Tamsui Oxford Journal of Information and Mathematical Sciences **29(4)** (2013) 487-504
Aletheia University

Estimates for the Initial Coefficients of Bi-univalent Functions *

S. Sivaprasad Kumar[†], Virendra Kumar[‡]
Department of Applied Mathematics Delhi Technological University Delhi-110042, India

and

V. Ravichandran[§]
Department of Mathematics University of Delhi Delhi–110007, India

Received January 12, 2014, Accepted February 27, 2014.

The work presented here was supported in parts by a Senior Research Fellowship from Delhi Technological University, New Delhi, and by a grant from University of Delhi. The authors are thankful to Prof. H. M. Srivastava for his useful comments.

Abstract

A bi-univalent function is a univalent function defined on the unit disk for which the inverse function has a univalent extension to the unit disk. The paper of H. M. Srivastava, A. K. Mishra and P. Gochhayat [Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23** (2010), no. 10, 1188–1192] renewed the investigation of the estimate on initial coefficients of bi-univalent functions. In this paper,

*2010 *Mathematics Subject Classification*. Primary 30C45, 30C80.
[†]Corresponding author. E-mail: spkumar@dce.ac.in
[‡]E-mail: vktmaths@yahoo.in
[§]E-mail: vravi@maths.du.ac.in

estimates for the initial coefficients of bi-univalent functions belonging to certain classes defined by subordination and of functions for which f and f^{-1} belong to different subclasses of univalent functions are derived. Improvement of the earlier known estimates were also pointed out.

Keywords and Phrases: *Univalent functions, Bi-univalent functions, Coefficient estimate, Subordination.*

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. A function $f \in \mathcal{A}$ has Taylor's series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The class of all univalent functions in the open unit disk $\mathbb{D}$ of the form (1.1) is denoted by $\mathcal{S}$. Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions in $\mathcal{S}$ gives the growth and distortion bounds as well as covering theorems.

Since univalent functions are one-to-one, it follows that they are invertible but their inverse functions need not be defined on the entire unit disk $\mathbb{D}$. In fact, the famous Koebe one-quarter theorem ensures that the image of the unit disk $\mathbb{D}$ under every function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus, inverse of every function $f \in \mathcal{S}$ is defined on a disk, which contains the disk $|z| < 1/4$. It can also be easily verified that

$$F(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{1.2}$$

in some disk of radius at least 1/4. A function $f \in \mathcal{A}$ is called *bi-univalent* in $\mathbb{D}$ if both f and the extension of f^{-1} to the unit disk are univalent in $\mathbb{D}$. In 1967, Lewin [21] introduced the class σ of bi-univalent analytic functions and showed that the second coefficient of every $f \in \sigma$ satisfy the inequality $|a_2| \leq 1.51$. Let σ_1 be the class of all functions $f = \phi \circ \psi^{-1}$ where ϕ, ψ map $\mathbb{D}$ onto a domain containing $\mathbb{D}$ and $\phi'(0) = \psi'(0)$. In 1969, Suffridge [38] gave a function in $\sigma_1 \subset \sigma$, satisfying $a_2 = 4/3$ and conjectured that $|a_2| \leq 4/3$ for all functions in σ. In 1969, Netanyahu [24] proved this conjecture for the subclass σ_1. Later in 1981, Styer and Wright [37] disproved the conjecture of Suffridge [38] by showing $a_2 > 4/3$ for some function in σ. Also see [5] for an example to show $\sigma \neq \sigma_1$. For results on bi-univalent polynomial, see [30, 18]. In 1967, Brannan [2] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. In 1985, Kedzierawski [17, Theorem 2] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. In 1985, Tan [39] obtained the bound for a_2 namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class σ. For some open problems and survey, see [12, 31]. In 1985, Kedzierawski [17] proved the following:

$$|a_2| \leq \begin{cases} 1.5894, & f \in \mathcal{S},\ f^{-1} \in \mathcal{S}; \\ \sqrt{2}, & f \in \mathcal{S}^*,\ f^{-1} \in \mathcal{S}^*; \\ 1.507, & f \in \mathcal{S}^*,\ f^{-1} \in \mathcal{S}; \\ 1.224, & f \in \mathcal{K},\ f^{-1} \in \mathcal{S}, \end{cases}$$

where $\mathcal{S}^*$ and $\mathcal{K}$ denote the well-known classes of starlike and convex functions in $\mathcal{S}$.

