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General System of Cubic Functional Equations in non–Archimedean Spaces *

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Abstract

In this paper, we introduce Menger probabilistic non-Archimedean normed spaces. we prove the generalized Hyers–Ulam–Rassias stability for a general system of cubic functional equations in non–Archimedean normed spaces and Menger probabilistic non–Archimedean normed spaces.

Keywords and Phrases: *Cubic Functional Equations, non-Archimedean Normed Spaces, Menger Probabilistic non-Archimedean Normed Spaces, Generalized Hyers–Ulam–Rassias Stability.*

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1. Introduction and Preliminaries

Hensel[13] has introduced a normed space which does not have the Archimedean property. During the last three decades theory of non–Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p–adic strings and superstrings [17]. Although many results in the classical normed space theory have a non–Archimedean counterpart, their proofs are different and require a rather new kind of intuition [3, 4, 8, 18, 25, 30]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non–Archimedean normed space; cf. [25]. These facts show that the non–Archimedean framework is of special interest.

Definition 1.1. Let $\mathbb{K}$ be a field. A valuation mapping on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq |a| + |b|$.

A field endowed with a valuation mapping will be called a valued field. If the condition (iii) in the definition of a valuation mapping is replaced with

$$(iii)' \ |a + b| \leq \max\{|a|, |b|\}$$

then the valuation $|\cdot|$ is said to be non–Archimedean. The condition $(iii)'$ is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from $(iii)'$ that $|n| \leq 1$ for each integer n. We always assume in addition that $|\cdot|$ is non trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.The most important examples of non-Archimedean spaces are p–adic numbers.

Example 1.2. Let p be a prime number. For any non–zero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p–adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a non–Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and is called the p–adic number field.

Definition 1.3. Let X be a linear space over a scalar field $\mathbb{K}$ with a non–Archimedean non–trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:

$(NA1)$ $\|x\| = 0$ if and only if $x = 0$;

$(NA2)$ $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

$(NA3)$ the strong triangle inequality (ultrametric); namely,
$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in X)$.

Then $(X, \|\cdot\|)$ is called a non–Archimedean space.

It follows from $(NA3)$ that

$$\|x_m - x_l\| \leq \max\{\|x_{\jmath+1} - x_\jmath\| : l \leq \jmath \leq m - 1\} \quad (m > l),$$

therefore a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non–Archimedean space.

Probabilistic normed spaces were first defined by Šerstnev in 1962 (see [28]). Their definition was generalized in [1]. We recall and apply the definition of Menger probabilistic normed spaces briefly as given in [27].

Definition 1.4. A distance distribution function (briefly, a d.d.f.) is a non-decreasing function F from $[0, +\infty]$ into $[0, 1]$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0, +\infty)$. The space of d.d.f.'s will be denoted by Δ^+; and the set of all F in Δ^+ for which $\lim_{t\to+\infty^-} F(t) = 1$ by D^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $[0, +\infty]$. For any $a \geq 0$, ε_a is the d.d.f. given by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ 1, & \text{if} \quad t > a. \end{cases}$$

Definition 1.5. A triangular norm (briefly t–norm) is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative, non–decreasing in each variable and has 1 as the unit element. Basic examples are the Łukasiewicz t–norm T_L, $T_L(a, b) = \max(a + b - 1, 0)$, the product t–norm T_P, $T_P(a, b) = ab$ and the strongest triangular norm T_M, $T_M(a, b) = \min(a, b)$.

Definition 1.6. A *Menger Probabilistic Normed space* is a triple (X, ν, T), where X is a real vector space, T is continuous t–norm and ν is a mapping (the *probabilistic norm*) from X into Δ^+, such that for every choice of p and q in X and a, s, t in $(0, +\infty)$, the following hold:

(PN1) $\nu(p) = \varepsilon_0$, if and only if, $p = \theta$ (θ is the null vector in X);

(PN2) $\nu(ap)(t) = \nu(p)(\frac{t}{|a|})$;

(PN3) $\nu(p+q)(s+t) \geq T\big(\nu(p)(s), \nu(q)(t)\big)$.

Now we introduce definition of a Menger probabilistic non–Archimedean normed space.

