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On LP-Sasakian Manifolds with a coefficient α Satisfying Certain Curvature Conditions *

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Abstract

The object of the present paper is to classify an LP-Sasakian manifold with a coefficient α satisfying certain curvature conditions. Also ξ-concircularly flat and ϕ-concircularly flat LP-Sasakian manifold with a coefficient α have been studied. Finally, we have shown the applications of such a manifold in general relativity and cosmology.

Keywords and Phrases: *LP-Sasakian manifold with a coefficient α, Einstein manifold, η-Einstein manifold, Concircular curvature tensor, ξ-concircularly flat, ϕ-concircularly flat.*

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1. Introduction

In 1989, Matsumoto [11] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [8] introduced the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have been studied by several authors ([1],[20],[12]). In a recent paper De, Shaikh and Sengupta [17]introduced the notion of LP-Sasakian manifolds with a coefficient α which generalizes the notion of LP-Sasakian manifolds. Lorentzian para-Sasakian manifold with a coefficient α have been studied by De et al ([19],[18]). In [17] it is shown that if a Lorentzian manifold admits a unit torse-forming vector field, then the manifold becomes an LP-Sasakian manifold with a coefficient α where α is a non-zero smooth function. Recently, T.Ikawa and his coauthors ([15],[16]) studied Sasakian manifolds with Lorentzian metric and obtained several results in this manifold. Motivated by the above studies we like to generalize LP-Sasakian manifold which is called an LP-Sasakian manifold with a coefficient α.

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\tilde{g}_{ij} = \psi^2 g_{ij}, \tag{1.1}$$

of the fundamental tensor g_{ij}, where ψ is a smooth function on the manifold. The transformation which preserves geodesic circles was first introduced by Yano [13].The conformal transformation satisfying the partial differential equation

$$\psi_{;i;j} = \phi g_{ij}, \tag{1.2}$$

changes a geodesic circle into a geodesic circle, where ϕ is a smooth function on the manifold. Such a transformation is known as the concircular transformation and the geometry which deals with such transformation is called the concircular geometry [13].

Let (M, g) be an n-dimensional Riemannian manifold. Then the concircular curvature tensor $\tilde{C}$ and the Weyl conformal curvature tensor C are defined by [14]

$$\tilde{C}(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}(g(Y,U)X - g(X,U)Y), \tag{1.3}$$

$$\begin{aligned} C(X,Y)U &= R(X,Y)U - \frac{1}{n-2}\{S(Y,U)X - S(X,U)Y + g(Y,U)QX \\ &- g(X,U)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,U)X - g(X,U)Y\} \end{aligned} \tag{1.4}$$

for all X, Y, $U \in TM$ respectively, where r is the scalar curvature of M and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S.

The importance of concircular transformation and concircular curvature tensor is very well known in the differential geometry of certain F-structure such as complex, almost complex, Kahler, almost Kahler, contact and almost contact structure etc. ([4],[21],[14]). In a recent paper Z.Ahsan and S.A.Siddiqui [22] studied the application of concircular curvature tensor in fluid space time.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). At each point $p \in M$, decompose the tangent space T_pM into direct sum $T_pM = \phi(T_pM) \oplus L(\xi_p)$, where $L(\xi_p)$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$. Thus the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus L(\xi_p), \qquad p \in M. \tag{1.5}$$

It may be natural to consider the following particular cases:
(1) $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow L(\xi_p)$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero.
(2) $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \phi(T_p(M))$, that is, the projection of the image of C in $L(\xi_p)$ is zero. This condition is equivalent to

$$C(X,Y)\xi = 0, \qquad \text{for all } X, Y, \in T_p(M). \tag{1.6}$$

(3) $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \longrightarrow L(\xi_p)$, that is, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0, \qquad \text{for all } X, Y, Z \in T_p(M). \tag{1.7}$$

