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Approximation of Entire Functions of Two Complex Variables Over Jordan Domains *

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Abstract

In the present paper, we study the polynomial approximation of entire functions of two complex variables over Jordan domains by using Faber polynomials. The coefficient characterizations of order and type of entire functions have been obtained in terms of the approximation errors.

Keywords and Phrases: *Entire function, Order, Type, Faber polynomial, Approximation error.*

1. Introduction

Let Γ_1 and Γ_2 be given Jordan curves in the complex plane C and D_j, E_j, j = 1,2., be the interior and exterior respectively, of Γ_j. Let φ_j map E_j conformally

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onto $\{w_j : |w_j| > 1\}$ such that $\varphi_j(\infty) = \infty$ and $\varphi_j^{'}(\infty) > 0$. Then by [4], for sufficiently large $|z_j|$, $\varphi_j(z_j)$ can be expressed as

$$w_1 \;=\; \varphi_1(z_1) = \frac{z_1}{d_1} + c_0 + \frac{c_1}{z_1} + \frac{c_2}{z_1^2} + ... \tag{1.1}$$

$$w_2 \;=\; \varphi_2(z_2) = \frac{z_2}{d_2} + c_0^{'} + \frac{c_1^{'}}{z_2} + \frac{c_2^{'}}{z_2^2} + ... \tag{1.2}$$

where $d_1 \text{and} d_2 > 0$. Let us put $D = D_1 \times D_2$ and $E = E_1 \times E_2$ in C^2 and let the function φ map E conformally onto the unit bidisc $U = \{|w_1| > 1, |w_2| > 1\}$ such that $\varphi(z_1, z_2) = \varphi_1(z_1)\ \varphi_2(z_2)$ satisfies the conditions $\varphi(\infty, \infty) = \infty$ and $\varphi^{'}(\infty, \infty) > 0$. Then for sufficiently large values of $|z_1|$and $|z_2|$, $\varphi(z_1, z_2)$ can be expressed as

$$w_1 w_2 \;=\; \varphi(z_1, z_2) = \frac{z_1}{d_1}\frac{z_2}{d_2} + \sum_{m,n=0}^{\infty} \frac{c_{m,n}}{z_1^m z_2^n}. \tag{1.3}$$

An arbitrary Jordan curve can be approximated from the inside as well as from the outside by analytic Jordan curves. Since Γ is analytic, φ is holomorphic on Γ as well. The (m,n) th Faber polynomial $F_{m,n}(z_1, z_2)$ of Γ is the principal part of $(\varphi(z_1, z_2))^{m+n}$ at (∞, ∞), so that

$$F_{m,n}(z_1, z_2) = \left(\frac{z_1}{d_1}\right)^m \left(\frac{z_2}{d_2}\right)^n + ...$$

Following Faber [3] for the one variable case, we can easily see that as $m, n \to \infty$,

$$F_{m,n}(z_1, z_2) \sim (\varphi_1(z_1))^m (\varphi_2(z_2))^n \tag{1.4}$$

uniformly for $z_1 \in E_1, z_2 \in E_2$ and

$$\lim_{m,n\to\infty} \left(\max_{z_1, z_2 \in \Gamma} |F_{m,n}(z_1, z_2)| \right)^{1/(m+n)} = 1. \tag{1.5}$$

A function f holomorphic in D can be represented by its Faber series

$$f(z_1, z_2) \;= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} F_{m,n}(z_1, z_2) \tag{1.6}$$

where

$$a_{m,n} = \frac{1}{(2\pi i)^2} \int_{|w_1|=r_1} \int_{|w_2|=r_2} f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2)) w_1^{-(m+1)} w_2^{-(n+1)} dw_1 dw_2$$

and $0 < r_1, r_2 < 1$ are sufficiently close to 1 so that for j = 1, 2., φ_j^{-1} are holomorphic and univalent in $|w_j| \geq r_j$ respectively, the series converging uniformly on compact subsets of D. Let $M(r_1, r_2) = \max_{|z_j|=r_j} |f(z_1, z_2)|$, $j = 1, 2$ be the maximum modulus of $f(z_1, z_2)$. The growth of an entire function $f(z_1, z_2)$ is measured in terms of its order ρ and type τ(see [1]) defined as under

$$\limsup_{r_1, r_2 \to \infty} \frac{\ln \ln M(r_1, r_2)}{\ln(r_1 r_2)} = \rho, \tag{1.7}$$

$$\limsup_{r_1, r_2 \to \infty} \frac{\ln M(r_1, r_2)}{r_1^\rho + r_2^\rho} = \tau, \tag{1.8}$$

for $0 < \rho < \infty$.

Let $L^p(D)$ denote the set of functions f holomorphic in D and such that

$$\|f\|_{L^p(D)} = \left(\frac{1}{A} \int \int_D |f(z_1, z_2)|^p dx_1 dy_1 dx_2 dy_2 \right)^{1/p} < \infty$$

where A is the area of D. For $f \in L^p(D)$, set

$$E^p_{m,n} = E^p_{m,n}(f; D) = \min_{\pi_{m,n}} \|f - \pi_{m,n}\|_{L^p(D)}$$

where $\pi_{m,n}$ is an arbitrary polynomial of degree at most $m + n$.