Definition 1.7. Let X be a vector space over a non–Archimedean field $\mathbb{K}$ and T be a continuous t–norm. A triple (X, ν, T) is said to be a Menger probabilistic non–Archimedean normed space if $(PN1)$ and $(PN2)$ (in Definition1.6) and

$$(PNA3)\ \nu(x+y)(\max\{s,t\}) \geq T\Big(\nu(x)(s), \nu(y)(t)\Big),$$

for all $x, y \in X$ and all $s, t \in \mathbb{K}$, are satisfied.

It follows from $\nu(x) \in \Delta^+$ that $\nu(x)$ is non–decreasing for every $x \in X$. So one can show that the condition $(PNA3)$ is equivalent to the following condition:

$$\nu(x+y)(t) \geq T\Big(\nu(x)(t), \nu(y)(t)\Big).$$

Definition 1.8. Let (X, ν, T) be a Menger probabilistic non–Archimedean normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \nu(x_n - x)(t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $\nu(x_{n+p} - x_n)(t) > 1 - \varepsilon$.

Let T be a given t–norm. Then (by associativity) a family of mappings $T^n : [0,1] \to [0,1]$, $n \in \mathbb{N}$, is defined as follows:

$$T^1(x) = T(x,x)\ ,\ T^n(x) = T(T^{n-1}(x), x)\ ,\ x \in [0,1].$$

For three important t–norms T_M, T_P and T_L we have

$$T_M^n(x) = x \ , \ T_P^n(x) = x^n \ , \ T_L^n(x) = \max\{(n+1)x - n, 0\} \ , \ n \in \mathbb{N}.$$

Definition 1.9. *(Hadzić[11])* A t–norm T is said to be of H–type if a family of functions $\{T^n(t)\}$; $n \in \mathbb{N}$, is equicontinuous at $t = 1$, that is,

$$\forall \varepsilon \in (0,1) \ \exists \delta \in (0,1) \ : \ t > 1 - \delta \ \Rightarrow \ T^n(t) > 1 - \varepsilon \ (n \geq 1).$$

The t-norm T_M is a trivial example of t–norm of H–type, but there are t-norms of H–type with $T \neq T_M$ (see, e.g., Hadzić[10]).

Lemma 1.10. *We consider the notations of the definition(1.8). Also assume that T is a t–norm of H–type. Then the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $\nu(x_{n+1} - x_n)(t) > 1 - \varepsilon$.*

Proof. Due to

$$\begin{aligned}
&\nu(x_{n+p} - x_n)(t) \geq T\Big(\nu(x_{n+p} - x_{n+p-1})(t), \nu(x_{n+p-1} - x_n)(t)\Big) \geq \\
&T\Big(\nu(x_{n+p} - x_{n+p-1})(t), T(\nu(x_{n+p-1} - x_{n+p-2})(t), \nu(x_{n+p-2} - x_n)(t))\Big) \geq \\
&\qquad \vdots \\
&\geq T\Big(\nu(x_{n+p} - x_{n+p-1})(t), T(\nu(x_{n+p-1} - x_{n+p-2})(t), \cdots , \\
&T(\nu(x_{n+2} - x_{n+1})(t), \nu(x_{n+1} - x_n)(t))) \cdots \Big),
\end{aligned}$$

and by the assumption of T, which is an H–type t–norm, the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $\nu(x_{n+1} - x_n)(t) > 1 - \varepsilon$. We will use this criterion in this paper.

It is easy to see that every convergent sequence in a (Menger probabilistic) non–Archimedean normed space is Cauchy. If each Cauchy sequence is convergent, then the (Menger probabilistic) non–Archimedean normed space is said to be complete and is called (Menger probabilistic) non–Archimedean Banach space.

The first stability problem concerning group homomorphisms was raised by Ulam[29] in 1940 and solved in the next year by Hyers[12]. Hyers' theorem was generalized by Aoki[2] for additive mappings and by Rassias[26] for linear mappings by considering an unbounded Cauchy difference. In 1994, a generalization of the Rassias?theorem was obtained by Găvruta[9] by replacing the unbounded Cauchy difference by a general control function.

