Tamsui Oxford Journal of Information and Mathematical Sciences **29(4)** (2013) 457-476
Aletheia University

On Some Classes of Weakly Projective Symmetric Manifolds *

Prajjwal Pal†

Chakdaha Co-operative Colony Vidyayatan(H.S),

P.O.- Chakdaha, Dist- Nadia, West Bengal, India

Received February 4, 2014, Accepted May 13, 2014.

Abstract

The object of the present paper is to study some classes of weakly projective symmetric manifolds $(WPS)_n$. At first some properties of the 1-forms of $(WPS)_n(n > 2)$ have been studied. Next we consider conformally flat $(WPS)_n(n > 2)$. Among others we obtain that, in a $(WPS)_n$ the integral curves of the vector ρ_3 defined by (1.3), are geodesics under certain condition. Next we consider $(WPS)_4$ perfect fluid spacetime. Finally, we give an example of a $(WPS)_n$.

Keywords and Phrases: *Weakly symmetric manifolds, Weakly projective symmetric manifolds.*

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [3], who, in particular, obtained a classification of those spaces. Let $(M^n, g), (n = dimM)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g, and let ∇ be the Levi-Civita connection of

---

*2000 *Mathematics Subject Classification.* Primary 53C25.

†E-mail: prajjwalpal@yahoo.in

(M^n, g). A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [21] . The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to different extent such as conformally symmetric manifolds by Chaki and Gupta [5], recurrent manifolds introduced by Walker [30], conformally recurrent manifolds by Adati and Miyazawa [1] , pseudo symmetric manifolds by Chaki [6], weakly symmetric manifolds by Tamassy and Binh [28] etc.

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric [28] if the curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned} &(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ &+C(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + E(V)R(Y, Z, U, X), \end{aligned} \tag{1.1}$$

where $R(Y, Z, U, V) = g(\mathcal{R}(Y, Z)U, V)$, $\mathcal{R}$ is the curvature tensor of type (1,3) and A, B, C, D and E are 1-forms respectively which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. It was proved in [9] that the 1-forms must be related as follows

$B = C$ and $D = E$.

That is, the weakly symmetric manifold is characterized by the condition

$$\begin{aligned} &(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ &+B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + D(V)R(Y, Z, U, X). \end{aligned} \tag{1.2}$$

The 1-forms A, B and D are called the associated 1-forms. If in (1.2) the 1-form A is replaced by $2A$; B and D are replaced by A, then the manifold (M^n, g) reduces to a pseudo symmetric manifold in the sense of Chaki [6].

Again if $A = B = D = 0$, the manifold defined by (1.2) reduces to a symmetric manifold in the sense of Cartan. The existence of a $(WS)_n$ was proved by Prvanović [24] and a concrete example is given by De and Bandyopadhyay ([9],[10]).

Weakly symmetric manifolds have been studied by several authors ([2], [7], [8], [11], [12], [13], [14], [16], [17], [18], [22], [23]) and many others.

Let ρ_1, ρ_2 and ρ_3 are the basic vectors corresponding to the 1-forms A, B and D respectively, that is

$$g(X,\rho_1) = A(X), \;\; g(X,\rho_2) = B(X)$$
$$and \;\; g(X,\rho_3) = D(X) \;\;\; and \;\; g(X,\rho_4) = E(X). \tag{1.3}$$

In 1993 Tamássy and Binh [29] introduced the notion of weakly Ricci symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)(Y,X), \tag{1.4}$$

where A, B, C are three non-zero 1-forms, and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such an n-dimensional manifold is denoted by $(WRS)_n$.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be an n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [26]

$$\mathcal{P}(Y,Z)U = \mathcal{R}(Y,Z)U - \frac{1}{n-1}[S(Z,U)Y - S(Y,U)Z], \tag{1.5}$$

for all $Y, \;\; Z, \;\; U \in T(M)$, where $\mathcal{R}$ is the curvature tensor and S is the Ricci tensor. In fact M is projectively flat if and only if it is a constant curvature [32]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Now (1.5) can be expressed as

$$P(Y,Z,U,V) = R(Y,Z,U,V) - \frac{1}{n-1}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V)], \tag{1.6}$$

where $P(Y,Z,U,V) = g(\mathcal{P}(Y,Z)U,V)$. Since the projective curvature tensor does not satisfy all the properties of Riemannian curvature tensor, therefore weakly projective symmetric manifold is characterized by the condition

$$(\nabla_X P)(Y,Z,U,V) = A(X)P(Y,Z,U,V) + B(Y)P(X,Z,U,V)$$
$$+C(Z)P(Y,X,U,V) + D(U)P(Y,Z,X,V) + E(V)P(Y,Z,U,X), \tag{1.7}$$