An almost contact metric manifold satisfying (1.6) and (1.7) are called ξ-conformally flat and ϕ-conformally flat respectively. An almost contact metric manifold satisfying the cases (1), (2) and (3) are considered in [5], [6] and [9] respectively. Furthermore in [10] and [3] the case (3) was considered in a (k, μ)- contact metric manifold and an LP-Sasakian manifold respectively. In an analogous way we define the following:

Definition 1.1. *An n-dimensional LP-Sasakian manifold with a coefficient α is said to be ξ-concircularly flat if*

$$\tilde{C}(X,Y)\xi = 0, \qquad \text{for any } X, Y \in TM. \tag{1.8}$$

Definition 1.2. *An n-dimensional LP-Sasakian manifold with a coefficient α is said to be ϕ-concircularly flat if*

$$g(\tilde{C}(\phi X, \phi Y)\phi Z, \phi W) = 0, \qquad \text{for any } X, Y, Z \in TM. \tag{1.9}$$

In the coordinate free method of differential geometry the spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-,+,+,+)$. The geometry of Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity. A non-zero vector $v \,\varepsilon\, T_pM$ is said to be timelike (resp; non-spacelike, null, spacelike) if it satisfies $g(v, v) < 0$ (resp; $\leq 0. = 0, > 0$) [2].
Here we consider a special type of spacetime which is called Lorentzian para-Sasakian type spacetime.

The present paper is organized as follows:
After preliminaries in section 3, we give some examples of LP-Sasakian manifolds with a coefficient α. In section 4, we find necessary and sufficient conditions for LP-Sasakian manifolds with a coefficient α satisfying the curvature conditions like $\tilde{C}(\xi, X) \cdot \tilde{C} = 0$, $\tilde{C}(\xi, X) \cdot S = 0$ and $\tilde{C}(\xi, X) \cdot C = 0$. Next we study ξ-concircularly flat LP-Sasakian manifolds with a coefficient α and prove that an n-dimensional, $n \geq 1$, LP-Sasakian manifold with a coefficient α is ξ-concircularly flat if and only if $r = (\alpha^2 - \sigma)n(n-1)$. Section 6 deals with the study of ϕ-concircularly flat LP-Sasakian manifolds with a coefficient α. Finally, we study Lorentzian para-Sasakian type spacetime.

2. Lorentzian Para-Sasakian Manifolds with a coefficient α

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor g_p:$T_pM \times T_pM \to \mathbb{R}$ is an inner product of signature $(-,+,+,.....,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X), g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y. Then such a structure (ϕ,ξ,η,g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ,ξ,η,g) is called Lorentzian almost paracontact manifold [11]. In the Lorentzian almost paracontact manifold M^n, the following relations hold [11] :

$$\phi\xi = 0, \eta(\phi X) = 0, \tag{2.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{2.4}$$

where $\Omega(X,Y) = g(X,\phi Y)$.
In the Lorentzian almost paracontact manifold M^n, if the relations

$$\begin{aligned}(\nabla_Z\Omega)(X,Y) &= \alpha[(g(X,Z) + \eta(X)\eta(Z))\,\eta(Y) \\ &\quad +(g(Y,Z) + \eta(Y)\eta(Z))\eta(X)],\end{aligned} \tag{2.5}$$

$$\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X\eta)(Y), \tag{2.6}$$

hold where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function, then M^n is called an LP-Sasakian manifold with a coefficient α [17]. An LP-Sasakian manifold with a coefficient 1 is an LP-Sasakian manifold [11].
If a vector field V satisfies the equation of the following form:

$$\nabla_X V = \alpha X + A(X)V, \tag{2.7}$$

where α is a non-zero scalar function and A is a non-zero 1-form, then V is called a torse-forming vector field [14].