Let us recall now various definitions required in sequel. An analytic function f is *subordinate* to another analytic function g, written $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f = g \circ w$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Let φ be an analytic univalent function in $\mathbb{D}$ with positive real part and $\varphi(\mathbb{D})$ be symmetric with respect to the real axis, starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Ma and Minda [23] gave a unified presentation of various subclasses of starlike and convex functions by introducing the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ of functions $f \in \mathcal{S}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$ and $1 + zf''(z)/f'(z) \prec \varphi(z)$ respectively, which includes several well-known classes as special case. For example, when $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$, the class $\mathcal{S}^*(\varphi)$ reduces to the class $\mathcal{S}^*[A, B]$ introduced by Janowski [16]. For $0 \leq \beta < 1$, the classes $\mathcal{S}^*(\beta) := \mathcal{S}^*((1+(1-2\beta)z)/(1-z))$ and $\mathcal{K}(\beta) := \mathcal{K}((1+(1-2\beta)z)/(1-z))$ are starlike and convex functions of order β. Further let $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the classes of starlike and convex functions respectively. The class of strongly starlike functions $\mathcal{S}^*_\alpha := \mathcal{S}^*(((1+z)/(1-z))^\alpha)$ of order α, $0 < \alpha \leq 1$. Denote by $\mathcal{R}(\varphi)$ the class of all functions satisfying $f'(z) \prec \varphi(z)$ and let $\mathcal{R}(\beta) := \mathcal{R}((1 + (1 - 2\beta)z)/(1 - z))$ and $\mathcal{R} := \mathcal{R}(0)$.

For $0 \leq \beta < 1$, a function $f \in \sigma$ is in the class $\mathcal{S}^*_\sigma(\beta)$ of *bi-starlike func-*

tion of order β, or $\mathcal{K}_\sigma(\beta)$ of *bi-convex function of order* β if both f and f^{-1} are respectively starlike or convex functions of order β. For $0 < \alpha \leq 1$, the function $f \in \sigma$ is *strongly bi-starlike function of order* α if both the functions f and f^{-1} are strongly starlike functions of order α. The class of all such functions is denoted by $\mathcal{S}^*_{\sigma,\alpha}$. These classes were introduced by Brannan and Taha [4] in 1985 (see also [3]). They obtained estimates on the initial coefficients a_2 and a_3 for functions in these classes. The work on bi-univalent functions gained much focus after Srivastava *et al.*[36], in 2010, derived the bounds of the initial coefficients of functions belonging to the classes $\mathcal{H}_\sigma(\beta) = \{f \in \sigma : \mathrm{Re}(f'(z)) > \beta \text{ and } \mathrm{Re}(F'(z)) > \beta, 0 \leq \beta < 1\}$ and $\mathcal{H}_{\sigma,\alpha} = \{f \in \sigma : |\arg f'(z)| \leq \alpha\pi/2 \text{ and } |\arg F'(z)| \leq \alpha\pi/2, 0 < \alpha \leq 1\}$. Further results motivated by [36] can be found in [6, 15, 7, 8, 9, 10, 11, 13, 14, 22, 25, 33, 34, 35, 26, 1, 29, 40, 41, 42] and references cited therein. Srivastava [32] has a recent survey of bi-univalent functions.

Motivated by Ali *et al.* [1] and Srivastava[36], the estimates on the initial coefficient a_2 of bi-univalent functions belonging to the class $\mathcal{R}_\sigma(\lambda,\varphi)$ as well as estimates on a_2 and a_3 for functions in classes $\mathcal{S}^*_\sigma(\varphi)$ and $K_\sigma(\varphi)$, defined later, are obtained. The estimates on initial coefficients a_2 and a_3 when f is in the some subclass of univalent functions and F belongs to some other subclass of univalent functions are also derived and connections and generalization of several well-known results in [1, 10, 17, 36] are also pointed out.

2. Coefficient estimates

Throughout this paper, we assume that φ is an analytic function in $\mathbb{D}$ of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \tag{2.1}$$

with $B_1 > 0$ and B_2 is any real number.

Definition 2.1. *Let $\lambda \geq 0$. A function $f \in \sigma$ given by (1.1) is in the class $\mathcal{R}_\sigma(\lambda,\varphi)$, if it satisfies*

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \varphi(z) \quad \textit{and} \quad (1-\lambda)\frac{F(w)}{w} + \lambda F'(w) \prec \varphi(w).$$

The class $\mathcal{R}_\sigma(\lambda,\varphi)$ includes many earlier classes, which are mentioned below:

