2} + 1} + \sqrt{2\left(\sqrt{\sqrt{2} + 1} - \sqrt{2}\right) - 1}\right). \tag{4.5}$$

Proof of (4.2)**.** Putting $n = 1/4$ in the equation (4.1) and using the equation (2.8), we find that

$$h_{2,4}^4 - 4(3 + \sqrt{2})h_{2,4}^3 + 4(5 + 2\sqrt{2})h_{2,4}^2 - 8(1 + \sqrt{2})h_{2,4} + 4 = 0. \tag{4.6}$$

Solving the above equation (4.6) and $0 < h_{2,4} < 1$, we obtain (4.2). □

Proof of (4.3)**.** Using the equation (4.2) in the equation (2.8) with $n = 4$, we obtain the equation (4.3). □

Proof of (4.4)**.** Putting $n = 1$ in the equation (4.1), we obtain

$$h_{2,16}^4-(20+12\sqrt{2})h_{2,16}^3+14\sqrt{2}h_{2,16}^2+24h_{2,16}^2-4\sqrt{2}h_{2,16}+2-8h_{2,16}=0. \quad (4.7)$$

Solving the above equation (4.7) and $0 < h_{2,16} < 1$, we obtain equation (4.4). □

Proof of (4.5)**.** Putting $n = 1$ in the equation (4.1) and using the fact that $h_{2,16} = 1/h_{2,1/16}$, we deduce that

$$\begin{aligned}&4\sqrt{2}h_{2,1/16}^4-6h_{2,1/16}^4+8h_{2,1/16}^3-4\sqrt{2}h_{2,1/16}^3-16h_{2,1/16}^2\\&+6\sqrt{2}h_{2,1/16}^2+12h_{2,1/16}-4\sqrt{2}h_{2,1/16}-3+2\sqrt{2}=0.\end{aligned} \quad (4.8)$$

Solving the above equation (4.8) and $h_{2,1/16} > 1$, we obtain (4.5). □

Remark: A different proof of the equations (4.2) and (4.3) can be found in [19].

Theorem 4.2. *If* $X = h_{2,n}h_{2,25n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,25n}}$, *then*

$$\begin{aligned}&Y^3-\frac{1}{Y^3}+20\left(Y^2+\frac{1}{Y^2}\right)+5\left(Y-\frac{1}{Y}\right)+32\left(X^2+\frac{1}{X^2}\right)+120\\&=40\sqrt{2}\left(Y+\frac{1}{Y}\right)\left(X+\frac{1}{X}\right).\end{aligned} \quad (4.9)$$

Proof. Employing the equation (1.3) with $k = 2$ in the equation (3.4), we obtain (4.9). □

Corollary 4.2. *We have,*

$$h_{2,5} = \left(\sqrt{2}-1\right)\left(2+\sqrt{5}\right)^{1/2}, \quad (4.10)$$

$$h_{2,1/5} = \left(\sqrt{2}+1\right)\left(-2+\sqrt{5}\right)^{1/2}, \quad (4.11)$$

$$h_{2,20} = \frac{(2+\sqrt{5})^{1/2}(\sqrt{2}+1)^2\left[\sqrt{2}-1-\sqrt{-3+\sqrt{10}}\right]}{(-3+\sqrt{10})}, \quad (4.12)$$

$$h_{2,1/20} = \frac{(-2+\sqrt{5})^{1/2}(\sqrt{2}-1)^2(-3+\sqrt{10})}{\left[\sqrt{2}-1-\sqrt{-3+\sqrt{10}}\right]}. \quad (4.13)$$

Proof of (4.10)**.** Putting $n = 1/5$ in the equation (4.9) and using the equation (2.8), we deduce that

$$(h_{2,5}^4 - 12h_{2,5}^2 + 8\sqrt{2}h_{2,5}^2 - 17 + 12\sqrt{2})(-h_{2,5}^4 + 4h_{2,5}^2 + 4\sqrt{2}h_{2,5}^2 - 3 - 2\sqrt{2})^2 = 0. \tag{4.14}$$

The first factor of the equation (4.14) vanishes for the specific value of $q = e^{-\pi\sqrt{5/2}}$, but the second factor does not vanish. Hence by solving the first factor and $0 < h_{2,5} < 1$, we obtain (4.10). □

Proof of (4.11)**.** Using the equation (4.10) in the equation (2.8) with $n = 5$, we obtain the equation (4.11). □

Proofs of (4.12) **and** (4.13)**.** Putting $n = 5$ in the equation (2.9) and using the equation (4.10), we will arrive at the equations (4.12) and (4.13). This completes the proof. □

Theorem 4.3. *If* $X = h_{2,n}h_{2,49n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,49n}}$, *then*

$$\begin{aligned} &Y^4 + \frac{1}{Y^4} + 56\left(Y^3 + \frac{1}{Y^3}\right) + 252\left(Y^2 + \frac{1}{Y^2}\right) + 56\left(Y + \frac{1}{Y}\right) \\ &\times\left(19 + 8\left(X^2 + \frac{1}{X^2}\right)\right) - 128\sqrt{2}\left(X^3 + \frac{1}{X^3}\right) + 224\left(X^2 + \frac{1}{X^2}\right) \\ &+ 1078 = 112\sqrt{2}\left(X + \frac{1}{X}\right)\left[2\left(Y^2 + \frac{1}{Y^2}\right) + 3\left(Y + \frac{1}{Y}\right) + 8\right]. \end{aligned} \tag{4.15}$$

Proof. Employing the equation (1.3) with $k = 2$ in the equation (3.7), we obtain (4.15). □

Corollary 4.3. *We have*

$$h_{2,7} = \sqrt{p_1 - \sqrt{q}_1}, \tag{4.16}$$

$$h_{2,1/7} = \sqrt{p_1 + \sqrt{q}_1}, \tag{4.17}$$

where

$$p_1 = -\sqrt{2}\left(5 + 4\sqrt{2} - \sqrt{65 + 46\sqrt{2}}\right)$$

and

$$q_1 = \left(\left(13 - 5\sqrt{2}\right) - 4\sqrt{7\left(-11 + 8\sqrt{2}\right)}\right)\left(\sqrt{2} + 1\right)^5.$$

Proofs of (4.16) **and** (4.17)**.** Putting $n = 1/7$ in the equation (4.15) and using the equation (2.8), we find that

$$(x^2 + 32x + 20x\sqrt{2} - 64 - 48\sqrt{2})(-x - 12 + 10\sqrt{2})^2 = 0, \tag{4.18}$$

where

$$x = h_{2,7}^2 + \frac{1}{h_{2,7}^2}.$$

The first factor of the equation (4.18) vanishes for specific value of $q = e^{-\pi\sqrt{7/2}}$, where as the second factor does not vanish. Hence by solving the first factor of the equation (4.18), we obtain

$$h_{2,7}^2 + \frac{1}{h_{2,7}^2} = 2\sqrt{2}\left(-2\sqrt{2} - 5 + \sqrt{65 + 46\sqrt{2}}\right). \tag{4.19}$$