where the 1-forms $A,\ \ B,\ \ C,\ \ D\ \ and\ \ E$ are not zero simultaneously. Such a manifold is denoted by $(WPS)_n$. In a recent paper Shaikh and Hui [25] proved that in a $(WPS)_n$, $B = C$. Hence (1.7) can be expressed as

$$(\nabla_X P)(Y,Z,U,V) = A(X)P(Y,Z,U,V) + B(Y)P(X,Z,U,V) \\ +B(Z)P(Y,X,U,V) + D(U)P(Y,Z,X,V) + E(V)P(Y,Z,U,X). \quad (1.8)$$

where $P(Y,Z,U,V) = g(\mathcal{P}(Y,Z)U,V)$.

Recently, Mantica and Molinari [16] have studied weakly-Z-symmetric manifolds. On the otherhand, Mantica and Suh ([17], [19]) have studied pseudo-Z-symmetric Riemannian manifolds with harmonic curvature tensors, pseudo-Q-symmetric Riemannian manifolds. Moreover Mantica and Suh investigated deeply pseudo-Z-symmetric spacetimes [20]. Motivated by the above studies in the present paper we have studied a type of non-flat Riemannian manifold defined by (1.7) and (1.8).

The paper is organized as follows:

After preliminaries, in Section 3, some properties of the 1-forms of a $(WPS)_n$ have been studied. In Section 4, we study conformally flat $(WPS)_n$. Section 5 deals with the property of a $(WPS)_n$, $D = E$ and $D(\rho_3) \neq 0$, with ρ_3 as a unit torse-forming vector field. Section 6 is devoted to the study of a $(WPS)_4$ perfect fluid spacetime. Finally, we give an example of a $(WPS)_n$.

2. Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively and L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S , that is ,

$$g(LX,Y) = S(X,Y). \quad (2.1)$$

In this section, some formulas are derived, which will be useful to the study of $(WPS)_n$. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$.

From (1.5) we can easily verify that the tensor $\mathcal{P}$ satisfies the following properties:

$$i)\mathcal{P}(Y,Z)U = -\mathcal{P}(Z,Y)U,$$

$$ii)\mathcal{P}(Y,Z)U+\mathcal{P}(Z,U)Y+\mathcal{P}(U,Y)Z=0. \tag{2.2}$$

Also from (1.6) we have

$$\Sigma_{i=1}^{n}P(Y,Z,e_i,e_i)=0=\Sigma_{i=1}^{n}P(e_i,e_i,U,V)=\Sigma_{i=1}^{n}P(e_i,Z,U,e_i) \tag{2.3}$$

and

$$\begin{aligned}&\Sigma_{i=1}^{n}P(Y,e_i,e_i,V)\\ =&\frac{n}{n-1}[S(Y,V)-\frac{r}{n}g(Y,V)],\end{aligned} \tag{2.4}$$

where $r=\Sigma_{i=1}^{n}S(e_i,e_i)$ is the scalar curvature.

From (1.6) it follows that

$$\begin{aligned}&(i)\;\; P(Y,Z,U,V)=-P(Z,Y,U,V),\\ &(ii)\;\; P(Y,Z,U,V)\neq -P(Y,Z,V,U),\\ &(iii)\;\; P(Y,Z,U,V)\neq P(U,V,Y,Z),\\ &(iv)\;\; P(X,Y,Z,U)+P(Y,Z,X,U)+P(Z,X,Y,U)=0.\end{aligned} \tag{2.5}$$

3. Some properties of the 1-forms of a $(WPS)_n(n>2)$

Let (M^n,g) be a $(WPS)_n$. Using (1.6) in (1.8) we get

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V)&-\frac{1}{n-1}[(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)]\\ &=A(X)[R(Y,Z,U,V)-\frac{1}{n-1}\{S(Z,U)g(Y,V)-S(Y,U)g(Z,V)\}]\\ &+B(Y)[R(X,Z,U,V)-\frac{1}{n-1}\{S(Z,U)g(X,V)-S(X,U)g(Z,V)\}]\\ &+B(Z)[R(Y,X,U,V)-\frac{1}{n-1}\{S(X,U)g(Y,V)-S(Y,U)g(X,V)\}]\\ &+D(U)[R(Y,Z,X,V)-\frac{1}{n-1}\{S(Z,X)g(Y,V)-S(Y,X)g(Z,V)\}]\\ &+E(V)[R(Y,Z,U,X)-\frac{1}{n-1}\{S(Z,U)g(Y,X)-S(Y,U)g(Z,X)\}].\end{aligned} \tag{3.1}$$

