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On a Type of Concircular φ-recurrent Trans-Sasakian Manifolds*†

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Abstract

The object of the present paper is to study on a type of concircular φ-recurrent trans-Sasakian manifolds.

Keywords and Phrases: *Trans-Sasakian manifold, Concircular curvature tensor, Concircular φ-recurrent manifold, Locally φ-symmetric trans-Sasakian manifold, Characteristic vector field.*

1. Introduction

The notion of locally φ-symmetric Sasakian manifold was introduced by T. Takahashi [11] in 1977. φ-recurrent Sasakian manifold was studied by the author [12].

Also J. A. Oubina in 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [2], α -Sasakian [5], Kenmotsu [5], β-Kenmotsu [5] and cosymplectic [5] manifolds, which was called trans-Sasakian manifold [6]. After him many authors ([3], [4], [5], [6], [8], [9],

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[10]) have studied various type of properties in trans-Sasakian manifold.

The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of concircular φ-recurrent trans-Sasakian manifold which satisfies $\varphi(grad\ \alpha) = (2n-1)grad\ \beta$ and proved that such a manifold is an Einstein manifold.
It is shown that in a concircular φ-recurrent trans-Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are opposite directional.

2. Preliminaries.

A $(2n+1)$ dimensional, $(n \geq 1)$ almost contact metric manifold M with almost contact metric structure (φ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta\ o\ \varphi = 0 \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in \chi(M)$

is called trans-Sasakian manifold [6] if and only if

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{2.4}$$

for some smooth functions α and β on M.
From (2.4) it follows that

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi) \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{2.6}$$

In [10], the authors obtained some results which shall be useful for next section. They are

$$R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y)+2\alpha\beta(\eta(Y)\varphi X-\eta(X)\varphi Y)$$
$$+(Y\alpha)\varphi X-(X\alpha)\varphi Y+(Y\beta)\varphi^2X-(X\beta)\varphi^2Y \tag{2.7}$$

$$R(\xi,X)\xi = (\alpha^2-\beta^2-\xi\beta)(\eta(X)\xi-X) \tag{2.8}$$

$$2\alpha\beta+\xi\alpha = 0 \tag{2.9}$$

$$S(X,\xi) = (2n(\alpha^2-\beta^2)-\xi\beta)\eta(X)-(2n-1)X\beta-(\varphi X)\alpha \tag{2.10}$$

$$Q\xi = (2n(\alpha^2-\beta^2)-\xi\beta)\xi-(2n-1)grad\ \beta+\varphi(grad\ \alpha). \tag{2.11}$$

When $\varphi(grad\ \alpha) = (2n-1)grad\ \beta$, then (2.10) and (2.11) reduces to

$$S(X,\xi) = 2n(\alpha^2-\beta^2)\eta(X) \tag{2.12}$$

$$Q\xi = 2n(\alpha^2-\beta^2)\xi. \tag{2.13}$$

Again a trans-Sasakian manifold is said to be locally φ-symmetric [11] if

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0 \tag{2.14}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Let us introduce concircular φ-recurrent manifold. A trans-Sasakian manifold is said to be concircular φ-recurrent manifold if there exists a non-zero 1-form A such that

$$\varphi^2((\nabla_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z \tag{2.15}$$

for $X, Y, Z, W \in \chi(M)$

where the 1-form A is defined as

$$g(X,\rho) = A(X), \quad \forall\, X \in \chi(M), \tag{2.16}$$

ρ being the vector field associated to the 1-form A and $\tilde{C}$ is a concircular curvature tensor given by [2]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \tfrac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{2.17}$$

where a, b are constants such that $a, b \neq 0$, R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature.
Also,

$$g(QX,Y) = S(X,Y) \tag{2.18}$$

Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S.

The above results will be useful in the next section.