In the Lorentzian manifold M^n, let us assume that ξ is a unit torse-forming vector field. Then we have

$$\nabla_X \xi = \alpha X + A(X)\xi. \tag{2.8}$$

Now $g(\xi, \xi) = -1$, which implies that $g(\nabla_X \xi, \xi) = 0$. Then using the equation (2.8) we get

$$A(X) = \alpha\eta(X). \tag{2.9}$$

Now

$$\begin{aligned} (\nabla_X \eta)(Y) &= \nabla_X \eta(Y) - \eta(\nabla_X Y) \\ &= g(Y, \nabla_X \xi). \end{aligned} \tag{2.10}$$

Using (2.7) and (2.9) in (2.10) yields

$$(\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)]. \tag{2.11}$$

Especially, if η satisfies

$$(\nabla_X \eta)(Y) = \epsilon[g(X, Y) + \eta(X)\eta(Y)], \;\; \epsilon^2 = 1 \tag{2.12}$$

then M^n is called an LSP-Sasakian manifold[11]. In particular, if α satisfies (2.11) and the equation of the following form:

$$\nabla_X \alpha = d\alpha(X) = \sigma\eta(X), \tag{2.13}$$

where σ is a smooth function and η is the 1-form, then ξ is called a concircular vector field.
Let us consider an LP-Sasakian manifold M^n (ϕ, ξ, η, g) with a coefficient α. Then we have the following relations [17]:

$$\eta(R(X, Y)Z) = (\alpha^2 - \sigma)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.14}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \sigma)\eta(X), \tag{2.15}$$

$$R(X, Y)\xi = (\alpha^2 - \sigma)[\eta(Y)X - \eta(X)Y], \tag{2.16}$$

$$R(\xi, Y)X = (\alpha^2 - \sigma)[g(X, Y)\xi - \eta(X)Y], \tag{2.17}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.18}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \sigma)g(X, Y), \tag{2.19}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold (M, g).
An n-dimensional LP-Sasakian manifold is said to be Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and η-Einstein if the Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions on the manifold .

3. Examples of LP-Sasakian manifolds with a coefficient α

We now give some examples of LP-Sasakian manifolds with a coefficient α both in odd and even dimensions.
Example 3.1: We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are standard coordinates of $\mathbb{R}^3$.
The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = e^{-2z}\frac{\partial}{\partial z}$$

are linearly independent at each point of M.
Let g be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1,$$

$$g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all smooth vector fields on M . Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.
Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.
Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g . Then we have

$$[e_1, e_2] = -e^{-z}e_2 \quad , [e_1, e_3] = e^{-2z}e_1 \quad and \quad [e_2, e_3] = e^{-2z}e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula [14] for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1}e_3 = e^{-2z}e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_1 = e^{-2z}e_3,$$

$$\nabla_{e_2}e_3 = e^{-2z}e_2, \quad \nabla_{e_2}e_2 = e^{-2z}e_3 - e^{-z}e_1, \quad \nabla_{e_2}e_1 = e^{-2z}e_2,$$

$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = 0.$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold with $\alpha = e^{-2z} \neq 0$.

Example 3.2: We consider the 4-dimensional manifold $M = \{(x, y, z, w) \in \mathbb{R}^4 \mid w \neq 0\}$, where (x, y, z, w) are standard coordinates of $\mathbb{R}^4$.

The vector fields

$$e_1 = w(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \quad e_2 = w\frac{\partial}{\partial y}, \quad e_3 = w(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}), \quad e_4 = (w)^3\frac{\partial}{\partial w}$$

are linearly independent at each point of M.
Let g be the Lorentzian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, g(e_4, e_4) = -1.$$

$g(e_i, e_j) = 0$ for $i \neq j$, $i, j = 1, 2, 3, 4$.
Let η be the 1-form defined by $\eta(Z) = g(Z, e_4)$ for any $Z \in \chi(M)$.
Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = e_3, \quad \phi(e_4) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_4) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_4,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.
Then for $e_4 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.
Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = -we_2 \quad , [e_1, e_4] = -(w)^2 e_1 \quad ,$$