Giroux [4] obtained the coefficient characterizations of order and type of entire function extensions of one complex variable of analytic functions over Jordan domains. He also obtained necessary and sufficient conditions in terms of approximation errors by using Faber polynomials for an entire function of one complex variable to be of required growth. To the best of our knowledge, coefficient characterization for order and type of entire functions of two complex variables over Jordan domain have not been obtained so far.

In this paper, we have made an attempt to bridge this gap. First we obtain coefficient characterization for order and type of entire functions of two complex variables over Jordan domains. Next we obtain necessary and sufficient conditions of order and type of entire functions of two complex variables in terms of approximation errors.

2. Order and Type

In this section we obtain the growth characterizations in terms of the coefficients $\{a_{m,n}\}$ of the Faber series (1.6). We first prove

Theorem 1. *The function f is the restriction to domain D of an entire function of finite order ρ if and only if*

$$\mu = \limsup_{m,n\to\infty} \frac{\ln m^m n^n}{-\ln|a_{m,n}|}. \tag{2.1}$$

is finite and then the order ρ of f is equal to μ.

Proof. Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} F_{m,n}(z_1, z_2)$ be an entire function of two complex variables in z_1 and z_2, where

$$a_{m,n} = \frac{1}{(2\pi i)^2} \int_{|w_1|=r_1} \int_{|w_2|=r_2} f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2)) w_1^{-(m+1)} w_2^{-(n+1)} dw_1 dw_2$$

with arbitrarily large r_1, r_2. Then

$$|a_{m,n}| = \left| \frac{1}{(2\pi i)^2} \int_{|w_1|=r_1} \int_{|w_2|=r_2} f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2)) w_1^{-(m+1)} w_2^{-(n+1)} dw_1 dw_2 \right|.$$

Since from (1.1) and (1.2), we have

$$\varphi_1(z_1) = w_1 \;\Rightarrow\; \varphi_1^{-1}(w_1) = z_1$$

$$\varphi_2(z_2) = w_2 \;\Rightarrow\; \varphi_2^{-1}(w_2) = z_2.$$

Hence

$$|a_{m,n}| \;\le M(r_1, r_2)\; r_1^{-m}\; r_2^{-n} \tag{2.2}$$

where $M(r_1, r_2) = \max_{|z_t|\le r_t} |f(z_1, z_2)|,\; t = 1, 2.$

Now first we show that $\rho \ge \mu$ and we assume that $\mu > 0$.Let $\varepsilon > 0$be chosen such that $\varepsilon < \mu < \infty$. Then from (2.1), we have

$$-(\mu - \varepsilon) \ln|a_{m,n}| \;\le\; \ln(m^m n^n)$$

$$\Rightarrow \ \ln|a_{m,n}| \ \geq \ -\frac{1}{(\mu-\varepsilon)}(m\ln m + n\ln n)$$

for an infinite sequence of values of m and n. From (2.2), we have

$$\ln M(r_1, r_2) \geq \ln|a_{m,n}| + \ln(r_1^m r_2^n)$$

$$\geq -\frac{1}{(\mu-\varepsilon)}(m\ln m + n\ln n) + m\ln r_1 + n\ln r_2$$

$$= m\left(\ln r_1 - \frac{1}{(\mu-\varepsilon)}\ln m\right) + n\left(\ln r_2 - \frac{1}{(\mu-\varepsilon)}\ln n\right).$$

Choosing

$$r_1 = (em)^{\frac{1}{(\mu-\varepsilon)}}, \quad r_2 = (en)^{\frac{1}{(\mu-\varepsilon)}}.$$

in the above inequality, we have

$$\ln M(r_1, r_2) \geq \frac{m}{(\mu-\varepsilon)} + \frac{n}{(\mu-\varepsilon)} = \frac{r_1^{\mu-\varepsilon} + r_2^{\mu-\varepsilon}}{e(\mu-\varepsilon)}.$$

Since $\mu - \varepsilon$ is independent of r_1 and r_2, therefore

$$\rho = \limsup_{r_1, r_2 \to \infty} \frac{\ln\ln M(r_1, r_2)}{\ln(r_1 r_2)} \geq \mu - \varepsilon.$$

and since ε is arbitrary, therefore we have

$$\rho \geq \mu. \tag{2.3}$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|a_{m,n}|} = \sigma.$$

Suppose $\sigma < \infty$. Then for every $\varepsilon > 0$, $\exists\ X(\varepsilon), Y(\varepsilon)$ such that for all $m \geq X$ and $n \geq Y$, we have

$$|a_{m,n}| \leq K\ m^{-\frac{m}{\sigma+\varepsilon}}\ n^{-\frac{n}{\sigma+\varepsilon}}.$$

Since $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, F_{m,n}(z_1, z_2)$, therefore

$$|f(z_1, z_2)| \le K \sum_{m,n=0}^{\infty} m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}}\, |F_{m,n}(z_1, z_2)|.$$