Contracting (3.1) over Y and V we get

$$\begin{aligned}(\nabla_X S)(Z,U) - \frac{1}{n-1}[(n-1)(\nabla_X S)(Z,U)] \\ = A(X)[S(Z,U) - S(Z,U)] + B(R(X,Z)U) \\ -\frac{1}{n-1}\{B(X)S(Z,U) - B(Z)S(X,U)\} \\ +B(Z)[S(X,U) - S(X,U)] + D(U)[S(X,Z) - S(X,Z)] \\ -E(R(U,X)Z) - \frac{1}{n-1}\{E(X)S(Z,U) - E(LU)g(Z,X)\}.\end{aligned}$$

or,

$$\begin{aligned}B(R(X,Z)U) - \frac{1}{n-1}\{B(X)S(Z,U) - B(Z)S(X,U)\} \\ -E(R(U,X)Z) - \frac{1}{n-1}\{E(X)S(Z,U) - E(LU)g(X,Z)\} = 0.\end{aligned} \tag{3.2}$$

Contracting (3.2) over X and U we get

$$B(LZ) = \frac{r}{n}B(Z). \tag{3.3}$$

Replaceing Z by X in (3.3) we get

$$B(LX) = \frac{r}{n}B(X), \tag{3.4}$$

which can be written as

$$S(X,\rho_2) = \frac{r}{n}g(X,\rho_2). \tag{3.5}$$

Hence we have the following theorem:

Theorem 3.1. *In a $(WPS)_n$, $\frac{r}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigenvector ρ_2 by (1.3).*

In a recent paper Shaikh and Hui [25] proved that

$$(B+E)(LX) = \frac{r}{n}(B+E)(X). \tag{3.6}$$

From (3.4) and (3.6) we can conclude that

$$E(LX) = \frac{r}{n}E(X), \tag{3.7}$$

which can be written as

$$S(X, \rho_4) = \frac{r}{n} g(X, \rho_4). \tag{3.8}$$

Hence we have the following theorem:

Theorem 3.2. *In a $(WPS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corre-spnding to the eigenvector ρ_4 defined by (1.3).*

Now contracting (3.1) over U and V we get

$$\begin{aligned} D(R(Y,Z)X) - \frac{1}{n-1}\{D(Y)S(X,Z) - D(Z)S(X,Y)\} \\ -E(R(Y,Z)X) - \frac{1}{n-1}\{E(LZ)g(X,Y) - E(LY)g(X,Z)\} = 0. \end{aligned} \tag{3.9}$$

Again contracting (3.9) over X and Z we get

$$D(LY) = \frac{r}{n} D(Y). \tag{3.10}$$

Replacing Y by X from (3.10) we get

$$D(LX) = \frac{r}{n} D(X). \tag{3.11}$$

or,

$$S(X, \rho_3) = \frac{r}{n} g(X, \rho_3). \tag{3.12}$$

Thus we have the following theorem:

Theorem 3.3. *In a $(WPS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corre-sponding to the eigenvector ρ_3 defined by (1.3).*

If possible we assume that in a $(WPS)_n$ $D = E$. Then from (3.9) we get

$$\begin{aligned} -D(Y)S(X,Z) + D(Z)S(X,Y) \\ -D(LZ)g(X,Y) + D(LY)g(X,Z) = 0. \end{aligned} \tag{3.13}$$

Now using (3.11) in (3.13) we get

$$\begin{aligned} D(Y)S(X,Z) - D(Z)S(X,Y) \\ +\frac{r}{n}D(Z)g(X,Y) - \frac{r}{n}D(Y)g(X,Z) = 0. \end{aligned}$$

or,

$$D(Y)[S(X,Z)-\frac{r}{n}g(X,Z)]$$
$$-D(Z)[S(X,Y)-\frac{r}{n}g(X,Y)]=0. \tag{3.14}$$

Putting $Y=\rho_3$ in (3.14) we get

$$D(\rho_3)[S(X,Z)-\frac{r}{n}g(X,Z)]$$
$$-D(Z)D(LX)+\frac{r}{n}D(X)D(Z)=0. \tag{3.15}$$

Again using (3.11) in (3.15) we get

$$D(\rho_3)[S(X,Z)-\frac{r}{n}g(X,Z)]=0. \tag{3.16}$$

Now let $D(\rho_3)\neq 0$ then from (3.16) we get

$$S(X,Z)=\frac{r}{n}g(X,Z). \tag{3.17}$$

Hence in this case the $(WPS)_n$ is an Einstein manifold. Thus we can state the following theorem:

Theorem 3.4. *If in a $(WPS)_n$, $D=E$ and $D(\rho_3)\neq 0$, then the manifold reduces to an Einstein manifold.*