$$[e_2, e_4] = -(w)^2 e_2 \quad and \quad [e_3, e_4] = -(w)^2 e_3.$$

Taking $e_4 = \xi$ and using Koszul's formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_4 = -(w)^2 e_1, \quad \nabla_{e_2} e_1 = we_2, \quad \nabla_{e_1} e_1 = -(w)^2 e_4,$$

$$\nabla_{e_2} e_4 = -(w)^2 e_2, \quad \nabla_{e_3} e_4 = -(w)^2 e_3,$$

$$\nabla_{e_3} e_3 = -(w)^2 e_4, \quad \nabla_{e_2} e_2 = -(w)^2 e_4 - we_1.$$

From the above it can be easily seen that $M^4(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold with $\alpha = -(w)^2 \neq 0$.

4. main result

In this section we obtain necessary and sufficient conditions for LP-Sasakian manifolds with a coefficient α satisfying the derivation conditions $\tilde{C}(\xi, X)\cdot\tilde{C} = 0$, $\tilde{C}(\xi, X)\cdot S = 0$ and $\tilde{C}(\xi, X)\cdot C = 0$, where $\tilde{C}(X, Y)$ is treated as a derivation of the tensor algebra for any tangent vectors X, Y. That is, $\tilde{C}(\xi, X)\cdot\tilde{C} = 0$, $\tilde{C}(\xi, X)\cdot S = 0$ and $\tilde{C}(\xi, X)\cdot C = 0$ mean $\tilde{C}$ operating on $\tilde{C}$, S and C respectively.
In an LP-Sasakian manifold with a coefficient α , we have

$$\tilde{C}(X, Y)\xi = ((\alpha^2 - \sigma) - \frac{r}{n(n-1)})\{\eta(Y)X - \eta(X)Y\}, \tag{4.1}$$

$$\tilde{C}(\xi, X)Y = ((\alpha^2 - \sigma) - \frac{r}{n(n-1)})\{g(X, Y)\xi - \eta(Y)X\}. \qquad (4.2)$$

Let us consider the condition $\tilde{C}(\xi, U) \cdot \tilde{C} = 0$, which implies that

$$\begin{aligned}\tilde{C}(\xi, U)\tilde{C}(X, Y)W &- \tilde{C}(\tilde{C}(\xi, U)X, Y)W - \tilde{C}(X, \tilde{C}(\xi, U)Y)W \\ &- \tilde{C}(X, Y)\tilde{C}(\xi, U)W = 0. \qquad (4.3)\end{aligned}$$

Putting $W = \xi$ in (4.3) we have

$$\begin{aligned}\tilde{C}(\xi, U)\tilde{C}(X, Y)\xi &- \tilde{C}(\tilde{C}(\xi, U)X, Y)\xi - \tilde{C}(X, \tilde{C}(\xi, U)Y)\xi \\ &- \tilde{C}(X, Y)\tilde{C}(\xi, U)\xi = 0, \qquad (4.4)\end{aligned}$$

which in view of (4.2) and (4.1) gives

$$((\alpha^2-\sigma)-\frac{r}{n(n-1)})\{\tilde{C}(X, Y)U+((\alpha^2-\sigma)-\frac{r}{n(n-1)})(g(X, U)Y-g(Y, U)X)\} = 0.$$

Therefore, either the scalar curvature $r = n(n-1)(\alpha^2 - \sigma)$ or,

$$\tilde{C}(X, Y)U + ((\alpha^2 - \sigma) - \frac{r}{2n(2n+1)})\{g(X, U)Y - g(Y, U)X\} = 0,$$

which in view of (1.3) gives

$$R(X, Y)U = (\alpha^2 - \sigma)(g(Y, U)X - g(X, U)Y). \qquad (4.5)$$

Using Bianchi 2nd identity the above equation implies that the manifold is of constant curvature $(\alpha^2 - \sigma)$.
Conversely, if the manifold has the scalar curvature $r = n(n-1)(\alpha^2 - \sigma)$, then from (4.2) it follows that $\tilde{C}(\xi, X) = 0$. Similarly, in the second case, since the manifold under consideration is of constant curvature, therefore we again get $\tilde{C}(\xi, X) = 0$. Therefore we state the following:

Theorem 4.1. *An n-dimensional LP-Sasakian manifold satisfies*

$$\tilde{C}(\xi, X) \cdot \tilde{C} = 0$$

if and only if either the scalar curvature of the manifold is $r = n(n-1)(\alpha^2 - \sigma)$ *or, the manifold is of constant curvature* $(\alpha^2 - \sigma)$.