By using (1.5), for all $z_1 \in E_1$ and $z_2 \in E_2$, we have

$$|F_{m,n}(z_1, z_2)| \le K\, (|\varphi_1(z_1)|)^m (|\varphi_2(z_2)|)^n$$

and by using (1.1) and (1.2), for all sufficiently large $|z_1|$ and $|z_2|$, we have

$$|\varphi_1(z_1)| \le \frac{|z_1|}{d_1 - \varepsilon}, \qquad \textit{and} \qquad |\varphi_2(z_2)| \le \frac{|z_2|}{d_2 - \varepsilon}.$$

By applying these inequalities, for all sufficiently large $|z_1|$ and $|z_2|$, we have

$$|f(z_1, z_2)| \le K \sum_{m,n=0}^{\infty} m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}} \left(\frac{|z_1|}{d_1 - \varepsilon}\right)^m \left(\frac{|z_2|}{d_2 - \varepsilon}\right)^n.$$

Hence

$$\begin{aligned} M(r_1, r_2) \le & \sum_{m=0}^{M} \sum_{n=0}^{N} m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}} \left(\frac{r_1}{d_1 - \varepsilon}\right)^m \left(\frac{r_2}{d_2 - \varepsilon}\right)^n \\ & + \sum_{m=M+1}^{\infty} \sum_{n=N+1}^{\infty} m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}} \left(\frac{r_1}{d_1 - \varepsilon}\right)^m \left(\frac{r_2}{d_2 - \varepsilon}\right)^n \\ & + \sum_{m=0}^{M} \sum_{n=N+1}^{\infty} m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}} \left(\frac{r_1}{d_1 - \varepsilon}\right)^m \left(\frac{r_2}{d_2 - \varepsilon}\right)^n \\ & + \sum_{m=M+1}^{\infty} \sum_{n=0}^{N} m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}} \left(\frac{r_1}{d_1 - \varepsilon}\right)^m \left(\frac{r_2}{d_2 - \varepsilon}\right)^n. \end{aligned}$$

We now proceed as in proof of Theorem IV of Bose and Sharma [1, pp. 221-223] and obtain

$$M(r_1, r_2) \le \; 0 \left\{ e^{\left(\frac{4 r_1 r_2}{(d_1 - \varepsilon)(d_1 - \varepsilon)}\right)^{\sigma + 2\varepsilon}} \right\}.$$

Proceeding to limits and since ε is arbitrary and independent of $r_1$and r_2, we have

$$\limsup_{r_1,r_2\to\infty} \frac{\ln\ln M(r_1,r_2)}{\ln(r_1r_2)} \ \leq \ \sigma. \tag{2.4}$$

From (2.3) and (2.4), we obtain the required result (2.1).This completes the proof of Theorem 1.

Next we prove

Theorem 2. *Let* $\alpha = \limsup_{m,n\to\infty}\left\{m^m n^n\left(\frac{|a_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{\frac{1}{m+n}}$. *If* $0 < \alpha < \infty$, *the function f is the restriction to domain D of an entire function of finite order* ρ *and type* τ *if and only if*

$$\alpha = e\tau\rho. \tag{2.5}$$

Proof. Since f is an entire function of finite order ρ and type τ, therefore

$$|f(\varphi_1^{-1}(w_1),\varphi_2^{-1}(w_2))| \ \leq \ e^{(\tau+\varepsilon)((d_1+\varepsilon)|w_1|)^\rho+((d_2+\varepsilon)|w_2|)^\rho},$$

and from Cauchy's inequality, we have

$$|a_{m,n}| \ \leq \ r_1^{-m}r_2^{-n}e^{(\tau+\varepsilon)((d_1+\varepsilon)|w_1|)^\rho+((d_2+\varepsilon)|w_2|)^\rho}$$

$$\leq \ r_1^{-m}r_2^{-n}e^{(\tau+\varepsilon)((d_1+\varepsilon)r_1)^\rho}e^{(\tau+\varepsilon)((d_2+\varepsilon)r_2)^\rho}$$

for all r_1, r_2 sufficiently large. To minimize the right hand side of this inequality, we select
$r_1 \ = \ \frac{1}{d_1+\varepsilon}\left[\frac{m}{\rho(\tau+\varepsilon)}\right]^{1/\rho}$, and $r_2 \ = \ \frac{1}{d_2+\varepsilon}\left[\frac{n}{\rho(\tau+\varepsilon)}\right]^{1/\rho}$.

