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N-Centralizing Generalized Derivations on Left Ideals *

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Abstract

Let R be a prime ring with center $Z(R)$, right Utumi quotient ring U and extended centroid C, S be a non-empty subset of R and $n \geq 1$ a fixed integer. A mapping $f : R \longrightarrow R$ is said to be n-centralizing on S if $[f(x), x^n] \in Z(R)$, for all $x \in S$. In this paper we will prove that if F is a non-zero generalized derivation of R, I a non-zero left ideal of R, $n \geq 1$ a fixed integer such that F is n-centralizing on the set $[I, I]$, then there exists $a \in U$ and $\alpha \in C$ such that $F(x) = xa$, for all $x \in R$ and $I(a - \alpha) = (0)$, unless when $x_1 s_4(x_2, x_3, x_4, x_5)$ is an identity for I.

Keywords and Phrases: *Prime ring, Generalized derivation.*

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Throughout the paper unless specifically stated, R always denotes a prime ring with center $Z(R)$ and extended centroid C, right Utumi quotient ring U. For any pair of elements $x, y \in R$, we denote $[x, y] = xy - yx$, the commutator of x, y and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A mapping $f : R \longrightarrow R$ is said to be n-centralizing (resp. n-commuting) on a non-empty subset S of R if $[f(x), x^n] \in Z(R)$ (resp. $[f(x), x^n] = 0$) for all $x \in S$ and n a fixed positive integer.
An additive mapping $d : R \longrightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A well known result of Posner (Theorem 4 in [23]) states that R must be commutative if there exists a nonzero derivation d on R such that $[d(x), x] \in Z(R)$ for all $x \in R$. Many related generalizations have been obtained by a number of authors in the literature (see [1], [16], [17], [22]). An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Obviously any derivation is a generalized derivation. One basic example of a generalized derivation is the mapping of the form $g(x) = ax + xb$ for all $x \in R$ and for some fixed $a, b \in R$. This kind of generalized derivations are called as inner generalized derivations of R. Many authors studied generalized derivations in context of prime and semiprime rings (see [11], [18], [19]). In [18] T.K. Lee extended the definition of a generalized derivation as follows: an additive mapping $F : J \longrightarrow U$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in J$, where U is the right Utumi quotient ring of R, J is a dense right ideal of R and d is a derivation from J to U. He also proved that every generalized derivation of R can be uniquely extended to a generalized derivation of U. In fact there exists a in U and a derivation d of U such that $F(x) = ax + d(x)$ for all $x \in U$ (Theorem 3 in [18]). A corresponding form to dense left ideals as follows: an additive mapping $F : I \longrightarrow U$ is called a generalized derivation if there exists a derivation $d : I \longrightarrow U$ such that $F(xy) = xF(y) + d(x)y$, for all $x, y \in I$, where U is the left Utumi quotient ring of R, I is a dense left ideal of R. Following the same methods as in [14], one can extend F uniquely to a generalized derivation of U. The extended generalized derivation of U can also be denoted by F and has the form $F(x) = xa + d(x)$ for all $x \in U$ and some $a \in U$, where d is a derivation of U. In this paper we shall prove some theorems for a generalized derivation which are in spirit of the above mentioned result of Posner and the results of Deng (Theorem 2 in [7]), Deng and Bell (Theorem 2 in [8]).
In the first section we will prove the following:

Theorem 1. *Let R be a prime ring, F a non-zero generalized derivation of R, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer such that F is n-centralizing on L. Then either $F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$ or R satisfies s_4, the standard identity of degree 4.*

Then we will extend the above result to the one-sided case, more precisely we will prove:

Theorem 2. *Let R be a prime ring, F a non-zero generalized derivation of R, I a non-zero left ideal of R, $n \geq 1$ a fixed integer such that F is n-centralizing on the set $[I, I]$. Then there exists $a \in U$ and $\alpha \in C$ such that $F(x) = xa$, for all $x \in R$ and $I(a - \alpha) = (0)$, unless when $x_1 s_4(x_2, x_3, x_4, x_5)$ is an identity for I.*