We also consider the condition $\tilde{C}(\xi, X) \cdot S = 0$, which implies that

$$S(\tilde{C}(\xi, X)Y, \xi) + S(Y, \tilde{C}(\xi, X)\xi) = 0,$$

which in view of (4.2) gives

$$((\alpha^2 - \sigma) - \frac{r}{n(n-1)})\{g(X, Y)S(\xi, \xi) - S(X, Y)\} = 0.$$

So by use of (2.14) and (2.1) we have

$$((\alpha^2 - \sigma) - \frac{r}{n(n-1)})\{-2ng(X, Y) - S(X, Y)\} = 0.$$

Therefore either the scalar curvature of (M, g) is $r = n(n-1)(\alpha^2 - \sigma)$ or, $S = -2ng$ which implies that the LP-Sasakian manifold is an Einstein manifold. The converse is trivial. So we can state the following:

Theorem 4.2. *An n-dimensional LP-Sasakian manifold with a coefficient α satisfies $\tilde{C}(\xi, X) \cdot S = 0$ if and only if either the manifold has the scalar curvature $r = n(n-1)(\alpha^2 - \sigma)$ or, the manifold is an Einstein manifold.*

Next, we consider an n-dimensional LP-Sasakian manifold with a coefficient α satisfying $\tilde{C}(\xi, X) \cdot C = 0$, which implies that

$$\begin{aligned} \tilde{C}(\xi, U)C(X, Y)W &- C(\tilde{C}(\xi, U)X, Y)W - C(X, \tilde{C}(\xi, U)Y)W \\ &- C(X, Y)\tilde{C}(\xi, U)W = 0, \end{aligned} \tag{4.6}$$

which in view of (4.2) we have

$$\begin{aligned} ((\alpha^2 - \sigma) &- \frac{r}{n(n-1)})\{C(X, Y, W, U)\xi - \eta(C(X, Y)W)U \\ &- g(U, W)C(X, Y)\xi + \eta(W)C(X, Y)U - g(U, X)C(\xi, Y)W \\ &+ \eta(X)C(U, Y)W - g(U, Y)C(X, \xi)W + \eta(Y)C(X, U)W\} = 0, \end{aligned}$$

where $C(X, Y, W, U) = g(C(X, Y)W, U)$.
So either the scalar curvature of (M, g) is $r = n(n-1)(\alpha^2 - \sigma)$ or, the equation

$$\begin{aligned} C(X, Y, W, U)\xi &- \eta(C(X, Y)W)U - g(U, W)C(X, Y)\xi + \eta(W)C(X, Y)U \\ &- g(U, X)C(\xi, Y)W + \eta(X)C(U, Y)W - g(U, Y)C(X, \xi)W \\ &+ \eta(Y)C(X, U)W) = 0 \end{aligned}$$

holds on the manifold.
Taking the inner product of the last equation with ξ we get

$$\begin{aligned}
-C(X,Y,W,U) &- \eta(C(X,Y)W)\eta(U) - g(U,W)\eta(C(X,Y)\xi) \\
&+ \eta(W)\eta(C(X,Y)U) - g(U,X)\eta(C(\xi,Y)W) \\
&+ \eta(X)\eta(C(U,Y)W) - g(U,Y)\eta(C(X,\xi)W) \\
&+ \eta(Y)\eta(C(X,U)W) = 0. \qquad (4.7)
\end{aligned}$$