Substitute r_1, r_2 in the above inequality, we have

$$|a_{m,n}| \leq \frac{(d_1+\varepsilon)^m\ (d_2+\varepsilon)^n\ [e\rho(\tau+\varepsilon)]^{(m+n)/\rho}}{(m^m n^n)^{1/\rho}}$$

or, $\left\{m^m n^n\left(\frac{|a_{m,n}|}{(d_1+\varepsilon)^m(d_2+\varepsilon)^n}\right)^\rho\right\}^{1/(m+n)} \leq e\rho(\tau+\varepsilon).$

Proceeding to limits, since ε is arbitrary, we obtain

$$\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{|a_{m,n}|}{d_1^m \, d_2^n} \right)^\rho \right\}^{1/(m+n)} \leq e\rho\tau. \tag{2.6}$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{|a_{m,n}|}{d_1^m \, d_2^n} \right)^\rho \right\}^{1/(m+n)} = \sigma.$$

Suppose $\sigma < \infty$. Then for given $\varepsilon > 0$, $\exists\ M(\varepsilon), N(\varepsilon)$ such that for all $m \geq M$ and $n \geq N$, we have

$$|a_{m,n}| \leq K\, m^{-m/\rho}\, n^{-n/\rho}\, d_1^m d_2^n\, [e\rho(\sigma+\varepsilon)]^{(m+n)/\rho}.$$

Since $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, F_{m,n}(z_1, z_2)$, therefore

$$|f(z_1, z_2)| \leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, d_1^m\, d_2^n\, [e\rho(\sigma+\varepsilon)]^{(m+n)/\rho}\, |F_{m,n}(z_1, z_2)|.$$

From (1.5), by using the estimate of $F_{m,n}(z_1, z_2)$ in the above inequality, we have

$$\begin{aligned}
&|f(z_1, z_2)| \\
&\leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, d_1^m\, d_2^n\, [e\rho(\sigma+\varepsilon)]^{(m+n)/\rho} \left(\frac{|z_1|}{d_1 - \varepsilon} \right)^m \left(\frac{|z_2|}{d_2 - \varepsilon} \right)^n \\
&\leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, [e\rho(\sigma+\varepsilon)]^{(m+n)/\rho} \left(\frac{d_1|z_1|}{d_1 - \varepsilon} \right)^m \left(\frac{d_2|z_2|}{d_2 - \varepsilon} \right)^n \\
&\leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, [e\rho(\sigma+\varepsilon)]^{(m+n)/\rho}\, r_1^m\, r_2^n.
\end{aligned}$$

To estimate the right hand side of the above inequality, we proceeded on the similar lines of proof of Theorem V of Bose and Sharma [1, p 224], and we obtain

$$|f(z_1, z_2)| \leq 0\{e^{(\sigma+\varepsilon)(r_1^\rho + r_2^\rho)}\}.$$

Hence

$$M(r_1, r_2) \leq 0\{e^{(\sigma+\varepsilon)(r_1^\rho + r_2^\rho)}\}.$$

$$\Rightarrow \frac{\ln M(r_1, r_2)}{r_1^\rho + r_2^\rho} \leq \sigma + \varepsilon.$$

On proceeding to limits, we obtain

$$\limsup_{m,n\to\infty} \frac{\ln M(r_1, r_2)}{r_1^\rho + r_2^\rho} \leq \sigma. \tag{2.7}$$

From (2.6) and (2.7), we get the required result. This completes the proof of Theorem 2.

3. L^p - Approximation

In this section we consider the approximations of an entire function over the domain D. Consider the polynomials

$$p_{m,n}(z_1, z_2) = \lambda_{m,n} z_1^m z_2^n + ...(\lambda_{m,n} > 0)$$

defined through the relation

$$\frac{1}{A} \int\int_D p_{m,n}(z_1, z_2)\, \overline{p_{k,l}(z_1, z_2)}\, dx_1 dy_1 dx_2 dy_2 = \delta_{m,n,k,l}.$$

By applying Carleman's result [2] independently on z_1 and z_2, we have

$$p_{m,n}(z_1, z_2) \sim \left(\frac{(m+1)(n+1)A_1 A_2}{\pi^2}\right)^{1/2} \varphi_1^{'}(z_1)(\varphi_1(z_1))^m \, \varphi_2^{'}(z_2)(\varphi_2(z_2))^n \tag{3.1}$$

as $m, n \to \infty$, uniformly for $z_1 \in E_1$ and $z_2 \in E_2$. Any function $f \in L^2(D)$ can be expanded in terms of these polynomials as

$$f(z_1, z_2) \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{m,n}\, p_{m,n}(z_1, z_2) \tag{3.2}$$

where

$$b_{m,n} = \frac{1}{A}\int\int_D f(z_1,z_2)\,\overline{p_{m,n}(z_1,z_2)}\,dx_1dy_1dx_2dy_2$$

and the series is uniformly convergent on compact subsets of D.

Applying Parseval's relation of one variable independently on m and n, we have

$$E^2_{m,n} \;=\; \left(\sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty}|b_{k,l}|^2\right)^{1/2}. \tag{3.3}$$

Before going to main results here we state and prove two lemmas which are more useful in the proof of main theorems. We now prove

Lemma 1.

$$\limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} = \limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln(E^2_{m,n})}. \tag{3.4}$$

Proof. From (3.3), we have

$$|b_{m+1,n+1}| \le E^2_{m,n},$$

$$\Rightarrow\; -\ln|b_{m+1,n+1}| \ge -\ln(E^2_{m,n}).$$

Proceeding to limits, we have

$$\limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln(E^2_{m,n})} \;\le\; \limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln|b_{m,n}|}. \tag{3.5}$$