On solving the equation (4.19), we arrive at (4.16) and (4.17). □

Theorem 4.4. *If* $X = h_{2,n}h_{2,64n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,64n}}$*, then*

$$\begin{aligned} &16Y^4 + \frac{1}{Y^4} + 8\left(8Y^3 - \frac{11}{Y^3}\right) + 64\left(4Y^2 + \frac{13}{Y^2}\right) + 16\left(74Y + \frac{113}{Y}\right) \\ &+ 4\left(X^4 + \frac{1}{X^4}\right) - 48\sqrt{2}\left(X^3 + \frac{1}{X^3}\right) + 640\left(X^2 + \frac{1}{X^2}\right) - 1072\sqrt{2}\left(X + \frac{1}{X}\right) \\ &+ 1936 = 4\sqrt{2}\left(X + \frac{1}{X}\right)\left[12\left(10Y + \frac{21}{Y}\right) + 2\left(20Y^2 + \frac{21}{Y^2}\right) + \left(8Y^3 + \frac{1}{Y^3}\right)\right] \\ &- 4\left(X^2 + \frac{1}{X^2}\right)\left[4\left(14Y + \frac{19}{Y}\right) + 3\left(4Y^2 + \frac{1}{Y^2}\right)\right] + 8\sqrt{2}\left(X^3 + \frac{1}{X^3}\right)\left(2Y + \frac{1}{Y}\right). \end{aligned} \tag{4.20}$$

Proof. Employing the equation (1.3) with $k = 2$ in the equation (3.12), we obtain (4.20). □

Corollary 4.4. *We have*

$$h_{2,8} = (\sqrt{2} + 1)^2\sqrt{\sqrt{2} - 2\sqrt{2\sqrt{2}}(\sqrt{2} - 1)}, \tag{4.21}$$

$$h_{2,1/8} = \sqrt{\frac{1 + 2\sqrt[4]{2}(\sqrt{2} - 1)}{\sqrt{2}}}. \tag{4.22}$$

Proof of (4.21)**.** Putting $n = 1/8$ in the equation (4.20) and using the equation (2.8), we deduce that

$$\begin{aligned}&(-h_{2,8}^4 + 48h_{2,8}^2 + 34\sqrt{2}h_{2,8}^2 - 34 - 24\sqrt{2})(-h_{2,8}^2 + \sqrt{2})^2 \\ &(h_{2,8}^4 - 20h_{2,8}^2 + 14\sqrt{2}h_{2,8}^2 - 6 + 4\sqrt{2})^2 = 0.\end{aligned} \tag{4.23}$$

We observe that the first factor of the equation (4.23) vanishes for specific value of $q = e^{-2\pi}$, but the other two factors does not vanish. Hence

$$h_{2,8}^4 - 48h_{2,8}^2 - 34\sqrt{2}h_{2,8}^2 + 24\sqrt{2} + 34 = 0. \tag{4.24}$$

On solving the equation (4.24) and $0 < h_{2,8} < 1$, we obtain (4.21). □

Proof of (4.22)**.** Using the equation (4.21) in the equation (2.8) with $n = 8$, we obtain the equation (4.22). □

Theorem 4.5. *If* $X = h_{2,n}h_{2,81n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,81n}}$, *then*

$$\begin{aligned}&Y^6 + \frac{1}{Y^6} - 136\left(Y^5 + \frac{1}{Y^5}\right) + 2458\left(Y^4 + \frac{1}{Y^4}\right) - 12264\left(Y^3 + \frac{1}{Y^3}\right) \\ &+ 40911\left(Y^2 + \frac{1}{Y^2}\right) - 73104\left(Y + \frac{1}{Y}\right) + 95532 = 1024\left(X^4 + \frac{1}{X^4}\right) \\ &\times\left[\left(Y + \frac{1}{Y}\right) - 1\right] + 384\sqrt{2}\left(X^3 + \frac{1}{X^3}\right)\left[6\left(Y^2 + \frac{1}{Y^2}\right) - 3\left(Y + \frac{1}{Y}\right) - 20\right] \\ &+ 1152\left(X^2 + \frac{1}{X^2}\right)\left[25\left(Y + \frac{1}{Y}\right) - 11\left(Y^2 + \frac{1}{Y^2}\right) + 3\left(Y^3 + \frac{1}{Y^3}\right) - 30\right] \\ &+ 192\sqrt{2}\left(X + \frac{1}{X}\right)\left[221\left(Y + \frac{1}{Y}\right) - 110\left(Y^2 + \frac{1}{Y^2}\right) + 35\left(Y^3 + \frac{1}{Y^3}\right)\right. \\ &\left. -5\left(Y^4 + \frac{1}{Y^4}\right) - 266\right].\end{aligned} \tag{4.25}$$

Poorf. Employing the equation (1.3) with $k = 2$ in the equation (3.13), we obtain (4.25). □

Corollary 4.5. *We have,*

$$h_{2,9} = (\sqrt{3}+\sqrt{2})(2-\sqrt{3}), \tag{4.26}$$

$$h_{2,1/9} = (\sqrt{3}-\sqrt{2})(\sqrt{3}+2), \tag{4.27}$$

$$h_{2,36} = (\sqrt{3}+\sqrt{2})(1+\sqrt{2})^3(2+\sqrt{3})^2\left[2-\sqrt{3}-\sqrt{5\sqrt{2}-7}\right], \tag{4.28}$$

$$h_{2,1/36} = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)^3(2-\sqrt{3})^2}{\left[2-\sqrt{3}-\sqrt{5\sqrt{2}-7}\right]}. \tag{4.29}$$

Proof of (4.26)**.** Putting $n = 1/3$ in the equation (4.25) and using the equation (2.8), we deduce that

$$(h_{2,9}^2 - 2(3-2\sqrt{2})h_{2,9} - 1)(-h_{2,9}^2 - 2(3-2\sqrt{2})h_{2,9} + 1)(h_{2,9}-1)^2(h_{2,9}+1)^2$$
$$(-h_{2,9}^8 + 8\sqrt{2}(3+2\sqrt{2})h_{2,9}^6 - 2(35+24\sqrt{2})h_{2,9}^4 + 8\sqrt{2}(3+2\sqrt{2})h_{2,9}^2 - 1)^2 = 0. \tag{4.30}$$

The second factor of the equation (4.30) vanishes for the specific value of $q = e^{-3\pi/\sqrt{2}}$ where as all the other factors does not vanish. Hence

$$h_{2,9}^2 + 6h_{2,9} - 4\sqrt{2}h_{2,9} - 1 = 0. \tag{4.31}$$

On solving the above equation (4.31) and $0 < h_{2,9} < 1$, we obtain equation (4.26). □

Proof of (4.27)**.** Using the equation (4.26) in the equation (2.8) with $n = 9$, we obtain the equation (4.27). □

Proofs of (4.28) **and** (4.29)**.** Putting $n = 9$ in the equation (2.9) and using the equation (4.27), we will arrive at the equations (4.28) and (4.29). This completes the proof. □

