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Euler's Number and Some Means *

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Abstract

We investigate several families of means, interpolating between the arithmetic, geometric and harmonic means, and find the one providing the best convergence of

$$\left(1+\frac{1}{n}\right)^{\mathcal{M}_t(n+1,n)}$$

to Euler's number.

1. Introduction

The following problem was proposed by Mihaly Bencze in the *Octagon* magazine [7]:

$$e < \left(1+\frac{1}{n}\right)^{\left(\frac{(n)^{1/3}+(n+1)^{1/3}}{2}\right)^3}. \tag{1}$$

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Although the solution to the problem has been known for a long time, its generalization seems interesting.
Let us use standard notation:

$$G(x,y)=\sqrt{xy},\quad L(x,y)=\frac{x-y}{\log x-\log y},\quad A(x,y)=\frac{x+y}{2}$$

for the geometric, logarithmic and arithmetic means of positive numbers x, y respectively. The inequalities

$$G(x,y)<L(x,y)<A(x,y) \tag{2}$$

hold for all $x \neq y$ (see [3] and references therein).
The logarithmic mean is linked with the Euler number by a simple and elegant formula

$$e=\left(1+\frac{1}{n}\right)^{L(n+1,n)}. \tag{3}$$

This, together with the basic property of means - lying in between - leads to the sequence of inequailities

$$\left(1+\frac{1}{n}\right)^{n}<\left(1+\frac{1}{n}\right)^{G(n+1,n)}<e<\left(1+\frac{1}{n}\right)^{A(n+1,n)}<\left(1+\frac{1}{n}\right)^{n+1}.$$

The following question arises: Suppose we have a continuous interpolation between the geometric and arithmetic means, i.e. a family of means

$$\mathcal{M}_t(x,y),\ 0\le t\le 1,\quad \mathcal{M}_0(x,y)=G(x,y) \text{ and } \mathcal{M}_1(x,y)=A(x,y)$$

that is monotone in t. For which value of t the sequence

$$\left(1+\frac{1}{n}\right)^{\mathcal{M}_t(n+1,n)}$$

assures the fastest convergence to e?
The aim of this paper is to answer this question for some known families of means.

Considering (3), our task reduces to finding t such that $\mathcal{M}_t(n+1,n)$ is closest to $L(n+1,n)$. Since all means we consider are homogeneous in x, y, it is enough to investigate the behaviour of $L(x,1)-\mathcal{M}_t(x,1)$ for $x\approx 1$, or that of $L(e^s,1)-\mathcal{M}_t(e^s,1)$ in a neighborhood of 0.

2. Generalized Heronian Means

Probably the most natural interpolation between A and G is their convex combination ([6]) given by

$$He_p(x,y) = (1-p)G(x,y) + pA(x,y), \quad 0 \le p \le 1,$$

named after Hero of Alexandria, who used $He_{2/3} = \frac{x+\sqrt{xy}+y}{3}$ to calculate the volume of a frustum of a piramid. The global inequality $L < He_{1/3}$ was obtained first by Carlsson ([3]) and then by Janous ([6]). In particular, the inequality

$$L(n+1,n) < He_p(n+1,n) \tag{4}$$

holds for all $p \ge 1/3$.

Theorem 1. *If $p \ge 1/3$, then (4) holds for all natural n. If $0 < p < 1/3$, then there exists $N = N(p)$ such that reversed (4) is valid for $n > N$.*

Proof.

$$He_p(e^{2t},1) - L(e^{2t},1) = (1-p)e^t + \frac{e^{2t}+1}{2} - \frac{e^{2t}-1}{2t} \tag{5}$$

$$= e^t \left(1 - p + p\cosh t - \frac{\sinh t}{t}\right) = e^t \sum_{n=1}^{\infty} \left(p - \frac{1}{2n+1}\right) \frac{t^{2n}}{(2n)!}. \tag{6}$$

Clearly, for $p \ge 1/3$ all the coefficients are positive and so is the left-hand side, which proofs (4), while for $0 < p < 1/3$ the difference is negative for small t and tends to infinity as t grows. Note that we obtain (4) or its reverse by setting $t = \log \sqrt{\frac{n+1}{n}}$.

3. Hölder Means

The Hölder mean [8], also known as power mean [4, 8] or generalized mean [2, 8], of order p is given as

$$M_p(x,y) = \begin{cases} \max(x,y) & p = \infty, \\ \left(\dfrac{x^p + y^p}{2}\right)^{1/p} & p \neq 0, \\ G(x,y) & p = 0, \\ \min(x,y) & p = -\infty. \end{cases} \tag{7}$$

With this definition and taking (3) into account, the original problem (1) can be rewritten as:
Show that if $p = 1/3$, then for all natural n

$$L(n+1, n) \leq M_p(n+1, n). \tag{8}$$

We shall prove the following:

Theorem 2. *If $p \geq 1/3$, then (8) holds for all natural n. If $p \leq 0$, then for all natural n reverse inequality holds. If $0 < p < 1/3$, then there exists $N = N(p)$ such that reversed (8) is valid for $n > N$.*