Jun and Kim [14] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation(1.1). They proved that a function $f : X \to Y$ where X and Y are real vector spaces, is a solution of (1.1) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$. Moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. The function C is given by

$$C(x,y,z)=\frac{1}{24}(f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)),$$

for all $x, y, z \in X$. Obviously, the function $f(x) = cx^3$ satisfies the functional equation(1.1) which is called the cubic functional equation. Jun et al.[15] investigated the solution and the Hyers?Ulam stability for the cubic functional equation

$$f(ax+by)+f(ax-by)=ab^2(f(x+y)+f(x-y))+2a(a^2-b^2)f(x)$$

where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$.

In recent years, many authors have proved the stability of various functional equations in various spaces (see for instance [4–7],[16],[19–25]). Using the method of our paper, one can investigate the stability of many general systems of various functional equations with n functional equations and n variables ($n \in \mathbb{N}$) and our paper notably generalizes previous papers in this area.

We assume that $f : X^n \to Y$ is a mapping and consider the following generalized system of cubic functional equations:

$$\begin{cases} f(a_1x_1+b_1y_1,x_2,...,x_n)+f(a_1x_1-b_1y_1,x_2,...,x_n)= \\ a_1b_1^2(f(x_1+y_1,x_2,...,x_n)+f(x_1-y_1,x_2,...,x_n)) \\ +2a_1(a_1^2-b_1^2)f(x_1,x_2,...,x_n); \\ \quad\vdots \\ f(x_1,x_2,...,x_{n-1},a_nx_n+b_ny_n)+f(x_1,x_2,...,x_{n-1},a_nx_n-b_ny_n)= \\ a_nb_n^2(f(x_1,x_2,...,x_{n-1},x_n+y_n)+f(x_1,x_2,...,x_{n-1},x_n-y_n)) \\ +2a_n(a_n^2-b_n^2)f(x_1,x_2,...,x_n); \end{cases} \tag{1.2}$$

for all $x_i, y_i \in X$ and $a_i, b_i \in \mathbb{K}\setminus\{0\}$, with $a_i \neq \pm 1, \pm b_i$, $i = 1, ..., n$.

In the section(4), we establish the generalized Hyers–Ulam–Rassias stability of system(1.2) in non–Archimedean Banach spaces. In the section(3), we establish the generalized Hyers–Ulam–Rassias stability of system(1.2) in Menger probabilistic non–Archimedean Banach spaces.

2. Stability of System(1.2) in non–Archimedean Banach Spaces

In this section, we prove the generalized Hyers–Ulam–Rassias stability of system(1.2) in non–Archimedean Banach spaces. Throughout this section, we assume that $i, k, m, n, p \in \mathbb{N}\cup\{0\}$, $\mathbb{K}$ is a non–Archimedean field, Y is a non–Archimedean Banach space over $\mathbb{K}$ and X is a vector space over $\mathbb{K}$. Also assume that $f : X^n \to Y$ is a mapping.

Theorem 2.1. *Let $\varphi_k : X^{n+1} \to [0,\infty)$ for $k \in \{1, ..., n\}$ be a function such that*

$$\lim_{m\to\infty}\frac{1}{|a_1^{3m}...a_n^{3m}|}\varphi_k(a_1^m x_1, ..., a_k^m x_k, a_k^m y_k, ..., a_n^m x_n) = 0, \tag{2.1}$$

and

$$\lim_{m\to\infty}\max\{\frac{1}{|2a_1^{3(m+1)}...a_k^{3(m+1)}a_{k+1}^{3m}...a_n^{3m}|} \varphi_k(a_1^{m+1}x_1, ..., a_{k-1}^{m+1}x_{k-1}, a_k^m x_k, 0, a_{k+1}^m x_{k+1}, ..., a_n^m x_n) \ : \ k = 1, ..., n\} = 0, \tag{2.2}$$