Contracting (3.1) over Z and U we get

$$\begin{aligned}
\frac{n}{n-1}(\nabla_X S)(Y,V)-\frac{dr(X)}{n-1}g(Y,V)\\
=A(X)[S(Y,V)-\frac{1}{n-1}\{rg(Y,V)-S(Y,V)\}]\\
+B(Y)[S(X,V)-\frac{1}{n-1}\{rg(X,V)-S(X,V)\}]\\
+B(R(Y,X)V)-\frac{1}{n-1}\{B(LX)g(Y,V)-B(LY)g(X,V)\}\\
+D(R(X,V)Y)-\frac{1}{n-1}D(LX)g(Y,V)+\frac{1}{n-1}D(V)S(X,Y)\\
+E(V)S(X,Y)-\frac{E(V)}{n-1}\{rg(X,Y)-S(X,Y)\}.
\end{aligned}$$

or,

$$\begin{aligned}(\nabla_X S)(Y,V) &= \frac{1}{n}dr(X)g(Y,V) + A(X)S(Y,V)\\ -\frac{r}{n}A(X)g(Y,V) &+ B(Y)S(X,V) - \frac{r}{n}B(Y)g(X,V)\\ -B(R(Y,X)V) &- \frac{1}{n}\{B(LX)g(Y,V) - B(LY)g(X,V)\}\\ +D(R(X,V)Y) &- \frac{1}{n}\{D(LX)g(Y,V) - D(V)S(X,Y)\}\\ &+E(V)S(X,Y) - \frac{r}{n}E(V)g(X,Y).\end{aligned} \tag{3.18}$$

Let in this $(WPS)_n$ r is a non-zero constant and the manifold is $(WRS)_n$ with the same 1-forms A, B and E then from (3.18) we get

$$\begin{aligned}-\frac{r}{n}A(X)g(Y,V) - \frac{r}{n}B(Y)g(X,V) - B(R(Y,X)V)\\ -\frac{1}{n}\{B(LX)g(Y,V) - B(LY)g(X,V)\} + D(R(X,V)Y)\\ -\frac{1}{n}\{D(LX)g(Y,V) - D(V)S(X,Y)\} - \frac{r}{n}E(V)g(X,Y) = 0.\end{aligned} \tag{3.19}$$

Again contracting (3.19) over Y and V we get

$$\begin{aligned}-rA(X) - \frac{r}{n}B(X) + \frac{1}{n}B(LX)\\ +\frac{1}{n}D(LX) - \frac{r}{n}E(X) = 0.\end{aligned} \tag{3.20}$$

Using (3.4) and (3.11) in (3.20) we get

$$n^2A(X) + (n-1)B(X) - D(X) + nE(X) = 0. \tag{3.21}$$

Thus we have the following theorem:

Theorem 3.5. *If a $(WPS)_n$ with non-zero constant scalar curvature is also $(WRS)_n$ with the same 1-forms A, B and E then $n^2A(X) + (n-1)B(X) - D(X) + nE(X) = 0$ holds for all X.*

4. Conformally flat $(WPS)_n (n > 3)$

In this section we assume that the manifold $(WPS)_n$ is conformally flat. Then $divC = 0$ where C denotes the Weyl's conformal curvature tensor and $'div'$ denotes divergence. Hence we have [15]

$$\begin{aligned} &(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) \\ &= \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \end{aligned} \tag{4.1}$$

Now replacing V by Z in (3.18) we get

$$\begin{aligned} (\nabla_X S)(Y,Z) &= \frac{1}{n}dr(X)g(Y,Z) \\ &+A(X)S(Y,Z) - \frac{r}{n}A(X)g(Y,Z) \\ &+B(Y)S(X,Z) - \frac{r}{n}B(Y)g(X,Z) \\ &-\frac{(n-1)}{n}B(R(X,Y)Z) - \frac{1}{n}\{B(LX)g(Y,Z) \\ &-B(LY)g(X,Z)\} + \frac{(n-1)}{n}D(R(X,Z)Y) \\ &-\frac{1}{n}\{D(LX)g(Y,Z) - D(Z)S(X,Y)\} \\ &+E(Z)S(X,Y) - \frac{r}{n}E(Z)g(X,Y). \end{aligned} \tag{4.2}$$

Contracting (4.2) over X and Z we get

$$\begin{aligned} (\frac{n-2}{2n})dr(Y) &= A(LY) \\ -\frac{r}{n}A(Y) + E(LY) &- \frac{r}{n}E(Y). \end{aligned} \tag{4.3}$$