Proof. The second part follows from inequality $M_0 = G < L$.
Tung-Po Lin proved in [5] that the inequality

$$L(x, y) < M_{1/3}(x, y)$$

holds for all $x \neq y$, which combined with monotonicity of power means completes the first part of the proof. For $0 < p < 1/3$, L and M_p are not comparable, but in a neighborhood of $x = 1$ they behave nicely. Consider the function

$$f_p(x) = \frac{p \ln x}{2^p} + \frac{(1 - x^p)}{(1 + x)^p}.$$

By simple computation, we see that $f_p(1) = f_p'(1) = f_p''(1) = 0$ and $f_p'''(1) = -\frac{p^2}{2^{p+2}}(p - 3)$, hence for $0 < p < 1/3$, there is a $\delta > 1$ such that $f_{1/p}(x) < 0$ if $x \in (1, \delta)$. Therefore, for sufficiently large n

$$f_{1/p}\left(\left(1 + \frac{1}{n}\right)^p\right) < 0,$$

which is equivalent to $L(n+1, n) > M_p(n+1, n)$.

4. Heinz Means

Heinz means are defined by

$$H_\alpha(x, y) = \frac{x^{\frac{1+\alpha}{2}} y^{\frac{1-\alpha}{2}} + x^{\frac{1-\alpha}{2}} y^{\frac{1+\alpha}{2}}}{2}, \quad 0 \leq \alpha \leq 1. \tag{9}$$

We have $H_0 = G$ and $H_1 = A$, and they increase with α. As above, we are asking if the inequality

$$\left(1+\frac{1}{n}\right)^{H_\alpha(n+1,n)} < e \tag{10}$$

or its reverse holds.
Comparison between Heinz means and the logarithmic mean was investigated by Pittenger in [1]. The proof below differs from the original one only in details.

Theorem 3. *If $\alpha \le \frac{\sqrt{3}}{3}$, then* (10) *holds for all n. If $\frac{\sqrt{3}}{3} < \alpha < 1$, then there exists $N = N(\alpha)$ such that reversed* (10) *holds for $n > N$.*

Proof. We shall show a stronger fact, that for $\alpha \le \frac{\sqrt{3}}{3}$ the logarithmic mean is always greater than the Heinz mean. As in case of Heronian means, we let $y = 1$ and $x = e^{2t}$. We have

$$\begin{aligned} L(e^{2t},1) - H_\alpha(e^{2t},1) &= \frac{e^{2t}-1}{2t} - \frac{e^{t(1+\alpha)} + e^{t(1-\alpha)}}{2} \\ &= e^t\left[\frac{\sinh t}{t} - \cosh \alpha t\right] = e^t \sum_{n=0}^{\infty} a_n(\alpha)\frac{t^{2n}}{(2n)!} \end{aligned} \tag{11}$$

where

$$a_n(\alpha) = \frac{1}{2n+1} - \alpha^{2n}. \tag{12}$$

For $\alpha \le \sqrt{3}/3$ we have $a_0(\alpha) = 0$, $a_1(\alpha) \ge a_1(\sqrt{3}/3) = 0$ and $a_n(\alpha) > 0$ for $n \ge 2$, which yields $L > H_\alpha$. To prove the second part, note that for $\alpha > \sqrt{3}/3$, $a_0 = 0$ and $a_1(\alpha) < 0$, which means that the left-hand side of (11) is negative in a neighborhood of 0. For n large enough, $\log\sqrt{\frac{n+1}{n}}$ falls into this neighborhood, and (11) becomes equivalent to reversed (10).

5. Geometric Interpolation

In section 2 we apply linear interpolation between A and G. In this section we deal with the family

$$G_\alpha(x,y) = G^{1-\alpha}(x,y)A^\alpha(x,y) \quad 0 \le \alpha \le 1.$$

Theorem 4. *If $\alpha \le 1/3$, then $G_\alpha(x,y) < L(x,y)$ for all $x \neq y$. If $\alpha > 1/3$, then there exists $N = N(\alpha)$ such that $G_\alpha(n+1,n) > L(n+1,n)$ for $n > N$.*

Proof. As above, setting $y = 1, x = e^{2t}$ reduces out task to comparison between two functions: $g_\alpha(t) = \cosh^\alpha t$ and $l(t) = t^{-1}\sinh t$. We have

$$\lim_{t\to 0^+} l(t)/g_\alpha(t) = 1 \quad \text{and} \quad \lim_{t\to\infty} l(t)/g_\alpha(t) = \infty. \tag{13}$$

Consider the function $h_\alpha(t) = \frac{\sinh t}{\cosh^\alpha t}$. Its second derivative equals

$$h''_\alpha(t) = (1-\alpha)^2 \sinh t \cosh^{-2-\alpha} t \left[\cosh^2 t - 1 - \frac{3\alpha - 1}{(1-\alpha)^2}\right]$$

and we see that h_α is convex for $\alpha \le 1/3$, hence its divided difference $t^{-1}h_\alpha(t) = l(t)/g_\alpha(t)$ increases, which, together with (13), completes the first part of our proof.
In case $\alpha > 1/3$, h_α is concave in some interval $(0,\delta)$ and its divided difference decreases, so we have $l(t) < g_\alpha(t)$ which yields the second part.