Theorem 4.6. *If* $X = h_{2,n}h_{2,121n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,121n}}$*, then*

$$Y^6 - \frac{1}{Y^6} - 286\left(Y^5 + \frac{1}{Y^5}\right) + 11660\left(Y^4 - \frac{1}{Y^4}\right) - 29766\left(Y^3 + \frac{1}{Y^3}\right)$$
$$+ 120021\left(Y^2 - \frac{1}{Y^2}\right) - 44\left(Y + \frac{1}{Y}\right)\left[4077 + 512\left(X^4 + \frac{1}{X^4}\right)\right]$$

$$+4096\left(X^5+\frac{1}{X^5}\right)+1408\sqrt{2}\left(X^3+\frac{1}{X^3}\right)\left[45+16\left(Y^2+\frac{1}{Y^2}\right)\right.$$
$$\left.-8\left(Y-\frac{1}{Y}\right)\right]+44\sqrt{2}\left(X+\frac{1}{X}\right)\left[3386+79\left(Y^4+\frac{1}{Y^4}\right)\right.$$
$$\left.-652\left(Y^3-\frac{1}{Y^3}\right)+1404\left(Y^2+\frac{1}{Y^2}\right)-1548\left(Y-\frac{1}{Y}\right)\right]$$
$$=352\left(X^2+\frac{1}{X^2}\right)\left[56\left(Y^3+\frac{1}{Y^3}\right)-129\left(Y^2-\frac{1}{Y^2}\right)+370\left(Y+\frac{1}{Y}\right)\right]. \tag{4.32}$$

Proof. Employing the equation (1.3) with $k=2$ in the equation (3.14), we obtain (4.32). □

Corollary 4.6. *We have,*

$$h_{2,11}=\sqrt{(5\sqrt{2}+7)(3\sqrt{11}-7\sqrt{2})}, \tag{4.33}$$
$$h_{2,1/11}=\sqrt{(5\sqrt{2}-7)(3\sqrt{11}+7\sqrt{2})}, \tag{4.34}$$
$$h_{2,44}=\frac{(3\sqrt{11}+7\sqrt{2})\left[\sqrt{3\sqrt{11}-7\sqrt{2}}-\sqrt{10-3\sqrt{11}}\right]}{(10-3\sqrt{11})(5\sqrt{2}-7)^{1/2}}, \tag{4.35}$$
$$h_{2,1/44}=\frac{(5\sqrt{2}-7)^{1/2}(3\sqrt{11}-7\sqrt{2})(10-3\sqrt{11})}{\left[\sqrt{3\sqrt{11}-7\sqrt{2}}-\sqrt{10-3\sqrt{11}}\right]}. \tag{4.36}$$

Proof of (4.33)**.** Putting $n=1/11$ in the equation (4.32) and using the equation (2.8), we find that

$$(h_{2,11}^4+14\sqrt{2}(5\sqrt{2}+7)h_{2,11}^2-99-70\sqrt{2})(h_{2,11}^4+2\sqrt{2}(\sqrt{2}+1)h_{2,11}^2-2\sqrt{2}-3)^2$$
$$(-h_{2,11}^4-10(7-5\sqrt{2})h_{2,11}^2-99+70\sqrt{2})^2(-h_{2,11}^2+1+\sqrt{2})^2=0. \tag{4.37}$$

By observing the behaviour of the factors, the first factor vanishes for specific value of $q=e^{-\pi\sqrt{11/2}}$, where as the other factors does not vanish. Hence

$$h_{2,11}^4+140h_{2,11}^2+98\sqrt{2}h_{2,11}^2-99-70\sqrt{2}=0. \tag{4.38}$$

On solving the above equation (4.38) and $0<h_{2,11}<1$, we obtain (4.33). □

Proof of (4.34)**.** Using the equation (4.33) in the equation (2.8) with $n = 11$, we obtain the equation (4.34). □

Proofs of (4.35) **and** (4.36)**.** Putting $n = 11$ in the equation (2.9) and using the equation (4.33), we will arrive at the equations (4.35) and (4.36). This completes the proof. □

Theorem 4.7. *If* $X = h_{2,n}h_{2,225n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,225n}}$, *then*

$$\begin{aligned}
&Y^{12} + \frac{1}{Y^{12}} + 1080\left(Y^{11} + \frac{1}{Y^{11}}\right) + 207540\left(Y^{10} + \frac{1}{Y^{10}}\right) + 2771560\left(Y^{9} + \frac{1}{Y^{9}}\right)\\
&+ 387029\left(Y^{8} + \frac{1}{Y^{8}}\right) + 57329160\left(Y^{7} + \frac{1}{Y^{7}}\right) - 27833756\left(Y^{6} + \frac{1}{Y^{6}}\right)\\
&+ 297924120\left(Y^{5} + \frac{1}{Y^{5}}\right) - 866201937\left(Y^{4} + \frac{1}{Y^{4}}\right) + 317655600\left(Y^{3} + \frac{1}{Y^{3}}\right)\\
&+ 4062669080\left(Y^{2} + \frac{1}{Y^{2}}\right) + 4943360\left(Y + \frac{1}{Y}\right) = 32\left\{32768\left(X^{8} + \frac{1}{X^{8}}\right)\right.\\
&+ 4096\sqrt{2}\left(X^{7} + \frac{1}{X^{7}}\right)\left[15 - 40\left(Y + \frac{1}{Y}\right) + \left(Y^{4} + \frac{1}{Y^{4}}\right)\right] + 1024\left(X^{6} + \frac{1}{X^{6}}\right)\\
&\times\left[1691 - 30\left(Y^{5} + \frac{1}{Y^{5}}\right) + 15\left(Y^{4} + \frac{1}{Y^{4}}\right) - 46\left(Y^{3} + \frac{1}{Y^{3}}\right) + 680\left(Y^{2} + \frac{1}{Y^{2}}\right)\right.\\
&\left.-570\left(Y + \frac{1}{Y}\right)\right] + 256\sqrt{2}\left(X^{5} + \frac{1}{X^{5}}\right)\left[1097 + 180\left(Y^{6} + \frac{1}{Y^{6}}\right) - 210\left(Y^{5} + \frac{1}{Y^{5}}\right)\right.\\
&\left.+735\left(Y^{4} + \frac{1}{Y^{4}}\right) - 3530\left(Y^{3} + \frac{1}{Y^{3}}\right) + 5020\left(Y^{2} + \frac{1}{Y^{2}}\right) - 15238\left(Y + \frac{1}{Y}\right)\right]\\
&+ 32\left(X^{4} + \frac{1}{X^{4}}\right)\left[645646 - 2200\left(Y^{7} + \frac{1}{Y^{7}}\right) + 4620\left(Y^{6} + \frac{1}{Y^{6}}\right)\right.\\
&-18900\left(Y^{5} + \frac{1}{Y^{5}}\right) + 55125\left(Y^{4} + \frac{1}{Y^{4}}\right) - 109260\left(Y^{3} + \frac{1}{Y^{3}}\right)\\
&\left.+333360\left(Y^{2} + \frac{1}{Y^{2}}\right) - 320360\left(Y + \frac{1}{Y}\right)\right] + 20\sqrt{2}\left(X^{3} + \frac{1}{X^{3}}\right)\left[1437\left(Y^{8} + \frac{1}{Y^{8}}\right)\right.
\end{aligned}$$