Hence using (2.14) and (1.4)in the equation (4.7)and by a suitable contraction we get

$$S(Y,W) = (\frac{r}{n-1} - (\alpha^2-\sigma))g(Y,W) + (\frac{r}{n-1} - n(\alpha^2-\sigma))\eta(Y)\eta(W), \quad (4.8)$$

which implies that the LP-Sasakian manifold is an η-Einstein manifold.
Now from (1.4) we obtain

$$\begin{aligned}
\eta(C(X,Y)W) &= \eta(R(X,Y)W) - \frac{1}{n-2}\{S(Y,W)\eta(X) - S(X,W)\eta(Y) \\
&+ g(Y,W)S(X,\xi) - g(X,W)S(Y,\xi)\} \\
&+ \frac{r}{(n-1)(n-2)}\{g(Y,W)\eta(X) - g(X,W)\eta(Y)\}. \qquad (4.9)
\end{aligned}$$

Using (2.14) and (4.8), (4.9) reduces to $\eta(C(X,Y)W) = 0$. Hence from (4.7) we have $C = 0$, that is, the manifold is conformally flat. Also the converse is trivial.
So we can state the following:

Theorem 4.3. *An n-dimensional LP-Sasakian manifold with a coefficient α satisfies $\tilde{C}(\xi,X)\cdot C = 0$ if and only if either the manifold has the scalar curvature $r = n(n-1)(\alpha^2-\sigma)$ or, the manifold is conformally flat*

5. ξ-concircularly flat LP-Sasakian manifolds with a coefficient α

In this section we study ξ-concircularly flat LP-Sasakian manifolds with a coefficient α. Let M be an n-dimensional, $n \geq 3$, ξ-concircularly flat LP-Sasakian manifolds with a coefficient α. Putting $U = \xi$ in (1.3) and applying

(1.7) and $g(X,\xi)=\eta(X)$, we have

$$R(X,Y)\xi=\frac{r}{n(n-1)}[\eta(Y)X-\eta(X)Y]. \tag{5.1}$$

Using (2.16) in (5.1), we obtain

$$((\alpha^2-\sigma)-\frac{r}{n(n-1)})[\eta(Y)X-\eta(X)Y]=0. \tag{5.2}$$

Now $[\eta(Y)X-\eta(X)Y]\neq 0$ in a paracontact metric manifold, in general. Therefore (5.2) gives

$$r=(\alpha^2-\sigma)n(n-1). \tag{5.3}$$

Now, we consider an n-dimensional LP-Sasakian manifold with a coefficient α with $r=(\alpha^2-\sigma)n(n-1)$. Then using (1.3) and (2.16) we easily obtain

$$\tilde{C}(X,Y)\xi=0. \tag{5.4}$$

In view of above discussions we state the following:

Theorem 5.1. *An n-dimensional LP-Sasakian manifold with a coefficient α is ξ-concircularly flat if and only if $r=(\alpha^2-\sigma)n(n-1)$.*

6. ϕ-concircularly flat LP-Sasakian manifolds with a coefficient α

This section is devoted to study ϕ-concircularly flat LP-Sasakian manifolds with a coefficient α. Let M be an n-dimensional ϕ-concircularly flat LP-Sasakian manifold with a coefficient α.
Using (1.8) in (1.3) we obtain

$$\begin{aligned} g(R(\phi X,\phi Y)\phi W,\phi V) &= \frac{r}{n(n-1)}[g(\phi Y,\phi W)g(\phi X,\phi V) \\ &\quad -g(\phi X,\phi W)g(\phi Y,\phi V)]. \end{aligned} \tag{6.1}$$

Let $\{e_1,e_2,...,e_{n-1},\xi\}$ be a local orthonormal basis of vector fields in M^n. Using that $\{\phi e_1,\phi e_2,...,\phi e_{n-1},\xi\}$ is also a local orthonormal basis, if we put