Conversely, let

$$\limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\varepsilon > 0$, $\exists\, M, N$ such that for all $m \ge M$, and $n \ge N$, we have

$$|b_{m,n}| \;\le K\, m^{-\frac{m}{\sigma+\varepsilon}}\; n^{-\frac{n}{\sigma+\varepsilon}}$$

so that

$$
\begin{aligned}
&(E^2_{m,n})^2 \\
\leq\ & K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-\frac{2k}{\sigma+\varepsilon}}\ l^{-\frac{2l}{\sigma+\varepsilon}} \leq K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} (m+1)^{-\frac{2k}{\sigma+\varepsilon}}\ (n+1)^{-\frac{2l}{\sigma+\varepsilon}} \\
\leq\ & K(m+1)^{-\frac{2(m+1)}{\sigma+\varepsilon}}\ (n+1)^{-\frac{2(n+1)}{\sigma+\varepsilon}}\ \left[1 - \frac{1}{(m+1)^{2/(\sigma+\varepsilon)}}\right]^{-1} \left[1 - \frac{1}{(n+1)^{2/(\sigma+\varepsilon)}}\right]^{-1} \\
\leq\ & O(1)K(m+1)^{-\frac{2(m+1)}{\sigma+\varepsilon}}\ (n+1)^{-\frac{2(n+1)}{\sigma+\varepsilon}}.
\end{aligned}
$$

Therefore

$$E^2_{m,n} \leq (m+1)^{-\frac{(m+1)}{\sigma+\varepsilon}}\ (n+1)^{-\frac{(n+1)}{\sigma+\varepsilon}}$$

$$\Rightarrow\ -\ln(E^2_{m,n})\ \geq \frac{1}{\sigma+\varepsilon}\ \ln((m+1)^{m+1})\ ((n+1)^{n+1}).$$

Proceeding to limits and since ε is arbitrary, therefore we have

$$\sigma = \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|}\ \geq\ \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E^2_{m,n})}. \tag{3.6}$$

From (3.5) and (3.6), we obtain the required result. This completes the proof of Lemma 1.

Next we prove

Lemma 2. For any $\rho > 0$,

$$\limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n} \right)^{\rho} \right\}^{1/(m+n)} = \limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{E^2_{m,n}}{d_1^m d_2^n} \right)^{\rho} \right\}^{1/(m+n)}. \tag{3.7}$$

Proof. From (3.3), we have

$$(|b_{m+1,n+1}|)^{\rho} \leq (E^2_{m,n})^{\rho}.$$

Since $d_1, d_2 > 0$, therefore for all $m, n > 0$, we have

$$\left(\frac{|b_{m+1,n+1}|}{d_1^m d_2^n}\right)^\rho \le \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho$$

$$\Rightarrow \left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \le \left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}$$

or $\frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \le \frac{1}{e\rho}\left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}.$

Proceeding to limits, we have

$$\limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \le \limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{E_{m,n}^2)}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}. \tag{3.8}$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\varepsilon > 0$, $\exists\ M(\varepsilon), N(\varepsilon)$ such that for all $m \ge M$ and $n \ge N$, we have

$$|b_{m,n}| \le \left\{(e\rho(\sigma+\varepsilon))^{m+n}\ m^{-m} n^{-n}\right\}^{1/\rho}\ d_1^m d_2^n$$

so that

$$(E_{m,n}^2)^2 \le K \sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} \left\{(e\rho(\sigma+\varepsilon))^{k+l}\ k^{-k} l^{-l}\right\}^{2/\rho}\ d_1^{2k} d_2^{2l}$$

$$\le K\left\{(e\rho(\sigma+\varepsilon))^{(m+1)+(n+1)}\ (s+1)^{-(s+1)}\right\}^{2/\rho}\ d_1^{2(m+1)}\ d_2^{2(n+1)}$$

$$\le O(1)K\left\{(e\rho(\sigma+\varepsilon))^{(m+1)+(n+1)}\ (s+1)^{-(s+1)}\right\}^{2/\rho}\ d_1^{2(m+1)}\ d_2^{2(n+1)}$$

for $m > 4d_1^\rho\ e\ \rho\ (\sigma+\varepsilon)$ and $n > 4d_2^\rho\ e\ \rho\ (\sigma+\varepsilon)$, where $(s+1)^{-(s+1)} = (m+1)^{-(m+1)}(n+1)^{-(n+1)}$, $X_1 = \left[1 - \left(\frac{(e\rho(\sigma+\varepsilon))^2 d_1^\rho}{(m+1)^{(m+1)}}\right)^{2/\rho}\right]^{-1}$, and $X_2 = \left[1 - \left(\frac{(e\rho(\sigma+\varepsilon))^2 d_2^\rho}{(n+1)^{(n+1)}}\right)^{2/\rho}\right]^{-1}$. Therefore

$$E_{m,n}^2 \leq O(1)K \ \left\{(e\rho(\sigma+\varepsilon))^{(m+1)+(n+1)} \ (s+1)^{-(s+1)}\right\}^{1/\rho} \ d_1^{(m+1)} \ d_2^{(n+1)}.$$

Proceeding to limits, we have

$$\sigma = \limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \geq \limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}. \tag{3.9}$$

From (3.8) and (3.9), we get the required result. This completes the proof of Lemma 2.