$$+769902 - 4656\left(Y^7+\frac{1}{Y^7}\right) + 28056\left(Y^6+\frac{1}{Y^6}\right) - 58216\left(Y^5+\frac{1}{Y^5}\right)$$
$$+171884\left(Y^4+\frac{1}{Y^4}\right) - 515688\left(Y^3+\frac{1}{Y^3}\right) + 468008\left(Y^2+\frac{1}{Y^2}\right)$$
$$-1643008\left(Y+\frac{1}{Y}\right)\Bigg] + \left(X^2+\frac{1}{X^2}\right)\Bigg[114518210 - 11886\left(Y^9+\frac{1}{Y^9}\right)$$
$$+39595\left(Y^8+\frac{1}{Y^8}\right) - 696270\left(Y^7+\frac{1}{Y^7}\right) + 662840\left(Y^6+\frac{1}{Y^6}\right)$$
$$-5205000\left(Y^5+\frac{1}{Y^5}\right) + 13248180\left(Y^4+\frac{1}{Y^4}\right) - 11892680\left(Y^3+\frac{1}{Y^3}\right)$$
$$+65473160\left(Y^2+\frac{1}{Y^2}\right) - 19525620\left(Y+\frac{1}{Y}\right)\Bigg] + \frac{1}{\sqrt{2}}\left(X+\frac{1}{X}\right)$$
$$\times\Bigg[2030\left(Y^{10}+\frac{1}{Y^{10}}\right) + 15615\left(Y^9+\frac{1}{Y^9}\right) + 524408\left(Y^8+\frac{1}{Y^8}\right)$$
$$+144775\left(Y^7+\frac{1}{Y^7}\right) + 5585270\left(Y^6+\frac{1}{Y^6}\right) - 8897204\left(Y^5+\frac{1}{Y^5}\right)$$
$$+16404960\left(Y^4+\frac{1}{Y^4}\right) - 79416580\left(Y^3+\frac{1}{Y^3}\right) + 12559260\left(Y^2+\frac{1}{Y^2}\right)$$
$$-226488270\left(Y+\frac{1}{Y}\right) + 13968720\Bigg]\Bigg\} = 6893052420. \tag{4.39}$$

Proof. Employing the equation (1.3) with $k = 2$ in the equation (3.15), we obtain (4.39). □

Corollary 4.7. *We have*

$$h_{2,15} = (\sqrt{2}+1)^2\sqrt{7\sqrt{6}(\sqrt{2}-1) - 5\sqrt{2} - m_1}, \tag{4.40}$$

$$h_{2,1/15} = (\sqrt{2}+1)^2\sqrt{7\sqrt{6}(\sqrt{2}-1) - 5\sqrt{2} + m_1}, \tag{4.41}$$

where $m_1 = \sqrt{(-5+175\sqrt{2}-140\sqrt{3})(\sqrt{2}-1)}$.

Proofs of (4.40) **and** (4.41)**.** Putting $n = 1/15$ in the equation (4.39) and using the equation (2.8), we find that

$$(h_{2,15}^8 + 480h_{2,15}^6 + 340\sqrt{2}h_{2,15}^6 - 1022h_{2,15}^4 - 720\sqrt{2}h_{2,15}^4 + 480h_{2,15}^2 + 340\sqrt{2}h_{2,15}^2 + 1)(-h_{2,15}^{12} - 268h_{2,15}^{10} + 190\sqrt{2}h_{2,15}^{10} - 1067h_{2,15}^8 + 752\sqrt{2}h_{2,15}^8 - 1592h_{2,15}^6 - 1 + 1132\sqrt{2}h_{2,15}^6 - 1067h_{2,15}^4 + 752\sqrt{2}h_{2,15}^4 - 268h_{2,15}^2 + 190\sqrt{2}h_{2,15}^2)^2(h_{2,15}^8 + 1 + 20\sqrt{2}h_{2,15}^6 - 62h_{2,15}^4 - 48\sqrt{2}h_{2,15}^4 + 32h_{2,15}^2 + 20\sqrt{2}h_{2,15}^2 + 32h_{2,15}^6)^2 = 0. \tag{4.42}$$

By observing the behaviour of the factors, the first factor vanishes for specific value of $q = e^{-\pi\sqrt{15/2}}$, where as the other factors does not vanish. Hence

$$h_{2,15}^8 + 20\sqrt{2}(12\sqrt{2}+17)h_{2,15}^6 - 2(511+360\sqrt{2})h_{2,15}^4 + 20\sqrt{2}(12\sqrt{2}+17)h_{2,15}^2 + 1. \tag{4.43}$$

On solving the above equation (4.43) and $0 < h_{2,15} < 1$, we obtain the equations (4.40) and (4.41). This completes the proof. □

Theorem 4.8. *If* $X = h_{2,n}h_{2,289n}$ *and* $Y = \dfrac{h_{2,n}}{h_{2,289n}}$, *then*

$$Y^9 + \frac{1}{Y^9} - 1938\left(Y^8 + \frac{1}{Y^8}\right) + 698649\left(Y^7 + \frac{1}{Y^7}\right) - 13577968\left(Y^6 + \frac{1}{Y^6}\right) - 138056524\left(Y^5 + \frac{1}{Y^5}\right) - 945602424\left(Y^4 + \frac{1}{Y^4}\right) - 3499343036\left(Y^3 + \frac{1}{Y^3}\right) - 8483099280\left(Y^2 + \frac{1}{Y^2}\right) - 14485100818\left(Y + \frac{1}{Y}\right) - 1048576\left(X^8 + \frac{1}{X^8}\right) + 4456448\left\{\sqrt{2}\left(X^7 + \frac{1}{X^7}\right)\left[2 + \left(Y + \frac{1}{Y}\right)\right] - \left(X^6 + \frac{1}{X^6}\right)\left[42 + 32\left(Y + \frac{1}{Y}\right) + 7\left(Y^2 + \frac{1}{Y^2}\right)\right] + \frac{1}{\sqrt{2}}\left(X^5 + \frac{1}{X^5}\right)\left[328 + 247\left(Y + \frac{1}{Y}\right) + 104\left(Y^2 + \frac{1}{Y^2}\right) + 13\left(Y^3 + \frac{1}{Y^3}\right)\right]\right\} - 17408\left(X^4 + \frac{1}{X^4}\right)\left[85519\left(Y + \frac{1}{Y}\right) + 43066\left(Y^2 + \frac{1}{Y^2}\right) + 11265\left(Y^3 + \frac{1}{Y^3}\right) + 879\left(Y^4 + \frac{1}{Y^4}\right) + 110030\right]$$