1. $\mathcal{R}_\sigma(\lambda, (1+(1-2\beta)z)/(1-z)) = \mathcal{R}_\sigma(\lambda, \beta)$ $(\lambda \geq 1;\ 0 \leq \beta < 1)$ [10, Definition 3.1]
2. $\mathcal{R}_\sigma(\lambda, ((1+z)/(1-z))^\alpha) = \mathcal{R}_{\sigma,\alpha}(\lambda)$ $(\lambda \geq 1;\ 0 < \alpha \leq 1)$ [10, Definition 2.1]
3. $\mathcal{R}_\sigma(1, \varphi) = \mathcal{R}_\sigma(\varphi)$ [1, p. 345].
4. $\mathcal{R}_\sigma(1, (1+(1-2\beta)z)/(1-z)) = \mathcal{H}_\sigma(\beta)$ $(0 \leq \beta < 1)$ [36, Definition 2]
5. $\mathcal{R}_\sigma(1, ((1+z)/(1-z))^\alpha) = \mathcal{H}_{\sigma,\alpha}$ $(0 < \alpha \leq 1)$ [36, Definition 1]

Our first result provides estimate for the coefficient a_2 of functions $f \in \mathcal{R}_\sigma(\lambda, \varphi)$.

Theorem 2.2. *If $f \in \mathcal{R}_\sigma(\lambda, \varphi)$, then*

$$|a_2| \leq \sqrt{\frac{B_1 + |B_1 - B_2|}{1+2\lambda}}. \tag{2.2}$$

Proof. Since $f \in \mathcal{R}_\sigma(\lambda, \varphi)$, there exist two analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi(r(z)) \text{ and } (1-\lambda)\frac{F(w)}{w} + \lambda F'(w) = \varphi(s(w)). \tag{2.3}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.4}$$

and

$$q(z) = \frac{1+s(z)}{1-s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots, \tag{2.5}$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right) \tag{2.6}$$

and

$$s(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left(q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right). \tag{2.7}$$

It is clear that p and q are analytic in $\mathbb{D}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{D}$, and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (2.3), (2.6) and (2.7), clearly

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \text{ and } (1-\lambda)\frac{F(w)}{w} + \lambda F'(w) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.8}$$

On expanding (2.1) using (2.6) and (2.7), it is evident that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\big(p_2 - \frac{1}{2}p_1^2\big) + \frac{1}{4}B_2 p_1^2\right) z^2 + \cdots. \tag{2.9}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1\big(q_2 - \frac{1}{2}q_1^2\big) + \frac{1}{4}B_2 q_1^2\right) w^2 + \cdots. \tag{2.10}$$

Since $f \in \sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $F = f^{-1}$ has the expansion given by (1.2). It follows from (2.8), (2.9) and (2.10) that

$$(1+\lambda)a_2 = \frac{1}{2}B_1 p_1,$$

$$(1+2\lambda)a_3 = \frac{1}{2}B_1\Big(p_2 - \frac{1}{2}p_1^2\Big) + \frac{1}{4}B_2 p_1^2, \tag{2.11}$$

$$-(1+\lambda)a_2 = \frac{1}{2}B_1 q_1,$$

$$(1+2\lambda)(2a_2^2 - a_3) = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2. \tag{2.12}$$

Now (2.11) and (2.12) yield

$$8(1+2\lambda)a_2^2 = 2(p_2+q_2)B_1 + (B_2 - B_1)(p_1^2 + q_1^2). \tag{2.13}$$

Finally an application of the known results, $|p_i| \leq 2$ and $|q_i| \leq 2$ in (2.13) yields the desired estimate of a_2 given by (2.2). $\square$

Remark 2.3. *Let $\varphi(z) = (1 + (1-2\beta)z)/(1-z)$, $0 \leq \beta < 1$. So $B_1 = B_2 = 2(1-\beta)$. When $\lambda = 1$, Theorem 2.2 gives the estimate $|a_2| \leq \sqrt{2(1-\beta)/3}$ for functions in the class $\mathcal{R}_\sigma(\beta)$ which coincides with the result [41, Corollary 2] of* Xu et al. *In particular if $\beta = 0$, then above estimate becomes $|a_2| \leq \sqrt{2/3} \approx$*

0.816 *for functions* $f \in \mathcal{R}_\sigma(0)$. *Since the estimate on* $|a_2|$ *for* $f \in \mathcal{R}_\sigma(0)$ *is improved over the conjectured estimate* $|a_2| \leq \sqrt{2} \approx 1.414$ *for* $f \in \sigma$, *the functions in* $\mathcal{R}_\sigma(0)$ *are not the candidate for the sharpness of the estimate in the class* σ.