4. Main Results

Now we prove

Theorem 3. *Let $2 \leq p \leq \infty$. Then f is restriction to the domain D of an entire function of finite order ρ if and only if*

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E_{m,n}^p)} = \rho. \tag{4.1}$$

Proof. We prove the result in two steps. First we consider the case p = 2. Let us assume that f is an entire function having finite order ρ. Then by Theorem 1, we have

$$|a_{m,n}| \leq K \ m^{-\frac{m}{\rho+\varepsilon}} \ n^{-\frac{n}{\rho+\varepsilon}}.$$

Now, by considering the property of orthonormality of polynomials $p_{m,n}(z_1,z_2)$, we have

$$b_{m,n} = \sum_{k=m+1}^{\infty} \ \sum_{l=n+1}^{\infty} a_{k,l} \ \frac{1}{A}\int\int_D F_{k,l}(z_1,z_2)\overline{p_{m,n}(z_1,z_2)} \ dx_1 dy_1 dx_2 dy_2.$$

Hence

$$|b_{m,n}| \leq \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} |a_{k,l}| \max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)|.$$

Since, by (1.5), we have

$$\max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)| \leq K(1+\varepsilon)^{(k+l)},$$

on substituting all these values the above inequality becomes,

$$|b_{m,n}| \leq K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-\frac{k}{\rho+\varepsilon}}\, l^{-\frac{l}{\rho+\varepsilon}} (1+\varepsilon)^{(k+l)}$$

$$\leq K m^{-\frac{m}{\rho+\varepsilon}}\, n^{-\frac{n}{\rho+\varepsilon}} (1+\varepsilon)^{(m+n)}$$

for all sufficiently large m and n. Therefore, we have

$$-\ln|b_{m,n}| \geq \frac{1}{(\rho+\varepsilon)} \ln(m^m n^n).$$

Proceeding to limits and since ε is arbitrary, we obtain

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} \leq \rho. \tag{4.2}$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\varepsilon > 0$, $\exists\, L(\varepsilon), Z(\varepsilon)$ such that for all $m > L$ and $n > Z$, we have

$$|b_{m,n}| \leq K\, m^{-\frac{m}{\sigma+\varepsilon}}\, n^{-\frac{n}{\sigma+\varepsilon}}.$$

From (3.1), we have

$$|p_{m,n}(z_1,z_2)| \leq K\, (m+1)^{1/2}(n+1)^{1/2}|\varphi_1'(z_1)||\varphi_1(z_1)|^m|\varphi_2'(z_2)||\varphi_2(z_2)|^n$$

and for all $z_1 \in E_1$ and $z_2 \in E_2$, we have

$$|\varphi_1'(z_1)| \;\leq\; K', \; |\varphi_2'(z_2)| \;\leq\; K"$$

where $K', K"$ are fixed positive constants, and

$$|\varphi_1(z_1)| \;\leq\; \frac{|z_1|}{d_1 - \varepsilon}, \; |\varphi_2(z_2)| \;\leq\; \frac{|z_2|}{d_2 - \varepsilon}$$

for all z_1, z_2 with sufficiently large modulus. Hence

$$\begin{aligned}|f(z_1, z_2)| \;&\leq K \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m^{-\frac{m}{\sigma+\varepsilon}} n^{-\frac{n}{\sigma+\varepsilon}} (m+1)^{1/2} (n+1)^{1/2} \left(\frac{|z_1|}{d_1 - \varepsilon}\right)^m \left(\frac{|z_2|}{d_2 - \varepsilon}\right)^n \\ &\leq K \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m^{-\frac{m}{\sigma+2\varepsilon}} n^{-\frac{n}{\sigma+2\varepsilon}} \left(\frac{|z_1|}{d_1 - \varepsilon}\right)^m \left(\frac{|z_2|}{d_2 - \varepsilon}\right)^n .\end{aligned}$$

To estimate the right hand side of above inequality, following the method used in Theorem 1, we have

$$\begin{aligned}M(r_1, r_2) &< \sum_1 + \sum_2 + \sum_3 + \sum_4 + 0\{e^{(\frac{2r_2}{d_2-\varepsilon})^{\sigma+2\varepsilon}}\} + 0\{e^{(\frac{2r_1}{d_1-\varepsilon})^{\sigma+2\varepsilon}}\} \\ &\leq 0\left\{e^{\left(\frac{2r_1}{d_1-\varepsilon}\right)^{\sigma+2\varepsilon} + \left(\frac{2r_2}{d_2-\varepsilon}\right)^{\sigma+2\varepsilon}}\right\} \leq \quad 0\left\{e^{\left(\frac{4r_1r_2}{(d_1-\varepsilon)(d_1-\varepsilon)}\right)^{\sigma+2\varepsilon}}\right\}.\end{aligned}$$

Now by applying limits, we obtain

$$\rho = \limsup_{r_1, r_2 \to \infty} \frac{\ln \ln M(r_1, r_2)}{\ln(r_1 r_2)} \leq \sigma. \tag{4.3}$$