1. N-centralizing Maps on Lie Ideals

Here we begin with the following:

Lemma 1. *Let R be a non-commutative prime ring, $a, b \in R$, I a two-sided ideal of R, $n \geq 1$ a fixed integer such that $[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]^n] \in Z(R)$, for any $r_1, r_2 \in I$. Then either $a, b \in Z(R)$ or R satisfies the standard identity s_4.*

Proof. Suppose that either $a \notin Z(R)$ or $b \notin Z(R)$. In both cases

$$\Big[[a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]^n], x_3\Big] \tag{1}$$

is a non-trivial generalized polynomial identity for I and so also for R (see [4]). Moreover, by Theorem 2 in [4], (1) is also an identity for RC. By Martindale's result in [21] RC is a primitive ring with non-zero socle. There exists a vectorial space V over a division ring D such that RC is dense of D-linear transformations over V.
Firstly we will prove that $dim_D V \leq 2$. By contradiction assume that $dim_D V \geq 3$. If $\{v, va\}$ is linearly D-independent for some $v \in V$, then by the density of RC, there exists $w \in V$ such that $\{w, v, va\}$ is linearly D-independent and

$x_0, y_0, z_0 \in RC$ such that $vx_0 = 0$, $vy_0 = 0$, $vz_0 = 0$, $(va)x_0 = w$, $(va)y_0 = 0$, $(va)z_0 = v$, $wy_0 = va$. This leads to the contradiction

$$0 = v\left[[a[x_0, y_0] + [x_0, y_0]b, [x_0, y_0]^n], z_0\right] = v.$$

Thus $\{v, va\}$ is linearly D-dependent, for all $v \in V$, which implies that $a \in C$. From this, RC satisfies

$$\left[[[x_1, x_2]b, [x_1, x_2]^n], x_3\right]. \tag{2}$$

As above suppose that there exists $v \in V$ such that $\{v, vb\}$ is linearly D-independent. Then there exists $w \in V$ such that $\{v, vb, w\}$ is linearly D-independent and there exist $x_0, y_0, z_0 \in RC$ such that $vx_0 = w$, $vy_0 = 0$, $vz_0 = vb$, $wy_0 = v$, $(vb)x_0 = v$, $(vb)y_0 = 0$, $(vb)z_0 = v$. This implies that

$$0 = v\left[[[x_0, y_0]b, [x_0, y_0]^n], z_0\right] = -v \neq 0,$$

a contradiction. Also in this case we conclude that $\{v, vb\}$ is linearly D-dependent, for all $v \in V$, and so $b \in C$.
The previous argument shows that if either $a \notin C$ or $b \notin C$, then $dim_D V \leq 2$. In this condition RC is a simple ring which satisfies a non-trivial generalized polynomial identity. By [24] (Theorem 2.3.29) $RC \subseteq M_t(K)$, for a suitable field K, moreover $M_t(K)$ satisfies the same generalized identity of RC, hence

$$\left[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]^n\right] \in Z(M_t(K))$$

for any $r_1, r_2 \in M_t(K)$. If $t \leq 2$, then R satisfies the standard identity s_4. If $t \geq 3$, by the above argument, we get $a, b \in Z(M_t(K))$. □

Now we will consider the n-centralizing condition on Lie ideals. We premit the following:

Fact 1. *Let R be a prime ring and L a non-central Lie ideal of R. Then either there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$ or $char(R) = 2$ and R satisfies s_4.*