3. Concircular φ - recurrent Trans-Sasakian Manifold

In this section we consider a trans-Sasakian manifold which is concircular φ-recurrent. Then by virtue of (2.1) and (2.15) we have

$$-(\nabla_W \tilde{C})(X,Y)Z + \eta((\nabla_W \tilde{C})(X,Y)Z)\xi = A(W)\tilde{C}(X,Y)Z. \tag{3.1}$$

From (3.1) it follows that

$$-g((\nabla_W \tilde{C})(X,Y)Z,U) + \eta((\nabla_W \tilde{C})(X,Y)Z)\eta(U)$$
$$= A(W)g(\tilde{C}(X,Y)Z,U). \tag{3.2}$$

Let $\{e_i\}$, $i = 1, 2, ..., 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$(\nabla_W S)(Y,Z) = \tfrac{dr(W)}{2n+1} g(Y,Z) - \tfrac{dr(W)}{2n(2n+1)}[g(Y,Z) - \eta(Y)\eta(Z)]$$
$$- A(W)[S(Y,Z) - \tfrac{r}{2n+1}\, g(Y,Z)]. \tag{3.3}$$

Replacing Z by ξ and using (2.1), (2.3) and (2.12) we obtain

$$(\nabla_W S)(Y,\xi) = \{\tfrac{dr(W)}{2n+1} - A(W)[2n(\alpha^2-\beta^2) - \tfrac{r}{2n+1}]\}\eta(Y). \tag{3.4}$$

Now we know

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{3.5}$$

Using (2.5) and (2.12) in the above relation (3.5) we get, after a brief calculation

$$(\nabla_W S)(Y,\xi) = 2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(W,Y)]$$
$$+ \alpha S(Y,\varphi W) - \beta S(Y,W). \tag{3.6}$$

Using (3.6) in (3.4) we have

$$2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(W,Y)] + \alpha S(Y,\varphi W) - \beta S(Y,W)$$
$$= \{\tfrac{dr(W)}{2n+1} - A(W)[2n(\alpha^2-\beta^2) - \tfrac{r}{2n+1}]\}\eta(Y). \tag{3.7}$$

Replacing Y and W by φY and φW respectively, we obtain

$$S(Y,W) = 2n(\alpha^2-\beta^2) g(Y,W) \tag{3.8}$$

and

$$S(\varphi Y,W) = 2n(\alpha^2-\beta^2) g(\varphi Y,W). \tag{3.9}$$

Hence we can state the following:

Theorem 3.1. *A concircular φ -recurrent trans - Sasakian manifold (M^{2n+1}, g) satisfying φ (gradα) = (2n - 1) grad β, is an Einstein manifold.*

Now from (3.1) and (2.16) we have

$$
\begin{aligned}
(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z \\
+ \tfrac{dr(W)}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y \\
- g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi] \\
+ \tfrac{r}{2n(2n+1)}A(W)[g(Y,Z)X - g(X,Z)Y]. \qquad (3.10)
\end{aligned}
$$

Using Bianchi's identity in (3.10) we obtain

$$
\begin{aligned}
A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\
= \tfrac{r}{2n(2n+1)}A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\
+ \tfrac{r}{2n(2n+1)}A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \\
+ \tfrac{r}{2n(2n+1)}A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \qquad (3.11)
\end{aligned}
$$

Putting $Y = Z = \{e_i\}$, where e_i be an orthonormal basis of the tangent space at any point of the manifold, in (3.11) and taking summation over $i,\ 1 \leq i \leq 2n+1$, we get

$$A(W) + A(X) = -\tfrac{1}{2n}[A(X)\eta(W) - A(W)\eta(X)]. \qquad (3.12)$$

Putting again $X = \xi$ and using (2.1) and (2.3) we obtain

$$A(W) = -\tfrac{1}{2n-1}\eta(\rho)\eta(W) \qquad (3.13)$$

for any vector field W and ρ being the vector field associated to the 1-form A, defined as (2.16). Thus we can state the following theorem:

Theorem 3.2. *In a concircular φ -recurrent trans-Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are opposite directional and the 1-form A is given by*

$$(2n-1)A(W) = -\eta(\rho)\eta(W) \qquad \forall \quad W \quad \in \quad \chi(M).$$

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