and

$$\Phi(x_1,...,x_n)=\lim_{p\to\infty}\max\Big\{\max\{\frac{1}{|2a_1^{3(m+1)}...a_k^{3(m+1)}a_{k+1}^{3m}...a_n^{3m}|}$$
$$\varphi_k(a_1^m x_1,...,a_{k-1}^{m+1}x_{k-1},a_k^m x_k,0,a_{k+1}^m x_{k+1},...,a_n^m x_n) \tag{2.3}$$
$$:\ k=1,...,n\}\ ,\ m=0,1,...,p\Big\}<\infty,$$

for all $x_i, y_i \in X,\ i=1,...,n$. *Let* $f: X^n \to Y$ *be a mapping satisfying*

$$\begin{cases}\|f(a_1x_1+b_1y_1,x_2,...,x_n)+f(a_1x_1-b_1y_1,x_2,...,x_n)\\ -a_1b_1^2(f(x_1+y_1,x_2,...,x_n)+f(x_1-y_1,x_2,...,x_n))\\ -2a_1(a_1^2-b_1^2)f(x_1,x_2,...,x_n)\|\le\varphi_1(x_1,y_1,x_2,...,x_n);\\ \quad\vdots\\ \|f(x_1,x_2,...,a_nx_n+b_ny_n)+f(x_1,x_2,...,a_nx_n-b_ny_n)\\ -a_nb_n^2(f(x_1,x_2,...,x_{n-1},x_n+y_n)+f(x_1,x_2,...,x_{n-1},x_n-y_n))\\ -2a_n(a_n^2-b_n^2)f(x_1,x_2,...,x_n)\|\le\varphi_n(x_1,x_2,...,x_n,y_n);\end{cases}$$

for all $x_i, y_i \in X,\ i=1,...,n$. *Then there exists a unique mapping* $T: X^n \to Y$ *satisfying (1.2) and*

$$\|f(x_1,...,x_n)-T(x_1,...,x_n)\|\le\Phi(x_1,...,x_n) \tag{2.4}$$

for all $x_i \in X,\ i=1,...,n$.

Proof. Fix $k\in\{1,2,...,n\}$ and consider the following inequality.

$$\begin{aligned}&\|f(x_1,x_2,...,a_kx_k+b_ky_k,...,x_n)+f(x_1,x_2,...,a_kx_k-b_ky_k,...,x_n)\\ &-a_kb_k^2(f(x_1,x_2,...,x_k+y_k,...,x_n)+f(x_1,x_2,...,x_k-y_k,...,x_n))\\ &-2a_k(a_k^2-b_k^2)f(x_1,x_2,...,x_n)\|\le\varphi_k(x_1,...,x_{k-1},x_k,y_k,x_{k+1},...,x_n).\end{aligned} \tag{2.5}$$

Let $y_k=0$ in (2.5). Then we get

$$\|f(x_1,...,x_k,...,x_n)-\frac{1}{a_k^3}f(x_1,x_2,...,a_kx_k,...,x_n)\|$$
$$\le\frac{1}{|2a_k^3|}\varphi_k(x_1,...,x_{k-1},x_k,0,x_{k+1},...,x_n).$$

Therefore one can obtain

$$\begin{aligned}&\|\frac{1}{a_1^3...a_{k-1}^3}f(a_1x_1,...,a_{k-1}x_{k-1},x_k,x_{k+1},...,x_n)\\&-\frac{1}{a_1^3...a_{k-1}^3a_k^3}f(a_1x_1,...,a_{k-1}x_{k-1},a_kx_k,x_{k+1},...,x_n)\|\\&\leq\frac{1}{|2a_1^3...a_{k-1}^3a_k^3|}\varphi_k(a_1x_1,...,a_{k-1}x_{k-1},x_k,0,x_{k+1},...,x_n).\end{aligned}$$

So we have

$$\begin{aligned}&\|f(x_1,x_2,...,x_n)-\frac{1}{a_1^3...a_n^3}f(a_1x_1,...,a_nx_n)\|\leq\\&\max\{\frac{1}{|2a_1^3...a_{k-1}^3a_k^3|}\varphi_k(a_1x_1,...,a_{k-1}x_{k-1},x_k,0,x_{k+1},...,x_n)\ :\ k=1,...,n\}\end{aligned}$$