Replacing Y by X in (4.3) we get

$$(A+E)(LX) - \frac{r}{n}(A+E)(X) = \frac{(n-2)}{2n}dr(X). \tag{4.4}$$

Again using (4.2) in (4.1) we get

$$
\begin{aligned}
A(X)\{S(Y,Z) - \frac{r}{n}g(Y,Z)\} \\
-A(Z)\{S(X,Y) - \frac{r}{n}g(X,Y)\} \\
-\frac{(n-1)}{n}\{B(R(Y,X)Z) - B(R(Y,Z)X)\} \\
-\frac{1}{n}\{B(LX)g(Y,Z) - B(LZ)g(X,Y)\} \\
+\frac{2(n-1)}{n}D(R(X,Z)Y) - \frac{1}{n}[D(LX)g(Y,Z) \\
-D(Z)S(X,Y) - D(LZ)g(X,Y) \\
+D(X)S(Y,Z)] + E(Z)S(X,Y) \\
-E(X)S(Y,Z) - \frac{r}{n}\{E(Z)g(X,Y) \\
-E(X)g(Y,Z)\} = -\frac{(n-2)}{2n(n-1)}[dr(X)g(Y,Z) - dr(Z)g(X,Y)]. \qquad (4.5)
\end{aligned}
$$

Contracting (4.5) over Y and Z we get

$$(A - D - E)(LX) - \frac{r}{n}(A - D - E)(X) = \frac{n-2}{2n}dr(X). \qquad (4.6)$$

From (4.4) and (4.6) we get

$$(D + 2E)(LX) = \frac{r}{n}(D + 2E)(X). \qquad (4.7)$$

Now (4.7) can be rewritten as

$$S(X, \rho_3 + 2\rho_4) = \frac{r}{n}g(X, \rho_3 + 2\rho_4), \qquad (4.8)$$

where ρ_3 and ρ_4 defined by (1.3). Also (4.7) can be rewritten as

$$S(X, \rho_5) = \frac{r}{n}g(X, \rho_5), \qquad (4.9)$$

where we assume that

$$\rho_5 = \rho_3 + \rho_4. \qquad (4.10)$$

Thus we have the following theorem:

Theorem 4.1. *In a conformally flat $(WPS)_n(n > 3)$, $\frac{r}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector ρ_5 defined by (4.10).*

Let (M^n, g) be a conformally flat $(WPS)_n(n > 3)$. Now using (3.7) and (3.11) in (4.6) we get

$$A(LX) - \frac{r}{n}A(X) = \frac{n-2}{2n}dr(X). \tag{4.11}$$

Again if in the $(WPS)_n$ the scalar curvature r is constant then from (4.11) we get

$$A(LX) = \frac{r}{n}A(X), \tag{4.12}$$

which can be written as

$$S(X, \rho_1) = \frac{r}{n}g(X, \rho_1), \tag{4.13}$$

where ρ_1 is defined by (1.3). Thus we have the following theorem:

Theorem 4.2. *In a conformally flat $(WPS)_n(n > 3)$, if the scalar curvature r is constant then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigen vector ρ_1 defined by (1.3).*

Also for a conformally flat $(WPS)_n(n > 3)$ if (4.12) holds then from (4.11) we can conclude that $r = constant$. Hence we have the following theorem:

Corollary 4.1. *In a conformally flat $(WPS)_n(n > 3)$, the scalar curvature r is constant if and only if (4.12) holds.*

5. The vector field ρ_3 as a torse-forming vector field

In this section we suppose that ρ_3 is a unit torse-forming vector field [31] defined by (1.3) and given by

$$\nabla_X \rho_3 = \lambda X + \omega(X)\rho_3, \tag{5.1}$$

where λ is a non-zero scalar and ω is a non-zero 1-form, called respectively the scalar and 1-form of the vector field ρ_3.

Now if in a $(WPS)_n$ $D = E$ and $D(\rho_3) \neq 0$ then from Theorem 3.4 we can conclude that the manifold is Einstein and so we can use (3.17). Hence from (3.17) we get

$$(\nabla_X S)(Y, \rho_3) = 0. \tag{5.2}$$

But

$$(\nabla_X S)(Y, \rho_3) = \nabla_X S(Y, \rho_3) - S(\nabla_X Y, \rho_3) - S(Y, \nabla_X \rho_3).$$

Therefore, using (3.17) we obtain

$$\frac{r}{n}(\nabla_X D)(Y) + S(Y, \nabla_X \rho_3) = 0. \tag{5.3}$$

By virtue of (5.1) we get from (5.3)

$$\frac{r}{n}(\nabla_X D)(Y) + S(Y, \lambda X + \omega(X)\rho_3) = 0.$$

or,

$$\frac{r}{n}(\nabla_X D)(Y) + \lambda S(Y, X) + \omega(X)S(Y, \rho_3) = 0. \tag{5.4}$$

Using (3.16) in (5.4) we get

$$\frac{r}{n}(\nabla_X D)(Y) + \lambda S(Y, X) + \frac{r}{n}\omega(X)D(Y) = 0. \tag{5.5}$$

Putting $Y = \rho_3$ in (5.5) we get

$$(\nabla_X D)(\rho_3) + \lambda D(X) + \omega(X) = 0, \tag{5.6}$$

since ρ_3 is a unit vector.