$X = V = e_i$ in (6.1) and sum up with respect to i, then

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi W, \phi e_i) = \sum_{i=1}^{n-1} \frac{r}{n(n-1)}[g(\phi Y, \phi W)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi W)g(\phi Y, \phi e_i)]. \quad (6.2)$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi W, \phi e_i) = S(\phi Y, \phi W) + g(\phi Y, \phi W), \quad (6.3)$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n + 1 \quad (6.4)$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi W)g(\phi Y, \phi e_i) = g(\phi Y, \phi W). \quad (6.5)$$

So by virtue of (6.3)-(6.5) the equation (6.2) can be written as

$$S(\phi Y, \phi W) = [\frac{r}{(n-1)} - 1]g(\phi Y, \phi W). \quad (6.6)$$

Then by making use of (2.2) and (2.19), the equation (6.6) takes the form

$$S(Y, W) = [\frac{r}{n-1} - 1 - (n-1)(\alpha^2 - \sigma)]g(Y, W) + [\frac{r}{n-1} - 1]\eta(Y)\eta(W). \quad (6.7)$$

In view of the equation (6.7) we state the following:

Theorem 6.1. *An n-dimensional ϕ-concircularly flat LP-Sasakian manifold with a coefficient α is an η-Einstein manifold.*

7. Lorentzian para-Sasakian type spacetime

In this section we study Lorentzian para-Sasakian type spacetime which is a 4- dimensional LP-Sasakian manifold with a constant coefficient α. Since α is a constant, we have from (2.13), $\sigma = 0$ and the equation (2.15) reduces to

$$S(X, \xi) = 3\alpha^2\eta(X). \quad (7.1)$$

Einstein's Field equation with cosmological constant λ is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{7.2}$$

for all vector fields X, Y where S is the Ricci tensor of the type $(0,2)$, r is the scalar curvature, k is the gravitational constant and T is the energy momentum tensor of type $(0,2)$.
The energy momentum tensor T is said to describe a perfect fluid [2] if

$$T(X,Y) = (\rho + p)A(X)A(Y) + pg(X,Y) \tag{7.3}$$

where ρ is the energy density function, p is the isotropic pressure function of the fluid, A is a non-zero 1-form such that $g(X,U) = A(X)$ for all X, U being the flow vector field of the fluid.
In a Lorentzian para- Sasakian type spacetime by considering the characteristic vector field ξ as the flow vector field of the fluid, the energy momentum tensor takes the form

$$T(X,Y) = (\rho + p)\eta(X)\eta(Y) + pg(X,Y). \tag{7.4}$$

Let us consider Einstein's Field equation with cosmological constant. Then putting $Y = \xi$ in (7.2) we have

$$\begin{aligned}
S(X,\xi) - \frac{r}{2}g(X,\xi) + \lambda g(X,\xi) &= k[(\rho+p)\eta(X)\eta(\xi) + pg(X,\xi)] \\
or, \quad 3\alpha^2\eta(X) - \frac{r}{2}\eta(X) + \lambda\eta(X) &= k[-(\rho+p)\eta(X) + p\eta(X)] \\
or, \quad 3\alpha^2 - \frac{r}{2} + \lambda &= -k\rho \\
or, \quad \rho &= \frac{r - 6\alpha^2 - 2\lambda}{2k}. \qquad (7.5)
\end{aligned}$$

Again contracting (7.2) we get

$$\begin{aligned}
r - 2r + 4\lambda &= k(3p - \rho) \\
or, \quad -r + 4\lambda &= 3kp - \frac{r}{2} + 3\alpha^2 + \lambda \\
or, \quad p &= \frac{-r - 6\alpha^2 + 6\lambda}{6k}. \qquad (7.6)
\end{aligned}$$