Therefore we get

$$\begin{aligned}&\|\frac{1}{a_1^{3m}...a_n^{3m}}f(a_1^mx_1,...,a_n^mx_n)-\frac{1}{a_1^{3(m+1)}...a_n^{3(m+1)}}f(a_1^{m+1}x_1,...,a_n^{m+1}x_n)\|\leq\\&\max\Big\{\frac{1}{|2a_1^{3(m+1)}...a_k^{3(m+1)}a_{k+1}^{3m}...a_n^{3m}|}\\&\varphi_k(a_1^{m+1}x_1,...,a_{k-1}^{m+1}x_{k-1},a_k^mx_k,0,a_{k+1}^mx_{k+1},...,a_n^mx_n)\ :\ k=1,...,n\Big\},\end{aligned}\tag{2.6}$$

for all $m\in\mathbb{N}\cup\{0\}$. It follows from (2.6) and (2.2) that the sequence

$$\{\frac{1}{a_1^{3m}...a_n^{3m}}f(a_1^mx_1,...,a_n^mx_n)\}$$

is Cauchy. Since the space Y is complete, it is convergent. Therefore we can define $T:X^n\to Y$ by

$$T(x_1,...,x_n):=\lim_{m\to\infty}\frac{1}{a_1^{3m}...a_n^{3m}}f(a_1^mx_1,...,a_n^mx_n),\tag{2.7}$$

for all $x_i \in X,\ i = 1, ..., n$. Using induction with (2.6) one can show that

$$\begin{aligned} &\|f(x_1, ..., x_n) - \frac{1}{a_1^{3(p+1)}...a_n^{3(p+1)}} f(a_1^{p+1}x_1, ..., a_n^{p+1}x_n)\| \leq \\ &\max\Big\{ \max\{\frac{1}{|2a_1^{3(m+1)}...a_k^{3(m+1)}a_{k+1}^{3m}...a_n^{3m}|} \\ &\varphi_k(a_1^{m+1}x_1, ..., a_{k-1}^{m+1}x_{k-1}, a_k^m x_k, 0, a_{k+1}^m x_{k+1}, ..., a_n^m x_n) \\ &\ :\ k = 1, ..., n\}\ ,\ m = 0, 1, ..., p\Big\}. \end{aligned} \tag{2.8}$$

for all $x_i \in X,\ i = 1, ..., n$ and $p \in \mathbb{N} \cup \{0\}$. By taking p to approach infinity in (2.8) and using (2.3) one obtains (2.4).

For $k \in \{1, 2, ..., n\}$ and by (2.5) and (2.7), we get

$$\begin{aligned} &\|T(x_1, x_2, ..., a_kx_k + b_ky_k, ..., x_n) + T(x_1, x_2, ..., a_kx_k - b_ky_k, ..., x_n) \\ &- a_kb_k^2(T(x_1, x_2, ..., x_k + y_k, ..., x_n) + T(x_1, x_2, ..., x_k - y_k, ..., x_n)) \\ &- 2a_k(a_k^2 - b_k^2)T(x_1, x_2, ..., x_n)\| \\ &= \lim_{m\to\infty} \|\frac{1}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_k^m(a_kx_k + b_ky_k), ..., a_n^m x_n) \\ &+ \frac{1}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_k^m(a_kx_k - b_ky_k), ..., a_n^m x_n) \\ &- \frac{a_kb_k^2}{a_1^{3m}...a_n^{3m}}(f(a_1^m x_1, ..., a_k^m x_k + a_k^m y_k, ..., a_n^m x_n) \\ &+ f(a_1^m x_1, ..., a_k^m x_k - a_k^m y_k, ..., a_n^m x_n)) \\ &- \frac{2a_k(a_k^2 - b_k^2)}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_n^m x_n)\| \\ &\leq \lim_{m\to\infty} \frac{1}{|a_1^{3m}...a_n^{3m}|}\varphi_k(a_1^m x_1, ..., a_k^m x_k, a_k^m y_k, ..., a_n^m x_n). \end{aligned} \tag{2.9}$$

By (2.1) and (2.9), we conclude that T satisfies (1.2).