Definition 2.4. *A function* $f \in \sigma$ *is in the class* ${\mathcal{S}^*}_\sigma(\varphi)$, *if it satisfies*

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad \textit{and} \quad \frac{wF'(w)}{F(w)} \prec \varphi(w).$$

Note that for a suitable choice of φ, the class ${\mathcal{S}^*}_\sigma(\varphi)$, reduces to the following well-known classes:

1. ${\mathcal{S}^*}_\sigma((1+(1-2\beta)z)/(1-z)) = {\mathcal{S}^*}_\sigma(\beta) \quad (0 \leq \beta < 1)$.
2. ${\mathcal{S}^*}_\sigma(((1+z)/(1-z))^\alpha) = \mathcal{S}^*_{\sigma,\alpha} \quad (0 < \alpha \leq 1)$.

Theorem 2.5. *If* $f \in {\mathcal{S}^*}_\sigma(\varphi)$, *then*

$$|a_2| \leq \min\left\{\sqrt{B_1 + |B_2 - B_1|}, \sqrt{\frac{B_1^2 + B_1 + |B_2 - B_1|}{2}}, \frac{B_1\sqrt{B_1}}{\sqrt{B_1^2 + |B_1 - B_2|}}\right\}$$

and

$$|a_3| \leq \min\left\{B_1 + |B_2 - B_1|, \frac{B_1^2 + B_1 + |B_2 - B_1|}{2}, R\right\},$$

where

$$R := \frac{1}{4}\left(B_1 + 3B_1 \max\left\{1; \left|\frac{B_1 - 4B_2}{3B_1}\right|\right\}\right).$$

Proof. Since $f \in \mathcal{S}^*_\sigma(\varphi)$, there are analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$\frac{zf'(z)}{f(z)} = \varphi(r(z)) \text{ and } \frac{wF'(w)}{F(w)} = \varphi(s(w)). \tag{2.14}$$

Let p and q be defined as in (2.4), then it is clear from (2.14), (2.6) and (2.7) that

$$\frac{zf'(z)}{f(z)} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad \frac{wF'(w)}{F(w)} = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.15}$$

It follows from (2.15), (2.9) and (2.10) that

$$a_2 = \frac{1}{2}B_1 p_1, \tag{2.16}$$

$$2a_3 = \frac{B_1 p_1}{2}a_2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2, \tag{2.17}$$

$$-a_2 = \frac{1}{2}B_1 q_1 \tag{2.18}$$

and

$$4a_2^2 - 2a_3 = -\frac{B_1 q_1}{2}a_2 + \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2. \tag{2.19}$$

The equations (2.16) and (2.18) yield

$$p_1 = -q_1, \tag{2.20}$$

$$8a_2^2 = (p_1^2 + q_1^2)B_1^2 \tag{2.21}$$

and

$$2a_2 = \frac{B_1(p_1 - q_1)}{2}. \tag{2.22}$$

From (2.17), (2.19) and (2.22), it follows that

$$8a_2^2 = 2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2). \tag{2.23}$$

Further a computation using (2.17), (2.19), (2.16) and (2.20) gives

$$16a_2^2 = 2B_1^2 q_1^2 + 2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2). \tag{2.24}$$

Similarly a computation using (2.17), (2.19), (2.22) and (2.21) yields

$$4(B_1^2 - B_2 + B_1)a_2^2 = B_1^3(p_2 + q_2). \tag{2.25}$$

Now (2.23), (2.24) and (2.25) yield the desired estimate on a_2 as asserted in the theorem. To find estimate for a_3 subtract (2.17) from (2.19), to get

$$-4a_3 = -4a_2^2 + \frac{B_1(q_2 - p_2)}{2}. \tag{2.26}$$

Now a computation using (2.24) and (2.26) leads to

$$16a_3 = 2B_1^2 q_1^2 + 4B_2 p_2 + (B_1 - B_2)(p_1^2 + q_1^2). \tag{2.27}$$

From (2.16), (2.17), (2.18) and (2.19), it follows that

$$4a_3 = \frac{B_1}{2}(3p_2 + q_2) + (B_2 - B_1)p_1^2 \tag{2.28}$$
$$= \frac{B_1 q_2}{2} + \frac{3B_1}{2}\left(p_2 - \frac{2(B_1 - B_2)}{3B_1}p_1^2\right). \tag{2.29}$$

On applying the result of Keogh and Merkes [19](see also [27]), that is for any complex number v, $|p_2 - vp_1^2| \leq 2\max\{1; |2v - 1|\}$, along with $|q_2| \leq 2$ in (2.29), we obtain

$$4|a_3| \leq B_1 + 3B_1 \max\left\{1; \left|\frac{B_1 - 4B_2}{3B_1}\right|\right\}. \tag{2.30}$$