$$+4352\sqrt{2}\left(X^3+\frac{1}{X^3}\right)$$
$$\left[810176+677050\left(Y+\frac{1}{Y}\right)+357624\left(Y^2+\frac{1}{Y^2}\right)+124911\left(Y^3+\frac{1}{Y^3}\right)\right.$$
$$\left.+20840\left(Y^4+\frac{1}{Y^4}\right)+1047\left(Y^5+\frac{1}{Y^5}\right)\right]-2176\left\{\left(X^2+\frac{1}{X^2}\right)\left[3807186\left(Y+\frac{1}{Y}\right)\right.\right.$$
$$+2192811\left(Y^2+\frac{1}{Y^2}\right)+816475\left(Y^3+\frac{1}{Y^3}\right)+197160\left(Y^4+\frac{1}{Y^4}\right)+4582352$$
$$\left.+19219\left(Y^5+\frac{1}{Y^5}\right)+645\left(Y^6+\frac{1}{Y^6}\right)\right]+\frac{1}{\sqrt{2}}\left(X+\frac{1}{X}\right)\left[8103555\left(Y+\frac{1}{Y}\right)\right.$$
$$+4816651\left(Y^2+\frac{1}{Y^2}\right)+1931229\left(Y^3+\frac{1}{Y^3}\right)+78927\left(Y^5+\frac{1}{Y^5}\right)+9749708$$
$$\left.\left.+2501\left(Y^6+\frac{1}{Y^6}\right)+81\left(Y^7+\frac{1}{Y^7}\right)+489418\left(Y^4+\frac{1}{Y^4}\right)\right]\right\}=169023853880. \tag{4.44}$$

Proof. Employing the equation (1.3) with $k = 2$ in the equation (3.16), we obtain (4.44). □

Corollary 4.8. *We have*

$$h_{2,17}=\sqrt{210-148\sqrt{2}+3\sqrt{17}(\sqrt{2}-1)^4-m_2}, \tag{4.45}$$

$$h_{2,1/17}=\sqrt{210-148\sqrt{2}+3\sqrt{17}(\sqrt{2}-1)^4+m_2}. \tag{4.46}$$

where $m_2=2\sqrt{44047-31146\sqrt{2}+3\sqrt{17}(3561-2518\sqrt{2})}$.

Proof of (4.45) **and** (4.46)**.** Putting $n = 1/17$ in the equation (4.44) and using the equation (2.8), we find that

$$(h_{2,17}^8-840h_{2,17}^6+592\sqrt{2}h_{2,17}^6-1490h_{2,17}^4+1056\sqrt{2}h_{2,17}^4-840h_{2,17}^2+1$$
$$+592\sqrt{2}h_{2,17}^2)(h_{2,17}^2+1)^2(-h_{2,17}^8+480h_{2,17}^6+344\sqrt{2}h_{2,17}^6-1030h_{2,17}^4-1$$
$$-720\sqrt{2}h_{2,17}^4+480h_{2,17}^2+344\sqrt{2}h_{2,17}^2)^2(h_{2,17}^2-6h_{2,17}+4\sqrt{2}h_{2,17}-1)^2$$
$$(-h_{2,17}^2-6h_{2,17}+4\sqrt{2}h_{2,17}+1)^2=0. \tag{4.47}$$

By observing the behaviour of the factors, the first factor vanishes for specific value of $q = e^{-\pi\sqrt{17/2}}$, where as the other factors does not vanish. Hence

$$h_{2,17}^8 - (840-592\sqrt{2})h_{2,17}^6 - (1490-1056\sqrt{2})h_{2,17}^4 - (840-592\sqrt{2})h_{2,17}^2 + 1 = 0. \tag{4.48}$$

On solving the above equation (4.48) and $0 < h_{2,17} < 1$, we obtain the equations (4.45) and (4.46). This completes the proof. □

5. Evaluations of Ramanujan–Göllnitz–Gordon Continued Fraction

On page 229 of his second notebook, Ramanujan recorded the continued fraction which is known as Ramanujan–Göllnitz–Gordon continued fraction along with the two identities as follows:

$$H(q) := \frac{q^{1/2}}{1+q} \; {}_{+} \; \frac{q^2}{1+q^3} \; {}_{+} \; \frac{q^4}{1+q^5} \; {}_{+} \; \frac{q^6}{1+q^7} \; {}_{+} \; \cdots, \qquad |\, q \,| < 1, \tag{5.1}$$

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)}, \tag{5.2}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{5.3}$$

Proofs of the above identities (5.1), (5.2) and (5.3) can be found in [15, p. 221] and for more details one can see, [6], [10] and [18]. In [3], the authors have established several theorems for the explicit evaluations of Ramanujan–Göllnitz–Gordon continued fraction by using some parameterizations for Ramanujan's theta-functions. They also establish the following identity

$$H^2(e^{-\pi\sqrt{n/2}}) = \frac{2^{1/4}h_{2,n} - 1}{2^{1/4}h_{2,n} + 1}, \quad \text{for any positive rational number n}, \tag{5.4}$$

which follows from the identity established by Chan and Huang in [6]. We establish the following lemma by using the values of $h_{2,n}$ in the identity (5.4).

Lemma 5.1. *We have*

$$H^2(e^{-\pi\sqrt{5/2}}) = \frac{2^{1/4}(\sqrt{2}-1)(2+\sqrt{5})^{1/2}-1}{2^{1/4}(\sqrt{2}-1)(2+\sqrt{5})^{1/2}+1}, \tag{5.5}$$

$$H^2(e^{-\pi/\sqrt{10}}) = \frac{2^{1/4}(\sqrt{2}+1)(-2+\sqrt{5})^{1/2}-1}{2^{1/4}(\sqrt{2}+1)(-2+\sqrt{5})^{1/2}+1}, \tag{5.6}$$

$$H^2(e^{-3\pi/\sqrt{2}}) = \frac{2^{1/4}(\sqrt{3}+\sqrt{2})(2-\sqrt{3})-1}{2^{1/4}(\sqrt{3}+\sqrt{2})(2-\sqrt{3})+1}, \tag{5.7}$$

$$H^2(e^{-\pi/3\sqrt{2}}) = \frac{2^{1/4}(\sqrt{3}-\sqrt{2})(2+\sqrt{3})-1}{2^{1/4}(\sqrt{3}-\sqrt{2})(2+\sqrt{3})+1}, \tag{5.8}$$

$$H^2(e^{-\pi\sqrt{11/2}}) = \frac{2^{1/4}\sqrt{(5\sqrt{2}+7)(3\sqrt{11}-7\sqrt{2})}-1}{2^{1/4}\sqrt{(5\sqrt{2}+7)(3\sqrt{11}-7\sqrt{2})}+1}, \tag{5.9}$$

$$H^2(e^{-\pi/\sqrt{22}}) = \frac{2^{1/4}\sqrt{(5\sqrt{2}-7)(3\sqrt{11}+7\sqrt{2})}-1}{2^{1/4}\sqrt{(5\sqrt{2}-7)(3\sqrt{11}+7\sqrt{2})}+1}, \tag{5.10}$$

$$H^2(e^{-\pi\sqrt{22}}) = \frac{2^{1/4}\left(\dfrac{\sqrt{3\sqrt{11}-7\sqrt{2}}-\sqrt{10-3\sqrt{11}}}{3\sqrt{11}-7\sqrt{2}}\right)-\dfrac{(10-3\sqrt{11})}{\sqrt{5\sqrt{2}+7}}}{2^{1/4}\left(\dfrac{\sqrt{3\sqrt{11}-7\sqrt{2}}-\sqrt{10-3\sqrt{11}}}{3\sqrt{11}-7\sqrt{2}}\right)+\dfrac{(10-3\sqrt{11})}{\sqrt{5\sqrt{2}+7}}}, \tag{5.11}$$