Suppose that there exists another mapping $T' : X^n \to Y$ which satisfies

(1.2) and (2.4). So we have

$$\|T(x_1, x_2, ..., x_n) - T'(x_1, x_2, ..., x_n)\| \leq$$
$$\frac{1}{|a_1^{3m}...a_n^{3m}|} \max\Big\{\|T(a_1^m x_1, ..., a_n^m x_n) - f(a_1^m x_1, ..., a_n^m x_n)\|,$$
$$\|f(a_1^m x_1, ..., a_n^m x_n) - T'(a_1^m x_1, ..., a_n^m x_n)\|\Big\} \leq$$
$$\frac{1}{|a_1^{3m}...a_n^{3m}|} \max\Big\{\Phi(a_1^m x_1, ..., a_n^m x_n)\ , \Phi(a_1^m x_1, ..., a_n^m x_n)\Big\},$$

which tends to zero as $m \to \infty$ by (2.3). Therefore $T = T'$. This completes the proof.

3. Stability of System(1.2) in Menger Probabilistic non–Archimedean Banach Spaces

In this section, we prove the generalized Hyers–Ulam–Rassias stability of system(1.2) in Menger probabilistic non–Archimedean Banach spaces. Throughout this section, we assume that $u \in \mathbb{R}$, $i, k, m, n \in \mathbb{N} \cup \{0\}$, $\mathbb{K}$ is a non–Archimedean field, T is a continuous t–norm of H–type, (Y, ν, T) is a Menger probabilistic non–Archimedean Banach space over $\mathbb{K}$, (Z, ω, T) is a Menger probabilistic non–Archimedean normed space over $\mathbb{K}$ and X is a vector space over $\mathbb{K}$. Also assume that $f : X^n \to Y$ is a mapping.

Theorem 3.1. *Let $\varphi_k : X^{n+1} \to Z$ for $k \in \{1, ..., n\}$ be a mappings such that*

$$\begin{cases} \tilde{\varphi}_k = \tilde{\varphi}_k(x_1, ..., x_n, u) = \\ \omega\Big(\frac{1}{|2a_1^3...a_{k-1}^3 a_k^3|}\varphi_k(a_1x_1, ..., a_{k-1}x_{k-1}, x_k, 0, x_{k+1}, ..., x_n)\Big)(u); \\ \Phi_1 = \Phi_1(x_1, ..., x_n, u) = \tilde{\varphi}_1(x_1, ..., x_n, u); \\ \Phi_k = \Phi_k(x_1, ..., x_n, u) = T\Big(\tilde{\varphi}_k(x_1, ..., x_n, u), \Phi_{k-1}(x_1, ..., x_n, u)\Big); \\ \lim_{m\to\infty} \Phi_n\Big(a_1^m x_1, ..., a_n^m x_n, |a_1^{3m}...a_n^{3m}|u\Big) = 1; \end{cases} \tag{3.1}$$

and

$$\lim_{m\to\infty} \omega\Big(\frac{1}{|a_1^{3m}...a_n^{3m}|}\varphi_k(a_1^m x_1, ..., a_k^m x_k, a_k^m y_k, ..., a_n^m x_n)\Big)(u) = 1 \tag{3.2}$$

and

$$\begin{cases} \Phi_m^* = \Phi_m^*(x_1, ..., x_n, u) = \Phi_n\Big(a_1^m x_1, ..., a_n^m x_n, |a_1^{3m}...a_n^{3m}|u\Big); \\ \Psi_0 = \Phi_0^*(x_1, ..., x_n, u) = \Phi_n(x_1, ..., x_n, u); \\ \Psi_m = \Psi_m(x_1, ..., x_n, u) = T\Big(\Phi_m^*(x_1, ..., x_n, u), \Psi_{m-1}(x_1, ..., x_n, u)\Big); \\ \Psi = \Psi\Big(x_1, ..., x_n, u\Big) = \lim_{m\to\infty} \Psi_m. \end{cases} \tag{3.3}$$

for all $u > 0$ *and* $x_i, y_i \in X,\ i = 1, ..., n$. *Let* $f : X^n \to Y$ *be a mapping satisfying*