6. Geometric Version of Heinz Means

To define the Heinz means we take the arithmetic mean and let the arguments vary between x, y and their geometric mean. Here we do a similar construction reversing the roles of the means:

$$K_p(x,y) = G\left(\frac{x+y}{2} + (1-p)\frac{x-y}{2}, \frac{x+y}{2} - (1-p)\frac{x-y}{2}\right), \quad 0 \le p \le 1.$$

Theorem 5. *For $p \ge 1 - \sqrt{2/3}$, the inequality $K_p(x,y) > L(x,y)$ holds for all $x \neq y$. If $p < 1 - \sqrt{2/3}$, then there exists $N = N(p)$ such that $K_p(n+1,n) < L(n+1,n)$ is valid for $n > N$.*

The proof is basically the same as in case of Heronian means and we leave it to the reader.

It is interesting to see if looking for optimal inequalities really makes sense, i.e. wheter the approximations are significantly better that the ones given by the geometric and arithmetic means. To learn that good approximation really makes the difference, consider Table 1, which shows the values of

$$\log_{10}\left|\left(1+\frac{1}{n}\right)^{\mathcal{M}_p(n+1,n)} - e\right|$$

or, in other words, the number of exact decimal digits of the Euler's number.

n	A	G	$He_{1/3}$	$M_{1/3}$	$H_{\frac{\sqrt{3}}{3}}$	$G_{1/3}$	$K_{\frac{1}{3+\sqrt{6}}}$
10^1	-2.68	-2.98	-7.46	-7.11	-6.51	-7.29	-2.70
10^2	-4.65	-4.95	-11.39	-11.04	-10.43	-11.22	-4.67
10^3	-6.65	-6.95	-15.39	-15.03	-14.43	-15.21	-6.67
10^4	-8.65	-8.95	-19.39	-19.04	-18.44	-19.22	-8.67
10^5	-10.65	-10.95	-23.40	-23.04	-22.44	-23.22	-10.67
10^6	-12.65	-12.95	-27.40	-27.05	-26.45	-27.23	-12.67
10^7	-14.66	-14.96	-31.40	-31.05	-30.45	-31.23	-14.68
10^8	-16.66	-16.96	-35.41	-35.06	-34.46	-35.23	-16.68
10^9	-18.66	-18.96	-39.41	-39.06	-38.46	-39.24	-18.68
10^{10}	-20.66	-20.96	-43.42	-43.06	-42.46	-43.24	-20.68
10^{11}	-22.66	-22.96	-47.43	-47.07	-46.47	-47.25	-22.69
10^{12}	-24.66	-24.97	-51.43	-51.07	-50.47	-51.25	-24.69
10^{13}	-26.67	-26.97	-55.43	-55.08	-54.48	-55.26	-26.69
10^{14}	-28.67	-28.97	-59.44	-59.09	-58.48	-59.26	-28.69
10^{15}	-30.67	-30.97	-63.44	-63.09	-62.49	-63.26	-30.70
10^{16}	-32.67	-32.98	-67.45	-67.10	-66.49	-67.27	-32.70
10^{17}	-34.68	-34.98	-71.45	-71.10	-70.50	-71.27	-34.70
10^{18}	-36.68	-36.98	-75.46	-75.10	-74.50	-75.28	-36.70
10^{19}	-38.68	-38.98	-79.46	-79.11	-78.50	-79.28	-38.70

Table 1: $\log_{10}\left|\left(1+\frac{1}{n}\right)^{\mathcal{M}_p(n+1,n)} - e\right|$

7. Record Breaker

We finish this discussion with a mean found trough some numerical experiments by the first author.

Theorem 6. *Let*

$$R(x,y) = \frac{14A(x,y) - H(x,y) + 32G(x,y)}{45}.$$

The inequality $L(x,y) < R(x,y)$ *holds for every* $x \neq y$ *.*

Proof.

$$R(e^t,1)-L(e^t,1)=\frac{1}{e^t+1}\sum_{k=1}^{\infty}a_k\frac{t^k}{(2k)!},$$

where $a_k=32\left(\frac{3^k+1}{2^k}\right)+12+\left(7-\frac{90}{k+1}\right)2^k$. One can easily calculate that $a_1=\ldots=a_5=0$, $a_6,\ldots,a_{12}>0$. For $k>12$ all a_k's are obviously positive, thus the proof is complete. Table 2 shows why we call it a record breaker:

n	R	n	R	n	R
10^1	-10.18	10^7	-46.09	10^{13}	-82.13
10^2	-16.07	10^8	-52.10	10^{14}	-88.14
10^3	-22.06	10^9	-58.10	10^{15}	-94.14
10^4	-28.07	10^{10}	-64.11	10^{16}	-100.15
10^5	-34.08	10^{11}	-70.12	10^{17}	-106.16
10^6	-40.08	10^{12}	-76.13	10^{18}	-112.16

Table 2: $\log_{10}\left|\left(1+\frac{1}{n}\right)^{R(n+1,n)}-e\right|$

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