$$H^2(e^{-\pi/2\sqrt{22}}) = \frac{\dfrac{2^{1/4}\sqrt{5\sqrt{2}-7}}{3\sqrt{11}+7\sqrt{2}}-\left(\dfrac{\sqrt{3\sqrt{11}-7\sqrt{2}}-\sqrt{10-3\sqrt{11}}}{10-3\sqrt{11}}\right)}{\dfrac{2^{1/4}\sqrt{5\sqrt{2}-7}}{3\sqrt{11}+7\sqrt{2}}+\left(\dfrac{\sqrt{3\sqrt{11}-7\sqrt{2}}+\sqrt{10-3\sqrt{11}}}{10-3\sqrt{11}}\right)}, \tag{5.12}$$

$$H^2(e^{-\pi\sqrt{15/2}}) = \frac{2^{1/4}(\sqrt{2}+1)^2\sqrt{7\sqrt{6}(\sqrt{2}-1)-5\sqrt{2}-m_1}-1}{2^{1/4}(\sqrt{2}+1)^2\sqrt{7\sqrt{6}(\sqrt{2}-1)-5\sqrt{2}-m_1}+1}, \tag{5.13}$$

$$H^2(e^{-\pi/\sqrt{30}}) = \frac{2^{1/4}(\sqrt{2}+1)^2\sqrt{7\sqrt{6}(\sqrt{2}-1)-5\sqrt{2}+m_1}-1}{2^{1/4}(\sqrt{2}+1)^2\sqrt{7\sqrt{6}(\sqrt{2}-1)-5\sqrt{2}+m_1}+1}, \tag{5.14}$$

$$\textit{where } m_1 = \sqrt{(-5+175\sqrt{2}-140\sqrt{3})(\sqrt{2}-1)}.$$

$$H^2(e^{-\pi\sqrt{17/2}}) = \frac{2^{1/4}\sqrt{210-148\sqrt{2}+3\sqrt{17}(\sqrt{2}-1)^4-m_2}-1}{2^{1/4}\sqrt{210-148\sqrt{2}+3\sqrt{17}(\sqrt{2}-1)^4-m_2}+1}, \tag{5.15}$$

$$H^2(e^{-\pi/\sqrt{34}}) = \frac{2^{1/4}\sqrt{210-148\sqrt{2}+3\sqrt{17}(\sqrt{2}-1)^4+m_2}-1}{2^{1/4}\sqrt{210-148\sqrt{2}+3\sqrt{17}(\sqrt{2}-1)^4+m_2}+1}, \tag{5.16}$$

$$\textit{where } m_2 = 2\sqrt{44047-31146\sqrt{2}+3\sqrt{17}(3561-2518\sqrt{2})}.$$

$$H^2(e^{-2\pi\sqrt{2}}) = \frac{2^{1/4}\left(4+3\sqrt{2}+(\sqrt{2}+1)^{5/2}\right)-a_5}{2^{1/4}\left(3\sqrt{2}(\sqrt{2}+1)+(\sqrt{2}+1)^{5/2}\right)+a_5}, \tag{5.17}$$

$$\textit{where } a_5 = \sqrt{76+54\sqrt{2}+(488\sqrt{2}+690)(\sqrt{2}-1)^{5/2}}.$$

$$H^2(e^{-\pi\sqrt{7/2}}) = \frac{2^{1/4}(\sqrt{p_1-\sqrt{q_1}})-1}{2^{1/4}(\sqrt{p_1-\sqrt{q_1}})+1}, \tag{5.18}$$

$$H^2(e^{-\pi/\sqrt{14}}) = \frac{2^{1/4}(\sqrt{p_1+\sqrt{q_1}})-1}{2^{1/4}(\sqrt{p_1+\sqrt{q_1}})+1}, \tag{5.19}$$

where

$$p_1 = -\sqrt{2}\left(5+4\sqrt{2}-\sqrt{65+46\sqrt{2}}\right) \tag{5.20}$$

and

$$q_1 = \left(\left(13-5\sqrt{2}\right)-4\sqrt{7\left(-11+8\sqrt{2}\right)}\right)\left(\sqrt{2}+1\right)^5.$$

$$H^2(e^{-\pi/4\sqrt{2}}) = \frac{\left(\frac{\sqrt{\sqrt{2}+1}}{\sqrt[4]{2}}\right)\left(\sqrt{\sqrt{2}+1}+\sqrt{2\left(\sqrt{\sqrt{2}+1}-\sqrt{2}\right)}-1\right)-1}{\left(\frac{\sqrt{\sqrt{2}+1}}{\sqrt[4]{2}}\right)\left(\sqrt{\sqrt{2}+1}+\sqrt{2\left(\sqrt{\sqrt{2}+1}-\sqrt{2}\right)}-1\right)+1}, \tag{5.21}$$

$$H^2(e^{-\pi\sqrt{10}}) = \frac{2^{1/4}a_1(\sqrt{2}-1-\sqrt{-3+\sqrt{10}})-(-3+\sqrt{10})}{2^{1/4}a_1(\sqrt{2}-1-\sqrt{-3+\sqrt{10}})+(-3+\sqrt{10})}, \tag{5.22}$$

$$H^2(e^{-\pi/2\sqrt{10}}) = \frac{2^{1/4}a_1(-3+\sqrt{10})-(\sqrt{2}-1-\sqrt{-3+\sqrt{10}})}{2^{1/4}a_1(-3+\sqrt{10})+(\sqrt{2}-1-\sqrt{-3+\sqrt{10}})}, \tag{5.23}$$

where $a_1 = \sqrt{2+\sqrt{5}}(\sqrt{2}+1)^2$.

$$H^2(e^{-3\pi\sqrt{2}}) = \frac{2^{1/4}a_3\left[2-\sqrt{3}-\sqrt{5\sqrt{2}-7}\right]-1}{2^{1/4}a_3\left[2-\sqrt{3}-\sqrt{5\sqrt{2}-7}\right]+1}, \tag{5.24}$$

$$H^2(e^{-\pi/6\sqrt{2}}) = \frac{2^{1/4}-a_3\left[2-\sqrt{3}-\sqrt{5\sqrt{2}-7}\right]}{2^{1/4}+a_3\left[2-\sqrt{3}-\sqrt{5\sqrt{2}-7}\right]}, \tag{5.25}$$

where $a_3 = (\sqrt{3}+\sqrt{2})(1+\sqrt{2})^3(2+\sqrt{3})^2$.

Remark 5.1. For a different proofs of the equations (5.5)–(5.8), one can see [3].