From (4.2) and (4.3), we have

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} = \rho.$$

By applying Lemma 1, we have

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E_{m,n}^2)} = \rho. \tag{4.4}$$

Now we consider the case for $p > 2$. Since

$$E^2_{m,n} \ \leq \ E^p_{m,n} \ \leq \ E^\infty_{m,n} \ for 2 \leq p \leq \infty, \tag{4.5}$$

it is sufficient to consider the case $p = \infty$. Suppose f is an entire function of order ρ. Then

$$E^\infty_{m,n} \leq \max_{z_1,z_2\in\Gamma} \left| f(z_1,z_2) - \sum_{k=0}^{m}\sum_{l=0}^{n} a_{k,l}\ F_{k,l}(z_1,z_2) \right|$$

$$\leq \sum_{k=0}^{m}\ \sum_{l=n+1}^{\infty} |a_{k,l}|\ \max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)| + \sum_{k=m+1}^{\infty}\ \sum_{l=0}^{n} |a_{k,l}|\ \max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)|$$

$$+ \sum_{k=m+1}^{\infty}\ \sum_{l=n+1}^{\infty} |a_{k,l}|\ \max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)|\,. \tag{4.6}$$

The first two summations in the above inequality (4.6) are bounded. It is sufficient to estimate the last summation. Since f is an entire function of finite order ρ, therefore by Theorem 1, we have
$|a_{m,n}| \ \leq K\ m^{-\frac{m}{\rho+\varepsilon}} n^{-\frac{n}{\rho+\varepsilon}}$ and $\max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)| \ \leq \ (1+\varepsilon)^{k+l}$.

Therefore the above inequality (4.6) becomes,

$$E^\infty_{m,n} \leq \sum_{k=m+1}^{\infty}\ \sum_{l=n+1}^{\infty} k^{-\frac{k}{\rho+\varepsilon}} l^{-\frac{l}{\rho+\varepsilon}}\ (1+\varepsilon)^{k+l}$$

$$\leq K \sum_{k=m+1}^{\infty}\ \sum_{l=n+1}^{\infty} \left(\frac{(1+\varepsilon)^{\rho+\varepsilon}}{m+1}\right)^{k/(\rho+\varepsilon)} \ \left(\frac{(1+\varepsilon)^{\rho+\varepsilon}}{n+1}\right)^{l/(\rho+\varepsilon)}$$

$$\leq K \left(\frac{(1+\varepsilon)^{\rho+\varepsilon}}{m}\right)^{m/(\rho+\varepsilon)} \ \left(\frac{(1+\varepsilon)^{\rho+\varepsilon}}{n}\right)^{n/(\rho+\varepsilon)}$$

$$\Rightarrow \frac{\ln(m^m n^n)}{-\ln(E^\infty_{m,n})} \ \leq \ \frac{\ln(m^m n^n)}{[1/(\rho+\varepsilon)]\ln(m^m n^n) - \ln K - (m+n)\ln(1+\varepsilon)}.$$

Proceeding to limits and since ε is arbitrary, we have

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E^\infty_{m,n})} \leq \rho.$$

In view of inequalities(4.5) and the fact that (4.1) holds for $p = 2$, this last inequality actually is an equality. Finally assuming (4.1) with $p = \infty$, we deduce from (4.5), that (4.1) will hold for $p = 2$ and hence that f is of order ρ. This completes the proof of Theorem 3.

Theorem 4. *Let $2 \leq p \leq \infty$. Then f is restriction to the domain D of an entire function having finite order ρ of type τ if and only if*

$$\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{E^p_{m,n}}{d_1^m d_2^n} \right)^\rho \right\}^{\frac{1}{m+n}} = e\rho\tau. \tag{4.7}$$

Proof. We prove the theorem in two steps. First we consider the case p = 2. Let us assume that f is an entire function having finite order ρ and finite type τ. Then by Theorem 2, we have

$$|a_{m,n}| \leq K\, m^{-(m/\rho)}\, n^{-(n/\rho)}\, d_1^m\, d_2^n\, (e\rho(\tau+\varepsilon))^{(m+n)/\rho}.$$

Now proceeding on the lines of Theorem 3, we have

$$|b_{m,n}| \leq K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-(k/\rho)}\, l^{-(l/\rho)}\, d_1^k\, d_2^l\, (e\rho(\tau+\varepsilon))^{(k+l)/\rho}(1+\varepsilon)^{(k+l)}$$

$$\leq Km^{-(m/\rho)}\, n^{-(n/\rho)}\, d_1^m\, d_2^n\, (e\rho(\tau+\varepsilon))^{(m+n)/\rho}(1+\varepsilon)^{(m+n)}$$

for all sufficiently large m and n. Therefore, we have

$$m^m n^n |b_{m,n}|^\rho \leq K(d_1^m\, d_2^n)^\rho\, (e\rho(\tau+\varepsilon))^{(m+n)}.$$

By applying limits, we have

$$\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n} \right)^\rho \right\}^{\frac{1}{m+n}} \leq e\rho\tau. \tag{4.8}$$