Now the desired estimate on a_3 follows from (2.27), (2.28) and (2.30) at once. □

Remark 2.6. *If $f \in \mathcal{S}^*{}_\sigma(\beta)$ $(0 \leq \beta < 1)$, then from Theorem 2.5 it is evident that*

$$|a_2| \leq \min\left\{\sqrt{2(1-\beta)}, \sqrt{(1-\beta)(3-2\beta)}\right\} = \begin{cases} \sqrt{2(1-\beta)}, & 0 \leq \beta \leq 1/2; \\ \sqrt{(1-\beta)(3-2\beta)}, & 1/2 \leq \beta < 1. \end{cases} \tag{2.31}$$

Recall Brannan and Taha's [3, Theorem 3.1] coefficient estimate, $|a_2| \leq \sqrt{2(1-\beta)}$ for functions $f \in \mathcal{S}^*{}_\sigma(\beta)$, who claimed that their estimate is better than the estimate $|a_2| \leq 2(1-\beta)$, given by Robertson [28]. But their claim is true only when $0 \leq \beta \leq 1/2$. Also it may noted that our estimate for a_2 given in (2.31) improves the estimate given by Brannan and Taha [3, Theorem 3.1].

Further if we take $\varphi(z) = ((1+z)/(1-z))^\alpha$, $0 < \alpha \leq 1$ in Theorem 2.5, we have $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Then we obtain the estimate on a_2 for functions $f \in \mathcal{S}^*{}_{\sigma,\alpha}$ as:

$$|a_2| \leq \min\left\{\sqrt{4\alpha - 2\alpha^2}, \sqrt{\alpha^2 + 2\alpha}, \frac{2\alpha}{\sqrt{1+\alpha}}\right\} = \frac{2\alpha}{\sqrt{1+\alpha}}.$$

Note that Brannan and Taha [3, Theorem 2.1] gave the same estimate $|a_2| \leq 2\alpha/\sqrt{1+\alpha}$ for functions $f \in \mathcal{S}^*{}_{\sigma,\alpha}$.

Definition 2.7. *A function f given by (1.1) is said to be in the class $K_\sigma(\varphi)$, if f and F satisfy the subordinations*

$$1+\frac{zf''(z)}{f'(z)} \prec \varphi(z) \ \ \textit{and}\ \ 1+\frac{wF''(w)}{F'(w)} \prec \varphi(w).$$

Note that $K_\sigma((1+(1-2\beta)z)/(1-z)) =: K_\sigma(\beta)$ $(0 \leq \beta < 1)$.

Theorem 2.8. *If $f \in K_\sigma(\varphi)$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{B_1^2+B_1+|B_2-B_1|}{6}}, \frac{B_1}{2}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{B_1^2+B_1+|B_2-B_1|}{6}, \frac{B_1(3B_1+2)}{12}\right\}.$$

Proof. Since $f \in K_\sigma(\varphi)$, there are analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, satisfying

$$1+\frac{zf''(z)}{f'(z)} = \varphi(r(z)) \text{ and } 1+\frac{wF''(w)}{F'(w)} = \varphi(s(w)). \tag{2.32}$$

Let p and q be defined as in (2.4), then it is clear from (2.32), (2.6) and (2.7) that

$$1+\frac{zf''(z)}{f'(z)} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \ \text{ and } \ 1+\frac{wF''(w)}{F'(w)} = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.33}$$

It follows from (2.33), (2.9) and (2.10) that

$$2a_2 = \frac{1}{2}B_1p_1, \tag{2.34}$$

$$6a_3 = B_1p_1a_2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2, \tag{2.35}$$

$$-2a_2 = \frac{1}{2}B_1q_1 \tag{2.36}$$

and

$$6(2a_2^2-a_3) = -B_1q_1a_2 + \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2. \tag{2.37}$$

Now (2.34) and (2.36) yield

$$p_1 = -q_1 \tag{2.38}$$

and

$$4a_2 = \frac{B_1(p_1 - q_1)}{2}. \tag{2.39}$$

From (2.35), (2.37), (2.38) and (2.34), it follows that

$$48a_2^2 = 2B_1^2 p_1^2 + 2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2). \tag{2.40}$$