Proof. See [10] (pp 4-5), Lemma 2 and Proposition 1 in [9], Theorem 4 in [13]. □

1.1 The Proof of Theorem 1.

Assume that R does not satisfy s_4. By Fact 1 we have that there exists a two-sided ideal I of R such that $[I, I] \subseteq L$. In this last case we get that $[F([r_1, r_2]), [r_1, r_2]^n] \in Z(R)$, for any $r_1, r_2 \in I$.
By [18] F has the form $F(x) = ax + d(x)$, for $a \in U$ and d a derivation of U. If d is an inner derivation induced by an element $c \in U$, it follows that

$$[(a+c)[r_1, r_2] - [r_1, r_2]c, [r_1, r_2]^n] \in Z(R)$$

for any $r_1, r_2 \in I$, and by Lemma 1 we have that $a, c \in C$, that is $d = 0$ and $F(x) = ax$, for all $x \in R$.
Assume now d is not an inner derivation of U. Notice that, if $d = 0$ then I satisfies

$$\Big[[a[x_1, x_2], [x_1, x_2]^n], x_3\Big]$$

and by Lemma 1 we get the conclusion $a \in C$ and $F(x) = ax$ for all $x \in U$ and so for all $x \in R$. Assume finally $d \neq 0$. Since

$$\Big[[a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]^n], x_3\Big]$$

is a differential identity for I, by Kharchenko's result in [12], it follows that I satisfies

$$\Big[[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]^n], x_3\Big]$$

and in particular

$$\Big[[[x_1, y_2], [x_1, x_2]^n], x_3\Big] \tag{3}$$

is a polynomial identity for I. This implies obviously that R is a PI-ring satisfying (3). Thus there exists a field K such that R and $M_t(K)$, the ring of all $t \times t$ matrices over K, satisfy the same polynomial identities. Since L is non-central, R must be non-commutative. Hence $t \geq 2$. In case $t = 2$, R satisfies s_4, a contradiction. Thus $t \geq 3$. Denote by e_{ij} the usual matrix unit with 1 in the (i, j)-entry and zero elsewhere. In (3) choose $x_1 = e_{12}$, $x_2 = e_{21}$, $x_3 = e_{33}$, $y_2 = e_{23}$, then it follows the contradiction

$$0 = \Big[[e_{13}, (e_{11} - e_{22})^n], e_{33}\Big] = -e_{13}.$$

2. N-centralizing Maps on Left Ideals

In this section we would like to extend Theorem 1 to left ideals in prime rings, more precisely we will prove Theorem 2.
For the remainder of the paper we assume that the conclusion

- I satisfies $x_1s_4(x_2, x_3, x_4, x_5)$

of Theorem 2 is false.
Thus there exist $a_1, a_2, a_3, a_4, a_5 \in I$ such that $a_1s_4(a_2, a_3, a_4, a_5) \neq 0$. Our goal is to ultimately arrive to prove that in this case there exists $a \in U$ such that $F(x) = xa$, for all $x \in R$ and $I[a, I] = (0)$.

Fact 2. In all that follows let $T = U *_C C\{X\}$ be the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $\{x_1, x_2, \ldots, x_n, \ldots\}$. The elements of T are called generalized polynomials with coefficients in U. I, IR and IU satisfy the same generalized polynomial identities with coefficients in U. We refer the reader to [2] and [4] for the definitions and the related properties of these objects.
Recall that, if B is a basis of U over C, then any element of $T = U *_C C\{x_1, \ldots, x_n\}$ can be written in the form $g = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials, that is $m_i = q_0 y_1 \cdots y_n q_n$, with $q_i \in B$ and $y_i \in \{x_1, \ldots, x_n\}$. In [4] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, if $a_1, a_2 \in U$ are linearly independent over C and $a_1g_1(x_1, \ldots, x_n) + a_2g_2(x_1, \ldots, x_n) = 0 \in T$, for some $g_1, g_2 \in T$, then both $g_1(x_1, \ldots, x_n)$ and $g_2(x_1, \ldots, x_n)$ are the zero element of T.

We begin with:

Lemma 2. *Either R is a ring satisfying a non-trivial generalized polynomial identity (GPI), or there exists $a \in U$ such that $F(x) = xa$, for all $x \in R$ and $I(a - \alpha) = (0)$ for some $\alpha \in C$.*

Proof. We know that F assumes the form $F(x) = ax + d(x)$ for all $x \in U$ and some $a \in U$, where d is a derivation on U. Suppose R does not satisfy any non-trivial GPI. We divide the proof into two cases:

Case 1: Suppose that d is an inner derivation induced by an element $q \in U$.