But

$$(\nabla_X D)(\rho_3) = D(\nabla_X \rho_3), \tag{5.7}$$

since ρ_3 is a unit vector.

Hence using (5.1) in (5.7) we get

$$(\nabla_X D)(\rho_3) = \lambda D(X) + \omega(X). \tag{5.8}$$

From (5.8) and (5.6) we get

$$\omega(X) = -\lambda D(X). \tag{5.9}$$

or,

$$\lambda = -\omega(\rho_3). \tag{5.10}$$

Hence (5.1) can be rewritten using (5.10) as

$$\nabla_X \rho_3 = -\omega(\rho_3)X + \omega(X)\rho_3.$$

Therefore $\nabla_{\rho_3}\rho_3 = 0$. Thus we have the following:

Theorem 5.1. *If in a* $(WPS)_n$, $D = E$ *and* $D(\rho_3) \neq 0$, *the vector field* ρ_3 *is a unit torse-forming vector field, then the integral curves of the vector* ρ_3 *are geodesics.*

6. Application of $(WPS)_4)$ perfect fluid spacetime with $D = E$ and $D(\rho_3) \neq 0$

A semi-Riemannian four-dimensional manifold (M^4, g) with Lorentzian metric g with signature (-,+,+,+) is called weakly projective symmetric spacetime if its projective curvature tensor satisfies (1.7) and (1.8), where the vector field ρ_3 is related by $g(X, \rho_3) = D(X)$ and also $D = E$ and $D(\rho_3) \neq 0$. In this section we consider $(WPS)_4$ relativistic spacetime that is, a 4-dimensional $(WPS)_4$ Lorentzian manifold as a perfect fluid spacetime with cosmological constant λ in which the associated vector field ρ_3 is the velocity vector field of the fluid.

For a perfect fluid spacetime, we have the Einstein's equation with cosmological constant [27] as

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{6.1}$$

where k is the Einstein's gravitational constant, T is the energy momentum tensor of type (0,2) given by

$$T(X,Y) = (\sigma + p)D(X)D(Y) + pg(X,Y), \tag{6.2}$$

where σ and p as the energy density and isotropic pressure of the fluid respectively and D being given by $g(X, \rho_3) = D(X)$ *for all* X, ρ_3 is the flow vector field of the fluid such that $g(\rho_3, \rho_3) = -1$. Using (6.2) in (6.1) we get

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k[(\sigma + p)D(X)D(Y) + pg(X,Y)]. \tag{6.3}$$

Putting $Y = \rho_3$ in (6.3) and using (3.17) we have

$$\sigma = \frac{r - 4\lambda}{4k}. \tag{6.4}$$

Taking a frame field and contracting (6.3) over X and Y we get

$$-r + 4\lambda = k(3p - \sigma). \tag{6.5}$$

Again if we use (6.4) in (6.5) for $(WPS)_4$ perfect fluid spacetime, we get

$$p = \frac{-r + 4\lambda}{4k}. \tag{6.6}$$

Since a $(WPS)_n$ with $D = E$ and $D(\rho_3) \neq 0$ is an Einstein manifold so for $n > 2$ the scalar curvature r of the manifold will be constant. Then from (6.4) and (6.6) we can conclude that p and σ are constants. Also from (6.4) and (6.6) we get $\sigma + \rho = 0$, which means the fluid behaves as a cosmological constant [27]. This is also termed as phantom barrier [4]. Now in a cosmology we know such a choice $\sigma = -p$ lead to rapid expansion of the spacetime which is now termed as inflation. Thus we have the follwing:

Theorem 6.1. *In a $(WPS)_n$ spacetime with the condition $D = E$ and $D(\rho_3) \neq 0$, the matter distribution is perfect fluid whose velocity vector field is ρ_3 defined by (1.3), then the spacetime represents inflation. In this case the isotropic pressure p and the energy density σ are constant. Also the fluid behaves as a cosmological constant. This is also termed as a phantom barrier.*

7. Example of a $(WPS)_4$

In this section we give an example of $(WPS)_n$, with the non-zero scalar curvature.