6. Evaluations of Ramanujan–Selberg Continued Fraction

The continued fraction identity

$$\begin{aligned} V(q) &:= \frac{q^{1/8}}{1+}\,\frac{q}{1+}\,\frac{q^2+q}{1+}\,\frac{q^3}{1+}\,\frac{q^4+q^2}{1+}\cdots \\ &= \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}, \qquad |q|<1, \end{aligned} \tag{6.1}$$

appears as Formula 5 [15, p.290] and was first proved by Selberg [17, eq.(54)]. Other proofs have been given by Ramanathan [16] and Andrews, Berndt, Jacobsen and Lamphere [2]. Adiga, Mahadeva Naika and Ramya Rao [1] have obtained two integral representations for $V(q)$, also derived a relation between $V(q)$ and $V(q^n)$ and some explicit evaluations of $V(q)$. Recently, Mahadeva Naika, Remy Y Denis and Sushan Bairy [11] have established several modular relations and explicit evaluations of Ramanujan-Selberg continued fraction.

Lemma 6.1. *We have*

$$\frac{\varphi^2(q^{1/2})}{\varphi^2(q)} = 1 + 4V^4(q). \tag{6.2}$$

Proof. Using the equations (2.1), (2.3), (2.4), (2.5) and (6.1), we arrive at the equation (6.2). □

Lemma 6.2. *For any positive rational number n, we have*

$$V^4(e^{-\pi\sqrt{2n}}) = \frac{\sqrt{2}h_{2,n}^2 - 1}{4}. \tag{6.3}$$

Proof. Using the equations (6.2) and (1.3) with $k = 2$, we obtain the equation (6.3). □

We establish the following lemma by using the values of $h_{2,n}$ in the identity (6.3).

Lemma 6.3. *We have*

$$V^4(e^{-\pi 2\sqrt{2}}) = \frac{4\sqrt{2}+5}{4} - \frac{(\sqrt{2}+1)^{3/2}}{\sqrt{2}}, \tag{6.4}$$

$$V^4(e^{-\pi/\sqrt{2}}) = \frac{\sqrt{\sqrt{2}-1}}{2\sqrt{2}}, \tag{6.5}$$

$$V^4(e^{-\pi/2\sqrt{2}}) = \frac{\sqrt{2}(\sqrt{2}+1)(\sqrt{\sqrt{2}+1}-1)\sqrt{2\sqrt{\sqrt{2}+1}-2\sqrt{2}}}{4}, \tag{6.6}$$

$$V^4(e^{-\pi\sqrt{10}}) = \frac{(3-2\sqrt{2})(\sqrt{10}-3)}{4}, \tag{6.7}$$

$$V^4(e^{-\pi\sqrt{2/5}}) = \frac{(3+2\sqrt{2})(\sqrt{10}-3)}{4}, \tag{6.8}$$

$$V^4(e^{-4\pi}) = 33 - 20(2^{3/4}) - 28(2^{1/4}) + 24\sqrt{2}, \tag{6.9}$$

$$V^4(e^{-\pi/2}) = \frac{\sqrt[4]{8}-\sqrt[4]{2}}{2}, \tag{6.10}$$

$$V^4(e^{-3\sqrt{2}\pi}) = \frac{(5\sqrt{2}-7)(7-4\sqrt{3})}{4}, \tag{6.11}$$

$$V^4(e^{-\sqrt{2}\pi/3}) = \frac{(5\sqrt{2}-7)(7+4\sqrt{3})}{4}, \tag{6.12}$$

$$V^4(e^{-\pi/3\sqrt{2}}) = (2-\sqrt{3})(5\sqrt{2}+4\sqrt{3})\sqrt{5\sqrt{2}-7}, \tag{6.13}$$

$$V^4(e^{-\pi\sqrt{22}}) = \frac{(10-3\sqrt{11})(3\sqrt{11}-7\sqrt{2})}{4}, \tag{6.14}$$

$$V^4(e^{-\pi\sqrt{2/11}}) = \frac{(10-3\sqrt{11})(3\sqrt{11}+7\sqrt{2})}{4}, \tag{6.15}$$

$$V^4(e^{-\pi/\sqrt{11}}) = \frac{(5\sqrt{2}+7)(\sqrt{11}-3)\sqrt{3\sqrt{11}-7\sqrt{2}}}{4}, \tag{6.16}$$

$$V^4(e^{-\pi\sqrt{30}}) = \frac{70\sqrt{6}+98\sqrt{3}-171-120\sqrt{2}-(17\sqrt{2}+24)v_1}{4}, \tag{6.17}$$

$$V^4(e^{-\pi\sqrt{2/15}}) = \frac{70\sqrt{6}+98\sqrt{3}-171-120\sqrt{2}+(17\sqrt{2}+24)v_1}{4}, \tag{6.18}$$

where

$$v_1 := \sqrt{(175\sqrt{2}-140\sqrt{3}-5)(\sqrt{2}-1)}.$$

$$V^4(e^{-\pi\sqrt{34}}) = \frac{210\sqrt{2}-297+51\sqrt{34}-72\sqrt{17}-\sqrt{2}v_2}{4}, \tag{6.19}$$

$$V^4(e^{-\pi\sqrt{2/17}}) = \frac{210\sqrt{2}-297+51\sqrt{34}-72\sqrt{17}+\sqrt{2}v_2}{4}, \tag{6.20}$$

where

$$v_2 := 2\sqrt{44047-31146\sqrt{2}+3\sqrt{17}(3561-2518\sqrt{2})}.$$

7. Evaluations of a Continued Fraction of Eisenstein

In [14], the authors have established the following continued fraction

$$E(q) := \frac{(q;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1} + \frac{2q}{1-q^2} + \frac{-q^3-q}{1+q^4} + \frac{q^5+q^3}{1-q^6} + \cdots, \quad \text{for } |q|<1. \tag{7.1}$$

They have also established several modular relations between a continued fraction of Eisenstein $E(q)$ and $E(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13,$ 15, 17, 19, 23, 25, 29, 31 and 55. They have also established several explicit evaluations for $E(e^{-\pi\sqrt{n}})$, where n is any positive rational.

Lemma 7.1. *We have*

$$\frac{\varphi^2(q)}{\varphi^2(q^2)} = \frac{2}{1+E^4(q)}. \tag{7.2}$$

Proof. Using the equations (2.1), (2.2), (2.6) and (2.7), we arrive at the equation (7.2). □

Lemma 7.2. *For any positive rational number n, we have*

$$E^4(e^{-\pi\sqrt{n/2}}) = \frac{\sqrt{2}-h_{2,n}^2}{h_{2,n}^2}. \tag{7.3}$$

Proof. Using the equations (7.2) and (1.3) with $k = 2$, we arrive at the equation (7.3). □

We establish the following lemma by using the values of $h_{2,n}$ in the identity (7.3).