$$\begin{cases} \nu\Big(f(a_1x_1 + b_1y_1, x_2, ..., x_n) + f(a_1x_1 - b_1y_1, x_2, ..., x_n) \\ -a_1b_1^2(f(x_1 + y_1, x_2, ..., x_n) + f(x_1 - y_1, x_2, ..., x_n)) \\ -2a_1(a_1^2 - b_1^2)f(x_1, ..., x_n)\Big)(u) \geq \omega\Big(\varphi_1(x_1, y_1, x_2, ..., x_n)\Big)(u); \\ \qquad \vdots \\ \nu\Big(f(x_1, x_2, ..., x_{n-1}, a_nx_n + b_ny_n) + f(x_1, x_2, ..., x_{n-1}, a_nx_n - b_ny_n) \\ -a_nb_n^2(f(x_1, x_2, ..., x_{n-1}, x_n + y_n) + f(x_1, x_2, ..., x_{n-1}, x_n - y_n)) \\ -2a_n(a_n^2 - b_n^2)f(x_1, ..., x_n)\Big)(u) \geq \omega\Big(\varphi_n(x_1, ..., x_n, y_n)\Big)(u); \end{cases}$$

for all $u > 0$ *and* $x_i, y_i \in X,\ i = 1, ..., n$. *Then there exists a unique mapping* $F : X^n \to Y$ *satisfying (1.2) and*

$$\nu\Big(f(x_1, ..., x_n) - F(x_1, ..., x_n)\Big)(u) \geq \Psi \tag{3.4}$$

for all $u > 0$ *and* $x_i \in X,\ i = 1, ..., n$.

Proof. Fix $k \in \{1, 2, ..., n\}$ and consider the following inequality.

$$\begin{aligned} &\nu\Big(f(x_1, x_2, ..., a_kx_k + b_ky_k, ..., x_n) + f(x_1, x_2, ..., a_kx_k - b_ky_k, ..., x_n) \\ &- a_kb_k^2(f(x_1, x_2, ..., x_k + y_k, ..., x_n) + f(x_1, x_2, ..., x_k - y_k, ..., x_n)) \\ &- 2a_k(a_k^2 - b_k^2)f(x_1, x_2, ..., x_n)\Big)(u) \geq \omega\Big(\varphi_k(x_1, ..., x_k, y_k, ..., x_n)\Big)(u). \end{aligned} \tag{3.5}$$

Let $y_k = 0$ in (3.5). Then we get

$$\begin{aligned} &\nu\Big(f(x_1, ..., x_n) - \frac{1}{a_k^3}f(x_1, x_2, ..., a_kx_k, ..., x_n)\Big)(u) \geq \\ &\omega\Big(\frac{1}{|2a_k^3|}\varphi_k(x_1, ..., x_k, 0, ..., x_n)\Big)(u). \end{aligned}$$

Therefore one can obtain

$$\nu\Big(\frac{1}{a_1^3...a_{k-1}^3}f(a_1x_1,...,a_{k-1}x_{k-1},x_k,x_{k+1},...,x_n)$$
$$-\frac{1}{a_1^3...a_{k-1}^3a_k^3}f(a_1x_1,...,a_{k-1}x_{k-1},a_kx_k,x_{k+1},...,x_n)\Big)(u)\geq$$
$$\omega\Big(\frac{1}{|2a_1^3...a_{k-1}^3a_k^3|}\varphi_k(a_1x_1,...,a_{k-1}x_{k-1},x_k,0,x_{k+1},...,x_n)\Big)(u)=\tilde{\varphi}_k.$$

Therefore we get

$$\nu\Big(f(x_1,...,x_n)-\frac{1}{a_1^3...a_n^3}f(a_1x_1,...,a_nx_n)\Big)(u)\geq\Phi_n.$$