Let $0 \neq b \in I$. Since R does not satisfy any non-trivial GPI, then

$$\Big[[a[x_1b, x_2b] + q[x_1b, x_2b] - [x_1b, x_2b]q, [x_1b, x_2b]^n], x_3\Big] \tag{4}$$

is the zero element in the free algebra T, for all $x_1, x_2, x_3 \in R$ (see Fact 2), that is

$$\begin{aligned}
&\Big((a+q)[x_1b, x_2b]^{n+1}\Big)x_3 \\
&+ \Big(-[x_1b, x_2b]q[x_1b, x_2b]^n - [x_1b, x_2b]^n(a+q)[x_1b, x_2b] + [x_1b, x_2b]^{n+1}q\Big)x_3 \\
&- x_3\Big((a+q)[x_1b, x_2b]^{n+1}\Big) \\
&+ x_3[x_1b, x_2b]q[x_1b, x_2b]^n \\
&- x_3\Big(-[x_1b, x_2b]^n(a+q)[x_1b, x_2b] + [x_1b, x_2b]^{n+1}q\Big) = 0 \in T.
\end{aligned} \tag{5}$$

If $a + q \notin C$, then $a + q$ and 1 are linearly C-independent and in this case from (5) we have $(a+q)[x_1b, x_2b]^{n+1}x_3 = 0 \in T$. This implies $a + q = 0$, a contradiction.
Hence $a+q \in C$. Thus $F(x) = (a+q)x - xq = x(a+q-q) = xa$ for all $x \in R$. Then (5) becomes

$$\begin{aligned}
&\big(-[x_1b, x_2b]a[x_1b, x_2b]^n - [x_1b, x_2b]^{n+1}a\big)x_3 \\
&- x_3\big([x_1b, x_2b]a[x_1b, x_2b]^n - [x_1b, x_2b]^{n+1}a\big) = 0 \in T.
\end{aligned}$$

If ba and b are linearly C-independent, then from above we have that R satisfies the non-trivial generalized polynomial identity $x_3[x_1b, x_2b]a[x_1b, x_2b]^n$, a contradiction. Hence we conclude that ba and a are linearly C-dependent for all $b \in I$. Thus there exists $\alpha \in C$ such that $I(a - \alpha) = (0)$.

Case 2: Suppose that d is not an inner derivation of U. Since R is not commutative, then there exists $0 \neq b \in I$, such that $b \notin C$. By our main assumption, R satisfies

$$\Big[[a[x_1b, x_2b] + [d(x_1)b + x_1d(b), x_2b] + [x_1b, d(x_2)b + x_2d(b)], [x_1b, x_2b]^n], x_3\Big]. \tag{6}$$

Since d is not inner and by [12], we have that R satisfies

$$\Big[[a[x_1b,x_2b]+[y_1b+x_1d(b),x_2b]+[x_1b,y_2b+x_2d(b)],[x_1b,x_2b]^n],x_3\Big] \quad (7)$$

and in particular

$$\Big[[y_1b,x_2b],[x_1b,x_2b]^n],x_3\Big] \quad (8)$$

is a generalized identity for R. Since $b \notin C$, then b and 1 are linearly C-independent, thus (8) is a non-trivial generalized polynomial identity for R, a contradiction. □

Lemma 3. *Without loss of generality, R is simple and equal to its own socle, $RI = I$.*

Proof. By Lemma 2, R is GPI (otherwise we are done). So U has non-zero socle H with non-zero left ideal $J = HI$ [21]. Note that H is simple, $J = HJ$ and J satisfies the same basic conditions as I (we refer to [15]). Just replace R by H, I by J and we are done. □