Example 7.1. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2] + (dx^3)^2 + x^1(dx^4)^2, \tag{7.1}$$

where $(i, j = 1, 2, 3, 4)$, x^1 and x^2 are non-zero. Here the only non-vanishing components of the Christoffel symbols and the curvature tensors are respectively :

$$\Gamma^2_{11} = -\frac{1}{x^2}, \quad \Gamma^1_{12} = \Gamma^2_{22} = \frac{1}{2x^2}, \quad \Gamma^1_{44} = -\Gamma^4_{14} = -\frac{1}{2x^2},$$

$$R_{1221} = -\frac{1}{2x^2}, \quad R_{1441} = -\frac{1}{4x^1}, \quad R_{1442} = -\frac{1}{4x^2}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are:

$$R_{11} = \frac{1}{2(x^2)^2} - \frac{1}{4(x^1)^2}, \quad R_{22} = -\frac{1}{2(x^2)^2}, \quad R_{44} = R_{12} = -\frac{1}{4x^1x^2},$$

It can be easily shown that the scalar curvature r of this $(\mathbb{R}^4, g)$ is $-\frac{1}{2(x^1)^2x^2} \neq 0$, which is non-vanishing and non-constant. Therefore the non-vanishing components of the projective curvature tensor and their covariant derivatives are respectively:

$$P_{1221} = -\frac{1}{3x^2}, \quad P_{1441} = -\frac{1}{6x^1}, \quad P_{1442} = -\frac{1}{4x^2};$$

$$P_{1221,2} = \frac{1}{3(x^2)^2}, \quad P_{1441,1} = \frac{1}{6(x^1)^2},, \quad P_{1442,2} = \frac{1}{4(x^2)^2}.$$

Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} -\frac{2}{x^1} & \textit{for i=1} \\ -\frac{2}{x^2} & \textit{for i=2} \\ 0 & \textit{otherwise}, \end{cases} \tag{7.2}$$

$$B_i(x) = \begin{cases} \frac{1}{2x^1} & \textit{for i=1} \\ \frac{1}{2x^2} & \textit{for i=2} \\ 0 & \textit{otherwise}, \end{cases} \tag{7.3}$$

$$D_i(x) = \begin{cases} \frac{1}{2x^2} & \textit{for i=2} \\ 0 & \textit{otherwise}, \end{cases} \tag{7.4}$$

$$E_i(x) = \begin{cases} \frac{1}{2x^1} & \textit{for i=1} \\ \frac{1}{x^2} & \textit{for i=2} \\ 0 & \textit{otherwise}, \end{cases} \tag{7.5}$$

at any point $x \in \mathbb{R}^4$. *Now the equation (1.8) reduces to the equations*

$$P_{1221,2} = A_2 P_{1221} + B_1 P_{2221} + B_2 P_{1221} + D_2 P_{1221} + E_1 P_{1222}, \tag{7.6}$$

$$P_{1441,1} = A_1 P_{1441} + B_1 P_{1441} + B_4 P_{1141} + D_4 P_{1411} + E_1 P_{1441}, \tag{7.7}$$

$$P_{1442,2} = A_2 P_{1442} + B_1 P_{2442} + B_4 P_{1242} + D_4 P_{1422} + E_2 P_{1442}, \tag{7.8}$$

since, for the other cases (1.8) holds trivially.

By (7.2), (7.3), (7.4) and (7.5) we get the following relation for the right hand side(R.H.S.) and the left hand side(L.H.S.) of (7.6)

$$\begin{aligned} \text{R.H.S. of (7.6)} &= A_2 P_{1221} + B_1 P_{2221} + B_2 P_{1221} + D_2 P_{1221} + E_1 P_{1222} \\ &= [A_2 + B_2 + D_2] P_{1221} \\ &= \{-\frac{1}{x^2}\}\{-\frac{1}{3x^2}\} \\ &= \frac{1}{3(x^2)^2} \\ &= P_{1221,2} \\ &= \text{L.H.S. of (7.6)}. \end{aligned}$$

By similar argument it can be shown that (7.7) and (7.8) are true. So, $\mathbb{R}^4$ *is a* $(WPS)_n$ *whose scalar curvature is non-zero and non-constant and the manifold* $(\mathbb{R}^4, g)$ *is neither projectively flat nor projectively symmetric.*

Acknowledgement. The author is thankful to the referees for their valuable suggestions towards the improvement of this paper.

References

[1] Adati, T. and Miyazawa, T., On a Riemannian space with recurrent conformal curvature, *Tensor(N.S.)*, **18** (1967), 348-354.

[2] Binh, T. Q., On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen*, **42** (1993), 103-107.

[3] Cartan, E., Sur une classes remarquable d'espaces de Riemannian, *Bull. Soc. Math. France*, **54** (1926), 214-264.

[4] Chakraborty, S., Mazumder, N. , and Biswas, R. , Cosmological Evolution Acroos Phantom Crossing and the Nature of the Horizon, *Astrophys. Space Sci.*, **334** nl (2011), 183-186.

[5] Chaki, M. C. and Gupta, B., On conformally symmetric spaces, *Indian J. Math.*, **5** (1963), 113-295.