Conversely ,let

$$\limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n} \right)^\rho \right\}^{\frac{1}{m+n}} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\varepsilon > 0$, $\exists\ H(\varepsilon), G(\varepsilon)$ such that for all $m > H$ and $n > G$, we have

$$|b_{m,n}| \leq L\, m^{-(m/\rho)}\, n^{-(n/\rho)}\, d_1^m d_2^n\, (e\rho(\sigma+\varepsilon))^{(m+n)/\rho}.$$

For sufficiently large r_1, r_2,

$$\begin{aligned} &|f(z_1,z_2)| \\ \leq\ & L \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g^{-(g/\rho)}\, d_1^m d_2^n\, (e\rho(\sigma+\varepsilon))^{(m+n)/\rho}(s+1)^{1/2}\left(\frac{|z_1|}{d_1-\varepsilon}\right)^m \left(\frac{|z_2|}{d_2-\varepsilon}\right)^n \\ \leq\ & L \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g^{-(g/\rho)}\, (e\rho(\sigma+2\varepsilon))^{(m+n)/\rho}\left(\frac{d_1|z_1|}{d_1-\varepsilon}\right)^m \left(\frac{d_2|z_2|}{d_2-\varepsilon}\right)^n \\ \leq\ & L \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g^{-(g/\rho)}\, (e\rho(\sigma+2\varepsilon))^{(m+n)/\rho}\, r_1^m\, r_2^n \end{aligned}$$

where $g^{-(g/\rho)} = m^{-(m/\rho)}\, n^{-(n/\rho)}$ and $(s+1)^{1/2} = (m+1)^{1/2}(n+1)^{1/2}$. To estimate the right hand side of above inequality we follow the same method as of Bose and Sharma [1, Theorem V, p 224], and we obtain

$$|f(z_1,z_2)| \leq 0\{e^{(\sigma+\varepsilon)(r_1^\rho + r_2^\rho)}\}.$$

Hence

$$M(r_1,r_2) \leq 0\{e^{(\sigma+\varepsilon)(r_1^\rho + r_2^\rho)}\}.$$

Now by applying limits, we have

$$\tau = \limsup_{r_1,r_2\to\infty} \frac{\ln M(r_1,r_2)}{r_1^\rho + r_2^\rho} \leq \sigma. \tag{4.9}$$

From (4.8) and (4.9), we have

$$\limsup_{m,n\to\infty}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{\frac{1}{m+n}} = e\rho\tau.$$

By applying above Lemma 2, we have

$$\limsup_{m,n\to\infty}\left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{\frac{1}{m+n}} = e\rho\tau.$$

Now we consider the case for $p > 2$. From (4.5), it is sufficient to consider the case $p = \infty$. Suppose f is an entire function having finite order ρ and of type τ. Then from (4.6), the first two summations of the above inequality are bounded. It is sufficient to estimate the last summation. Since f is an entire function of finite type τ, therefore by Theorem 2, we have

$$|a_{m,n}| \ \leq K \ m^{-(m/\rho)} \ n^{-(n/\rho)} \ d_1^m \ d_2^n (e\rho(\tau+\varepsilon))^{(m+n)/\rho}.$$

By using above inequality and from (4.6), we have

$$E_{m,n}^{\infty} \leq K \sum_{k=m+1}^{\infty} \ \sum_{l=n+1}^{\infty} k^{-(k/\rho)} \ l^{-(l/\rho)} \ d_1^k \ d_2^l \ (e\rho(\tau+\varepsilon))^{(k+l)/\rho} \ (1+\varepsilon)^{k+l}$$

$$\leq K \sum_{k=m+1}^{\infty} \ \sum_{l=n+1}^{\infty} \left(\frac{(1+\varepsilon)^{\rho}}{m+1}\right)^{k/\rho} \ \left(\frac{(1+\varepsilon)^{\rho}}{n+1}\right)^{l/\rho} \ d_1^k \ d_2^l \ (e\rho(\tau+\varepsilon))^{k+l}$$

$$\leq K \left(\frac{(1+\varepsilon)^{\rho}}{m+1}\right)^{m/\rho} \ \left(\frac{(1+\varepsilon)^{\rho}}{n+1}\right)^{n/\rho} \ d_1^m \ d_2^n \ (e\rho(\tau+\varepsilon))^{m+n}$$

$$\Rightarrow \left(m^m n^n \left(\frac{E_{m,n}^{\infty}}{d_1^m d_2^n}\right)^{\rho}\right)^{1/(m+n)} \ \leq \ (1+\varepsilon) \ (e\rho(\tau+\varepsilon))$$

Hence $\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{E_{m,n}^{\infty}}{d_1^m d_2^n}\right)^{\rho} \right\}^{\frac{1}{m+n}} \leq e\rho\tau.$

In view of inequalities(4.5) and the fact that (4.7) holds for $p = 2$, this last inequality actually is an equality. Finally assuming (4.7) with $p = \infty$, we deduce from (4.5), that (4.7) will hold for $p = 2$ and hence that f is of type τ. This completes the proof of Theorem 4.

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