Lemma 4. *Let R be a prime ring, $0 \neq c \in R$, I a non-zero left ideal of R, $m \geq 1$ a fixed integer such that $c[r_1,r_2]^m \in Z(R)$, for all $r_1,r_2 \in I$. Then $x_1s_4(x_2,x_3,x_4,x_5)$ is an identity for I.*

Proof. Firstly we notice that if $c[x_1,x_2]^m$ is a generalized polynomial identity for I, then by [6] and since $c \neq 0$, we have $r_1[r_2,r_3] = 0$ for all $r_1,r_2,r_3 \in I$, and a fortiori $x_1s_4(x_2,x_3,x_4,x_5)$ is an identity for I. Therefore we may assume there exist $a_1,a_2 \in I$ such that $0 \neq c[a_1,a_2]^m \in Z(R)$. By Theorem 1 in [3] R is a PI-ring and so RC is a finite dimensional central simple C-algebra. By Wedderburn-Artin theorem $RC \cong M_k(D)$ for some $k \geq 1$ and D a finite-dimensional central division C-algebra. By Theorem 2 in [14] $c[r_1,r_2]^m \in C$ for all $r_1,r_2 \in CI$. Without loss of generality we may replace R with RC and assume that $R = M_k(D)$. Let E be a maximal subfield of D, so that $E \otimes_C M_k(D) \cong M_t(E)$ where $t = k \cdot [E:C]$. Hence $c[r_1,r_2]^m \in C$ for all $r_1,r_2 \in Z(M_t(E))$ for any $r_1,r_2 \in E \otimes_C I$ (Lemma 2 in [14] and Proposition in [20]). Therefore we may assume that $R \cong M_t(E)$ and replace I with $E \otimes_C I$. Moreover $0 \neq c[b_1,b_2]^m \in Z(M_t(E))$, for $b_1 = 1_E \otimes_C a_1$, $b_2 = 1_E \otimes_C a_2$. Then I contains an invertible element of R, and so $I = R = M_t(E)$ and $c[r_1,r_2]^m \in Z(R)$, for all $r_1,r_2 \in R$. Consider the following subset of R,

$$G = \{a \in R \mid a[r_1,r_2]^m \in Z(R), \quad \forall r_1,r_2 \in R\}$$

and notice that G is a subgroup of R, which is invariant under the action of all automorphisms of R, moreover $c \in G$. By a Theorem of Chuang ([5]), one of the following holds:

- R satisfies s_4 and $char(R) = 2$ (in this case we are done);
- $G \subseteq Z(R)$ and since $c \neq 0$, it follows $[r_1, r_2]^m \in Z(R)$, for all $r_1, r_2 \in R$;
- $[R, R] \subseteq G$, which implies $[s_1, s_2][r_1, r_2]^m \in Z(R)$, for all $s_1, s_2, r_1, r_2 \in R$.

In order to conclude our proof, we may assume that in any case $[r_1, r_2]^{2m} \in Z(R)$, for all $r_1, r_2 \in R$. This implies easily that R must satisfy s_4. □

We are now ready for the following:

2.1 The Proof of Theorem 2.

By the regularity of R, there exists $e^2 = e \in RI$ such that $Re = Ra_1 + Ra_2 + Ra_3 + Ra_4 + Ra_5$ and $a_i e = a_e$, for $i = 1, \ldots, 5$. In view of Kharchenko's Theorem in [12], we divide the proof into two cases:

Case 1. If d is an inner derivation induced by the element $q \in U$, then I satisfies the

$$\Big[[a[x_1, x_2] + q[x_1, x_2] - [x_1, x_2]q, [x_1, x_2]^n], x_3\Big]. \tag{9}$$