In view of $|p_i| \leq 2$ and $|q_i| \leq 2$ together with (2.39) and (2.40) yield the desired estimate on a_2 as asserted in the theorem. In order to find a_3, we subtract (2.35) from (2.37) and use (2.38) to obtain

$$-12a_3 = -12a_2^2 + \frac{B_1(q_2 - p_2)}{2}. \tag{2.41}$$

Now a computation using (2.40) and (2.41) leads to

$$-48a_3 = 2B_1^2 p_1^2 - 4B_2 p_2 + (B_1 - B_2)(p_1^2 + q_1^2). \tag{2.42}$$

From (2.39) and (2.41), it follows that

$$-12a_3 = \frac{B_1(q_2 - p_2)}{2} - \frac{3(p_1 - q_1)^2 B_1^2}{16}. \tag{2.43}$$

Now (2.42) and (2.43) yield the desired estimate on a_3 as asserted in the theorem. □

Remark 2.9. *If $f \in K_\sigma(\beta)$ $(0 \leq \beta < 1)$, then Theorem 2.8 gives*

$$|a_2| \leq \min\left\{\sqrt{\frac{(1-\beta)(3-2\beta)}{3}}, 1-\beta\right\} = 1-\beta$$

and

$$|a_3| \leq \min\left\{\frac{(1-\beta)(3-2\beta)}{3}, \frac{(1-\beta)(4-3\beta)}{3}\right\} = \frac{(1-\beta)(3-2\beta)}{3},$$

which improves the Brannan and Taha's [3, Theorem 4.1] estimates $|a_2| \leq \sqrt{1-\beta}$ and $|a_3| \leq 1-\beta$ for functions $f \in K_\sigma(\beta)$.

Theorem 2.10. *Let $f \in \sigma$ be given by (1.1). If $f \in \mathcal{K}(\varphi)$ and $F \in \mathcal{R}(\varphi)$, then*

$$|a_2| \leq \sqrt{\frac{3[B_1 + |B_2 - B_1|]}{8}}$$

and

$$|a_3| \leq \frac{5[B_1 + |B_2 - B_1|]}{12}.$$

Proof. Since $f \in \mathcal{K}(\varphi)$ and $F \in \mathcal{R}(\varphi)$, there exist two analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{zf''(z)}{f'(z)} = \varphi(r(z)) \text{ and } F'(w) = \varphi(s(w)). \tag{2.44}$$

Let the functions p and q are defined by (2.4). It is clear that p and q are analytic in $\mathbb{D}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{D}$, and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. Proceeding as in the proof of Theorem 2.2 it follow from (2.44), (2.9) and (2.10) that

$$\begin{aligned} 2a_2 &= \frac{1}{2}B_1 p_1, \\ 6a_3 - 4a_2^2 &= \frac{1}{2}B_1\Big(p_2 - \frac{1}{2}p_1^2\Big) + \frac{1}{4}B_2 p_1^2, \\ -2a_2 &= \frac{1}{2}B_1 q_1 \end{aligned} \tag{2.45}$$

and

$$3(2a_2^2 - a_3) = \frac{1}{2}B_1\Big(q_2 - \frac{1}{2}q_1^2\Big) + \frac{1}{4}B_2 q_1^2. \tag{2.46}$$

A computation using (2.45) and (2.46), leads to

$$a_2^2 = \frac{2(p_2 + 2q_2)B_1 + (p_1^2 + 2q_1^2)(B_2 - B_1)}{32}. \tag{2.47}$$

and

$$a_3 = \frac{2(3p_2 + 2q_2)B_1 + (3p_1^2 + 2q_1^2)(B_2 - B_1)}{48}. \tag{2.48}$$

Now the desired estimates on a_2 and a_3, follow from (2.47) and (2.48) respectively. □

Remark 2.11. *If $f \in \mathcal{K}(\beta)$ and $F \in \mathcal{R}(\beta)$, then from Theorem 2.10 we see that*

$$|a_2| \leq \sqrt{3(1-\beta)}/2 \quad \text{and} \quad |a_3| \leq 5(1-\beta)/6.$$

In particular if $f \in \mathcal{K}$ and $F \in \mathcal{R}$, then $|a_2| \leq \sqrt{3}/2 \approx 0.867$ and $|a_3| \leq 5/6 \approx 0.833$.