Lemma 7.3. *We have*

$$E^4(e^{-\pi\sqrt{2}}) = (2-\sqrt{2})\sqrt{\sqrt{2}+1}, \tag{7.4}$$

$$E^4(e^{-\pi/2\sqrt{2}}) = 5+4\sqrt{2}-2\sqrt{14+10\sqrt{2}}, \tag{7.5}$$

$$E^4(e^{-\pi\sqrt{5/2}}) = (\sqrt{10}-3)(\sqrt{2}+1)^2, \tag{7.6}$$

$$E^4(e^{-\pi/\sqrt{10}}) = (\sqrt{10}-3)(\sqrt{2}-1)^2, \tag{7.7}$$

$$E^4(e^{-\pi\sqrt{10}}) = (2\sqrt{2}+\sqrt{10})(\sqrt{2}-1)\sqrt{\sqrt{10}-3}, \tag{7.8}$$

$$E^4(e^{-\pi/2\sqrt{10}}) = (1662777 + 1175760\sqrt{2} + 743616\sqrt{5} + 525816\sqrt{10})^{1/2} - (1662776 + 1175760\sqrt{2} + 743616\sqrt{5} + 525816\sqrt{10})^{1/2}, \tag{7.9}$$

$$E^4(e^{-2\pi}) = 2^{5/4}(\sqrt{2} - 1), \tag{7.10}$$

$$E^4(e^{-\pi/4}) = 33 - 28\sqrt[4]{2} + 24\sqrt{2} - 20\sqrt[4]{8}, \tag{7.11}$$

$$E^4(e^{-3\pi/\sqrt{2}}) = (5\sqrt{2} - 7)(4\sqrt{3} + 7), \tag{7.12}$$

$$E^4(e^{-\pi/3\sqrt{2}}) = (5\sqrt{2} - 7)(7 - 4\sqrt{3}), \tag{7.13}$$

$$E^4(e^{-3\pi\sqrt{2}}) = (2 - \sqrt{3})(4\sqrt{3} + 5\sqrt{2})\sqrt{5\sqrt{2} - 7}, \tag{7.14}$$

$$E^4(e^{-\pi\sqrt{11/2}}) = (3\sqrt{11} + 7\sqrt{2})(10 - 3\sqrt{11}), \tag{7.15}$$

$$E^4(e^{-\pi/\sqrt{22}}) = (3\sqrt{11} - 7\sqrt{2})(10 - 3\sqrt{11}), \tag{7.16}$$

$$E^4(e^{-\pi\sqrt{22}}) = (\sqrt{11} - 3)(5\sqrt{2} + 7)\sqrt{3\sqrt{11} - 7\sqrt{2}}, \tag{7.17}$$

$$E^4(e^{-\pi\sqrt{15/2}}) = 98\sqrt{3} - 120\sqrt{2} + 70\sqrt{6} - 171 + (17\sqrt{2} + 24)e_1, \tag{7.18}$$

$$E^4(e^{-\pi/\sqrt{30}}) = 98\sqrt{3} - 120\sqrt{2} + 70\sqrt{6} - 171 - (17\sqrt{2} + 24)e_1, \tag{7.19}$$

where

$$e_1 = \sqrt{(175\sqrt{2} - 140\sqrt{3} - 5)(\sqrt{2} - 1)}.$$

$$E^4(e^{-\pi\sqrt{17/2}}) = -297 + 51\sqrt{34} + 210\sqrt{2} - 72\sqrt{17} + e_2\sqrt{2}, \tag{7.20}$$

$$E^4(e^{-\pi/\sqrt{34}}) = -297 + 51\sqrt{34} + 210\sqrt{2} - 72\sqrt{17} - e_2\sqrt{2}, \tag{7.21}$$

where

$$e_2 = 2\sqrt{44047 - 31146\sqrt{2} + 3\sqrt{17}(3561 - 2518\sqrt{2})}.$$

Acknowledgement. The authors are grateful to Prof. H. M. Srivastava for his valuable suggestions to improve the quality of the paper.

References

[1] C. Adiga, M. S. Mahadeva Naika, and Ramya Rao, Integral representations and some explicit evaluations of a continued fraction of Ramanujan, *JP J. Algebra Number Theory Appl.*, **2** no. 1 (2002), 5-2.

[2] G. E. Andrews, B. C. Berndt, L. Jacobsen, and R. L. Lamphere, The continued fractions found in the unorganized portions of Ramanujan's notebooks, *Memoir No.477, Amer. Math. Soc. 99*, Providence, Rhode Island, 1992.

[3] N. D. Baruah and Nipen Saikia, Explicit evaluations of Ramanujan–Göllnitz–Gordon continued fraction, Monatsh. Math., **154** no. 4 (2008), 271-288.

[4] B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, 1991.

[5] B. C. Berndt, *Ramanujan's Notebooks*, Part V, Springer-Verlag, New York, 1998.

[6] H. H. Chan and S.-S. Haung, On the Ramanujan-Göllnitz-Gordon continued fraction, *Ramanujan J.*, **1** (1997), 75-90.

[7] M. S. Mahadeva Naika, S. Chandankumar , and M. Manjunatha, On some new modular equations and their applications to continued fractions, *Int. Math. Forum*, **6** no. 58 (2011), 2881-2905.

[8] M. S. Mahadeva Naika, S. Chandankumar , and K. Sushan Bairy, Modular equations for the ratios of Ramanujan's theta function ψ and evaluations, *New Zealand J. Math.*, **40** (2010), 33-48.

[9] M. S. Mahadeva Naika, S. Chandankumar , and K. Sushan Bairy, New identities for Ramanujan's cubic continued fraction, *Funct. Approx. Comment. Math.*, **46** part 1 (2012), 29-44.

[10] M. S. Mahadeva Naika, B. N. Dharmendra , and S. Chandankumar, Some identities for Ramanujan–Göllnitz–Gordon Continued fraction, *Aust. J. Math. Anal. Appl.*, **10** no. 1, Art. 2, (2013), 1-36 .

[11] M. S. Mahadeva Naika, R. Y. Denis, and K. S. Bairy, On some Ramanujan–Selberg continued fraction, *Indian J. Math.*, **51** no. 3 (2009), 585-596.

[12] M. S. Mahadeva Naika, K. Sushan Bairy , and S. Chandankumar, *On some explicit evaluation of the ratios of Ramanujan's theta–function*, (communicated).

[13] M. S. Mahadeva Naika, K. Sushan Bairy , and M. Manjunatha, Some new modular equations of degree four and their explicit evaluations, *Eur. J. Pure Appl. Math.*, **3** no. 6 (2010), 924-947.

[14] M. S. Mahadeva Naika, K. Sushan Bairy, M. Manjunatha, Certain identities for a continued fraction of Eisenstein, *Far East J. Appl. Math., Far East J. Math. Sci.*, **57** no. 2, 205-226.

[15] S. Ramanujan, Notebooks (2 volumes), *Tata Institute of Fundamental Research*, Bombay, 1957.

[16] K. G. Ramanathan, Ramanujan's continued fraction, *Indian J. Pure Appl. Math.*, **16** (1985), 695-724

[17] A. Selberg, Uber einige arithmetische identitaten, *Avh. Norske Vid. -Akad. Oslo I. Mat. -Naturv. Kl.*, (1936) 2-23.

[18] K. R. Vasuki and B. R. Srivatsa Kumar, Certain identities for Ramanujan–Göllnitz–Gordon continued fraction, *J. Comput. Appl. Math.*, **187** (2006), 87-95.

[19] J. Yi, Theta-function identities and the explicit formulas for theta-function and their applications, *J. Math. Anal. Appl.*, **292** (2004), 381-400.