Thus for all $r, s, t \in R$

$$\Big[[a[re, se] + q[re, se] - [re, se]q, [re, se]^n], t\Big] = 0. \tag{10}$$

In particular for $t = 1 - e$ and left multiplying by e, we have

$$e \cdot \Big[[(a + q)[re, se] - [re, se]q, [re, se]^n], 1 - e\Big] = 0 \tag{11}$$

that is $e[re, se]^{n+1}q(1 - e) = 0$, for all $r, s \in R$. This implies $[er, es]^{n+1}eq(1 - e) = 0$. By [6], either $[eR, eR]e = (0)$ or $eq(1-e) = 0$. Since $s_4(eRe) \neq 0$, then a fortiori $[eRe, eRe] \neq 0$, therefore we have $eq = eqe \in Re$ and $F(Re) \subseteq Re$. Let $\lambda = Re$, $\overline{\lambda} = \frac{\lambda}{\lambda \cap r_R(\lambda)}$, where $r_R(\lambda)$ is the right annihilator of λ in R.

Therefore the prime ring $\overline{\lambda}$ satisfies the generalized polynomial identity (9) and by Lemma 1 it follows $s_4(\overline{\lambda}) = \overline{0}$ or both $[a, \overline{\lambda}] = \overline{0}$ and $[q, \overline{\lambda}] = \overline{0}$.
Since $s_4(\overline{\lambda}) = \overline{0}$ implies the contradiction $a_1 s_4(a_2, a_3, a_4, a_5) = 0$, we may assume that $\lambda[a, \lambda] = 0$ and $\lambda[q, \lambda] = 0$. In this case, standard arguments show that there exist $\alpha, \gamma \in C$ such that $I(a - \alpha) = (0)$ and $I(q - \gamma) = (0)$. Denote $a' = a - \alpha$, $q' = q - \gamma$ and notice that, in light of (9), we also have that

$$\Big[[a'[x_1, x_2] + q'[x_1, x_2] - [x_1, x_2]q', [x_1, x_2]^n], x_3\Big] \tag{12}$$

is satisfies by I, that is $(a' + q')[x_1, x_2]^{n+1}$ is a generalized identity for I. By Lemma 4 and since $a_1 s_4(a_2, a_3, a_4, a_5) \neq 0$, it follows $a' + q' = 0$, i.e. $a + q \in C$. Therefore $F(x) = ax + qx - xq = xa$ and we are done.

Case 2. Now assume that d is not inner. By our main assumption, R satisfies

$$\Big[[a[x_1e, x_2e] + [d(x_1)e + x_1 d(e), x_2e] + [x_1e, d(x_2)e + x_2 d(e)], [x_1e, x_2e]^n], x_3\Big]. \tag{13}$$

Since d is not inner and by [12], we have that

$$\Big[[a[x_1e, x_2e] + [y_1e + x_1 d(e), x_2e] + [x_1e, y_2e + x_2 d(e)], [x_1e, x_2e]^n], x_3\Big] \tag{14}$$

is a generalized identity for R. In particular R satisfies both

$$\Big[\big[[y_1e, x_2e], [x_1e, x_2e]^n\big], x_3\Big] \tag{15}$$

and

$$\Big[\big[[x_1e, y_2e], [x_1e, x_2e]^n\big], x_3\Big]. \tag{16}$$

By replacing in (15) y_1 with $(1-e)y_1$ and x_3 wih x_3e it follows that R satisfies $(1 - e)y_1 e x_2 e[x_1e, x_2e]^n x_3 e$ and by the primeness of R we have

$$er_2e[r_1e, r_2e]^n = 0, \qquad \forall r_1, r_2 \in R. \tag{17}$$

Analogously, by replacing in (16) y_2 with $(1 - e)y_2$ and x_3 wih x_3e it follows that R satisfies $-(1 - e)y_2 e x_1 e[x_1e, x_2e]^n x_3 e$ and by the primeness of R we have

$$er_1e[r_1e, r_2e]^n = 0, \qquad \forall r_1, r_2 \in R. \tag{18}$$

In light of (17) and (18) we finally have that $[r_1e, r_2e]^{n+1} = 0$, for all $r_1, r_2 \in R$. Again by [6] we get $e[Re, Re] = 0$, a contradiction.
The proof of Theorem is now complete.

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