Theorem 2.12. *Let $f \in \sigma$ be given by (1.1). If $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{R}(\varphi)$, then*

$$|a_2| \leq \frac{\sqrt{5[B_1 + |B_2 - B_1|]}}{3}, \quad \text{and} \quad |a_3| \leq \frac{7[B_1 + |B_2 - B_1|]}{9}.$$

Proof. Since $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{R}(\varphi)$, there exist two analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$\frac{zf'(z)}{f(z)} = \varphi(r(z)) \text{ and } F'(w) = \varphi(s(w)). \tag{2.49}$$

Let the functions p and q be defined as in (2.4). Then

$$\frac{zf'(z)}{f(z)} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad F'(w) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.50}$$

It follow from (2.50), (2.9) and (2.10) that

$$a_2 = \frac{1}{2}B_1p_1,$$

$$2a_3 - a_2^2 = \frac{1}{2}B_1\Big(p_2 - \frac{1}{2}p_1^2\Big) + \frac{1}{4}B_2p_1^2, \tag{2.51}$$

$$-2a_2 = \frac{1}{2}B_1q_1,$$

$$3(2a_2^2 - a_3) = \frac{1}{2}B_1\Big(q_2 - \frac{1}{2}q_1^2\Big) + \frac{1}{4}B_2q_1^2. \tag{2.52}$$

A computation using (2.51) and (2.52) leads to

$$a_2^2 = \frac{2(3p_2 + 2q_2)B_1 + (3p_1^2 + 2q_1^2)(B_2 - B_1)}{36} \tag{2.53}$$

and

$$a_3 = \frac{2(6p_2 + q_2)B_1 + (6p_1^2 + q_1^2)(B_2 - B_1)}{36}. \tag{2.54}$$

Now the bounds for a_2 and a_3 are obtained from (2.53) and (2.54) respectively using the fact that $|p_i| \leq 2$ and $|q_i| \leq 2$. $\square$

Remark 2.13. *If $f \in \mathcal{S}^*(\beta)$ and $F \in \mathcal{R}(\beta)$, then from Theorem 2.12 it is easy to see that*

$$|a_2| \leq \sqrt{10(1-\beta)}/3 \quad \text{and} \quad |a_3| \leq 14(1-\beta)/9.$$

In particular if $f \in \mathcal{S}^*$ and $F \in \mathcal{R}$, then $|a_2| \leq \sqrt{10}/3 \approx 1.054$ and $|a_3| \leq 14/9 \approx 1.56$.

Theorem 2.14. *Let $f \in \sigma$ given by (1.1). If $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{K}(\varphi)$, then*

$$|a_2| \leq \sqrt{\frac{B_1 + |B_2 - B_1|}{2}}$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{2}.$$

Proof. Assuming $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{K}(\varphi)$ and proceeding in the similar way as in the proof of Theorem 2.10, it is easy to see that

$$a_2 = \frac{1}{2}B_1 p_1,$$

$$3a_3 - a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2, \tag{2.55}$$

$$-2a_2 = \frac{1}{2}B_1 q_1,$$

$$8a_2^2 - 6a_3 = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2. \tag{2.56}$$

A computation using (2.55) and (2.56) leads to

$$a_2^2 = \frac{2(2p_2 + q_2)B_1 + (2p_1^2 + q_1^2)(B_2 - B_1)}{24} \tag{2.57}$$

and

$$a_3 = \frac{2(8p_2 + q_2)B_1 + (8p_1^2 + q_1^2)(B_2 - B_1)}{72}. \tag{2.58}$$

Now using the result $|p_i| \leq 2$ and $|q_i| \leq 2$, the estimates on a_2 and a_3 follow from (2.57) and (2.58) respectively. $\square$

Remark 2.15. *Let $f \in \mathcal{S}^*(\beta)$ and $F \in \mathcal{K}(\beta)$, $0 \leq \beta < 1$. Then from Theorem 2.14, it is easy to see that*

$$|a_2| \leq \sqrt{1-\beta} \ \text{ and } \ |a_3| \leq 1-\beta.$$

In particular if $f \in \mathcal{S}^*$ and $F \in \mathcal{K}$, then $|a_2| \leq 1$ and $|a_3| \leq 1$.

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, Cofficient estimates for bi-univalent function Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, **25** (2012), 344-351.

[2] A. Brannan and J. G. Clunie, *Aspects of contemporary complex analysis Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham*, July 120, 1979, Academic Press New York, London, 1980.

[3] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in Mathematical analysis and its applications (Kuwait, 1985), 53-60,*KFAS Proc. Ser.*, 3 Pergamon, Oxford.

[4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babeş-Bolyai Math.*, **31** no. 2 (1986), 70-77.

[5] D. Bshouty, W. Hengartner and G. Schober, Estimates for the Koebe constant and the second coefficient for some classes of univalent functions, *Canad. J. Math.*, **32** no. 6 (1980), 1311-1324.

[6] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, *Novi Sad J. Math.*, **43** no. 2 (2013), 59-65.

[7] M. Çağlar, H. Orhan and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat*, **27** (2013), 1165-1171.

[8] O. Crişan, Coefficient estimates for certain subclasses of bi-univalent functions, *Gen. Math. Notes*, **16** (2013), 93-102.