[6] Chaki, M. C., On pseudo symmetric manifolds, *Ann. St. Univ. "Al I Cuza" Iasi*, **33** (1987), 53-58.

[7] De, U. C., On weakly symmetric structures on Riemannian manifolds, *Facta Univ. Ser. Mech. Automat. Control Robot.*, **3** no. 14 (2003), 805-819.

[8] De, U. C., On weakly conformally symmetric Ricci-recurrent spaces, *Publ. Math. Debrecen*, **62** (2003), 379-386.

[9] De, U. C. and Bandyopadhyay, S., On weakly symmetric spaces, *Publ. Math. Debrecen*, **54** (1999), 377-381.

[10] De, U. C. and Bandyopadhyay, S., On weakly symmetric spaces, *Acta Math. Hungarica*, **83** (2000), 205-212.

[11] De, U. C. and Bandyopadhyay, S., On weakly conformally symmetric spaces, *Publ. Math. Debrecen*, **57** (2000), 71-78.

[12] De, U. C. and Ghosh, S. K., On weakly Ricci symmetric spaces, *Publ. Math. Debrecen*, **60** (2002), 201-208.

[13] De, U. C. and Mallick, S., On weakly symmetric spacetimes, *Kragujevac J. Math.*, **36** (2012), 299-308.

[14] De, U. C. and Sengupta, J., On a weakly symmetric Riemannian manifold admitting a special type of semi-symmetric metric connection, *Novi Sad. J. Math.*, **29** (1999), 89-95.

[15] Eisenhart, L. P., *Riemannian Geometry*, Princeton University Press, 1949.

[16] Mantica, C. A. and Molinari, L. G., Weakly Z-Symmetric manifolds, *Acta Math. Hungarica*, **135** no. 1-2 (2012), 80-96.

[17] Mantica, C. A. and Y. J. Suh, Pseudo-Z-symmetric Riemannian manifolds with harmonic curvature tensors, *Int. J. Geom. Meth. Mod. Phys.*, **9** no. 1 (2012), 1250004(21 pages).

[18] Mantica, C. A. and Y. J. Suh, Recurrent Z forms on Riemannian and Kaehler manifolds, *Int. J. Geom. Meth. Mod. Phys.*, **9** no. 7 (2012), 1250059(26 pages).

[19] Mantica, C. A. and Y. J. Suh, Pseudo-Q-symmetric Riemannian manifolds, *Int. J. Geom. Meth. Mod. Phys.*, **10**(5) (2013), 1350013(25 pages).

[20] Mantica, C. A. and Suh, Y. J., Pseudo-Z-symmetric space-times, *Journal of Math Phys*, **55** n4, (2014), 042502(12 pages).

[21] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York-London, 1983.

[22] Ozen, F. and Altay, S., Weakly and pseudo-symmetric Riemannian spaces, *Indian J. Pure appl. Math.*, **33** no. 10 (2002), 1477-1488.

[23] Ozen, F. and Altay, S., On weakly and pseudo-concircular symmetric structures on a Riemannian manifold, *Acta Univ. Palack. Olomuc., Fac. Rerum. natur. Math.*, **47** (2008), 129-138.

[24] Prvanović, M., On weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen*, **46** (1995), 19-25.

[25] Shaikh, A. A. and Hui, S. K., On weakly projective symmetric manifolds, *Acta Math. Acad. Paeda. Nyiregy.*, **25** (2009), 247-269.

[26] Sooś, G., *Ü*ber die geodätischen Abbildungen von Riemannaschen Räumen auf projektiv symmetrische Riemannsche Räume, *Acta. Math. Acad. Sci. Hungar. Tom*, **9** (1958), 359-361.

[27] Stephani, H. , Kramer, D., MacCallum, M. , Hoenselaers, C. , and Hertl, E. , *Exact Solutions of Einstein's Field equations*, 2^{nd}*ed.*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2003).

[28] Tamássy, L. and Binh, T. Q., On weakly symmetric and weakly projectively symmetric Riemannian manifolds, *Colloq. Math. Soc. Janos Bolyai*, **56** (1989), 663-670.

[29] Tamássy, L. and Binh, T. Q., On weak symmetries of Einstein and Sasakian manifolds, *Tensor, (N. S.)*, **53**(1993), 140-148.

[30] Walker, A. G., On Ruse's space of recurrent curvature, *Proc. of London Math. Soc.*, **52** (1950), 36-54.

[31] Yano, K., On the torse-forming directions in Riemannian spaces, *Proc. Imp. Acad. Tokyo*, **20** (1944), 340-345.

[32] Yano, K. and Bochner, S., *Curvature and Betti numbers*, Annals of mathematics studies, **32**(Princeton university press)(1953).