So we have

$$\nu\Big(\frac{1}{a_1^{3m}...a_n^{3m}}f(a_1^mx_1,...,a_n^mx_n)-\frac{1}{a_1^{3(m+1)}...a_n^{3(m+1)}}f(a_1^{m+1}x_1,...,a_n^{m+1}x_n)\Big)(u)$$
$$\geq\Phi_n\Big(a_1^mx_1,...,a_n^mx_n,|a_1^{3m}...a_n^{3m}|u\Big). \tag{3.6}$$

for all $m\in\mathbb{N}\cup\{0\}$. Therefore by (3.1) the sequence

$$\{\frac{1}{a_1^{3m}...a_n^{3m}}f(a_1^mx_1,...,a_n^mx_n)\}$$

is Cauchy. By completeness of Y, we conclude that it is convergent. Therefore we can define $F:X^n\to Y$ by

$$\lim_{m\to\infty}\nu\Big(F(x_1,...,x_n)-\frac{1}{a_1^{3m}...a_n^{3m}}f(a_1^mx_1,...,a_n^mx_n)\Big)(u)=1, \tag{3.7}$$

for all $u>0$ and $x_i\in X,\ i=1,...,n$. Using induction with (3.6) one can show that

$$\nu\Big(f(x_1,...,x_n)-\frac{1}{a_1^{3(m+1)}...a_n^{3(m+1)}}f(a_1^{m+1}x_1,...,a_n^{m+1}x_n)\Big)(u)\geq\Psi_m. \tag{3.8}$$

By taking m to approach infinity in (3.8) and using (3.3) one obtains (3.4).

For $k \in \{1, 2, ..., n\}$ and by (3.5) and (3.7), we get

$$\begin{aligned}
&\nu\Big(F(x_1, x_2, ..., a_k x_k + b_k y_k, ..., x_n) + F(x_1, x_2, ..., a_k x_k - b_k y_k, ..., x_n) \\
&- a_k b_k^2 (F(x_1, x_2, ..., x_k + y_k, ..., x_n) + F(x_1, x_2, ..., x_k - y_k, ..., x_n)) \\
&- 2a_k(a_k^2 - b_k^2) F(x_1, x_2, ..., x_n)\Big)(u) \\
&= \lim_{m\to\infty} \nu\Big(\frac{1}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_k^m(a_k x_k + b_k y_k), ..., a_n^m x_n) \\
&+ \frac{1}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_k^m(a_k x_k - b_k y_k), ..., a_n^m x_n) \\
&- \frac{a_k b_k^2}{a_1^{3m}...a_n^{3m}} (f(a_1^m x_1, ..., a_k^m x_k + a_k^m y_k, ..., a_n^m x_n) \\
&+ f(a_1^m x_1, ..., a_k^m x_k - a_k^m y_k, ..., a_n^m x_n)) \\
&- \frac{2a_k(a_k^2 - b_k^2)}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_n^m x_n)\Big)(u) \\
&\geq \lim_{m\to\infty} \omega\Big(\frac{1}{|a_1^{3m}...a_n^{3m}|} \varphi_k(a_1^m x_1, ..., a_k^m x_k, a_k^m y_k, ..., a_n^m x_n)\Big)(u).
\end{aligned} \tag{3.9}$$

By (3.2) and (3.9), we conclude that F satisfies (1.2).

Suppose that there exists another mapping $F' : X^n \to Y$ which satisfies (1.2) and (3.4). So we have

$$\begin{aligned}
&\nu\Big(F(x_1, x_2, ..., x_n) - F'(x_1, x_2, ..., x_n)\Big)(u) = \\
&\nu\Big(\frac{1}{a_1^{3m}...a_n^{3m}} F(a_1^m x_1, ..., a_n^m x_n) - \frac{1}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_n^m x_n) \\
&+ \frac{1}{a_1^{3m}...a_n^{3m}} f(a_1^m x_1, ..., a_n^m x_n) - \frac{1}{a_1^{3m}...a_n^{3m}} F'(a_1^m x_1, ..., a_n^m x_n)\Big)(u) \\
&\geq T\Big\{\Psi\Big(a_1^m x_1, ..., a_n^m x_n, |a_1^{3m}...a_n^{3m}|u\Big), \Psi\Big(a_1^m x_1, ..., a_n^m x_n, |a_1^{3m}...a_n^{3m}|u\Big)\Big\},
\end{aligned}$$

which tends to 1 as $m \to \infty$ by (3.1) and (3.3). Therefore $F = F'$. This completes the proof.

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