[9] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Class. Anal.*, **2** (2013), 49-60.

[10] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.*, **24** no. 9 (2011), 156–1573.

[11] G. Singh, Coefficient estimates for bi-univalent functions with respect to symmetric points, *J. Nonlinear Funct. Anal.*, **2013** (2013), 1-9.

[12] A. W. Goodman, An invitation to the study of univalent and multivalent functions,*Internat. J. Math. Math. Sci.*, **2** no. 2 (1979), 163-186.

[13] S. P. Goyal and P. Goswami, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, *J. Egyptian Math. Soc.*, **20** no. 3 (2012), 179-182.

[14] T. Hayami and S. Owa, *Coefficient bounds for bi-univalent functions, Panamer. Math. J.*, **22** no. 4 (2012), 15-26.

[15] J. M. Jahangiri and S. G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, *Int. J. Math. Math. Sci.*, **2013** (2013), Art. ID 190560, 4 pp.

[16] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, **23** (1970/1971), 159-177.

[17] A. W. Kedzierawski, Some remarks on bi-univalent functions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **39** (1985), 77-81 (1988).

[18] A. W. Kedzierawski and J. Waniurski, *Bi-univalent polynomials of small degree, Complex Variables Theory Appl.*, **10** no. 2-3 (1988), 97-100.

[19] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, **20** (1969), 8-12.

[20] L. S. Keong, V. Ravichandran and S. Supramaniam, *Initial coefficients of bi-univalent functions*, preprint.

[21] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18** (1967), 63–68.

[22] X.-F. Li and A.-P. Wang, Two new subclasses of bi-univalent functions, *Int. Math. Forum*, **7** no. 29-32 (2012), 1495-1504.

[23] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157-169, Conf. Proc. Lecture Notes Anal., I Int. Press, Cambridge, MA.

[24] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.*, **32** (1969), 100-112.

[25] Z. Peng and Q. Han, On the Coefficients of Several Classes of Bi-Univalent Functions, *Acta Math. Sci. Ser. B Engl. Ed.*, **34** no. 1 (2014),, 228-240.

[26] S. Porwal and M. Darus, On a new subclass of bi-univalent functions, *J. Egyptian Math. Soc.*, **21** no. 3 (2013), 190-193.

[27] V. Ravichandran, Y. Polatoglu, M. Bolcal and A. Sen, Certain subclasses of starlike and convex functions of complex order, *Hacet. J. Math. Stat.*, **34** (2005), 9-15.

[28] M. I. S. Robertson, On the theory of univalent functions, *Ann. of Math.*, (2) **37** no. 2 (1936), 374-408.

[29] Y. J. Sim and O. S. Kwon, Notes on analytic functions with a bounded positive real part, *J. Inequal. Appl.*, **2013**, 2013:370, 9 pp.

[30] H. V. Smith, Bi-univalent polynomials, *Simon Stevin*, **50** no. 2 (1976/77), 115-122.

[31] H. V. Smith, Some results/open questions in the theory of bi-univalent functions, *J. Inst. Math. Comput. Sci. Math. Ser.*, **7** no. 3 (1994), 185-195.

[32] H. M. Srivastava, Some inequalities and other results associated with certain subclasses of univalent and bi-univalent analytic functions, in *Nonlinear analysis*, 607-630, Springer Optim. Appl., 68, Springer, New York

[33] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat*, **27** (2013), 831-842.

[34] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, *Global J. Math. Anal.*, **1** (2) (2013), 67-73.

[35] H. M. Srivastava, G. Murugusundaramoorthy and K. Vijaya, Coefficient estimates for some families of bi-Bazilevič functions of the Ma-Minda type involving the Hohlov operator, *J. Class. Anal.*, **2** (2013), 167-181.

[36] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** no. 10 (2010), 1188-1192.

[37] D. Styer and D. J. Wright, Results on bi-univalent functions, *Proc. Amer. Math. Soc.*, **82** no. 2 (1981), 243-248.

[38] T. J. Suffridge, A coefficient problem for a class of univalent functions, *Michigan Math. J.*, **16** (1969), 33-42.

[39] D. L. Tan, Coefficient estimates for bi-univalent functions, *Chinese Ann. Math. Ser. A*, **5** no. 5 (1984), 559-568.

[40] H. Tang, G.-T. Deng and S.-H. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, *J. Inequal. Appl.*, **2013** (2013), Article ID 317, 10 pp.

[41] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.*, **25** (2012), 990-994.

[42] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.*, **218** (2012), 11461-11465.