If r is constant, then it follows from (7.5) and (7.6) that ρ and p are constant. Since $\rho > 0$, from (7.5) we have

$$\lambda < \frac{r - 6\alpha^2}{2}. \tag{7.7}$$

Since $p > 0$, we have from (7.6)

$$\lambda > \frac{r + 6\alpha^2}{6}. \tag{7.8}$$

From (7.7) and (7.8) we obtain

$$\frac{r + 6\alpha^2}{6} < \lambda < \frac{r - 6\alpha^2}{2}. \tag{7.9}$$

Since div $T = 0$, we get the energy and force equations as follows [2]:

$$\xi.\rho = -(\rho + p)\text{div } \xi \qquad \text{[Energy equation]} \tag{7.10}$$

$$(\rho + p)\nabla_\xi \xi = -\text{grad } p - (\xi p)\xi \qquad \text{[Force equation]}\ . \tag{7.11}$$

Since ρ is constant, it follows from (7.10) that div $\xi = 0$, because $(\rho + p) \neq 0$. Again since p is constant, it follows from (7.11) that $\nabla_\xi \xi = 0$. It is known that div ξ represents the expansion scalar and $\nabla_\xi \xi$ represents the acceleration vector. Thus in this case both the expansion scalar and the acceleration vector are zero. Hence we can state the following:

Theorem 7.1. *If in a Lorentzian para-Sasakian type spacetime of non-zero constant scalar curvature the matter distribution is perfect fluid whose velocity vector field is the characteristic vector field ξ of the spacetime, then the acceleration vector of the fluid must be zero and the expansion scalar also so. Moreover the cosmological constant λ satisfies the relation $\frac{r+6\alpha^2}{6} < \lambda < \frac{r-6\alpha^2}{2}$.*

Next we take Einstein's field equation without cosmological constant. Then (7.2) can be written as

$$S(X, Y) - \frac{r}{2} g(X, Y) = kT(X, Y) \tag{7.12}$$

From (7.12) and (7.4) we have

$$S(X, Y) - \frac{r}{2} g(X, Y) = k[(\rho + p)\eta(X)\eta(Y) + pg(X, Y)]. \tag{7.13}$$

Taking a frame field and contracting (7.13) over X and Y we obtain

$$r = k(\rho - 3p). \tag{7.14}$$

In view of (7.14), (7.13) yields

$$S(X,Y) = k[(\rho + p)\eta(X)\eta(Y) + \frac{1}{2}(\rho - p)g(X,Y)]. \tag{7.15}$$

Let Q be the Ricci operator, that is, $g(QX,Y) = S(X,Y)$. Then setting $X = QX$ in (7.15) we get

$$S(QX,Y) = k[(\rho + p)\eta(QX)\eta(Y) + \frac{1}{2}(\rho - p)S(X,Y)]. \tag{7.16}$$

Contracting (7.16) over X and Y we have

$$\parallel Q \parallel^2 = k[(\rho + p)S(\xi,\xi) + \frac{1}{2}(\rho - p)r]. \tag{7.17}$$

Using (7.1) and (7.14) in (7.17) we obtain

$$\parallel Q \parallel^2 = k[(\rho + p)(-3\alpha^2) + \frac{1}{2}(\rho - p)(\rho - 3p)]. \tag{7.18}$$

Again setting $X = Y = \xi$ in (7.15) we get

$$-3\alpha^2 = \frac{k}{2}(\rho + 3p). \tag{7.19}$$

By virtue of (7.19) we obtain from (7.18) that

$$\parallel Q \parallel^2 = k^2(\rho^2 + 3p^2). \tag{7.20}$$

We now suppose that the length of the Ricci operator of the perfect fluid Lorentzian para-Sasakian type spacetime is $\frac{1}{3}r^2$, where r is the scalar curvature of the spacetime. Then from (7.20) we have

$$\frac{1}{3}r^2 = k^2(\rho^2 + 3p^2),$$

which yields by virtue of (7.14) that $k^2\rho(\rho + 3p) = 0$. Since $k \neq 0$, either $\rho = 0$ or $\rho + 3p = 0$. If possible let $\rho + 3p = 0$, then from (7.19) it follows that

$\alpha = 0$ which is not possible. Then $\rho = 0$ which is not possible as when the pure matter exists, ρ is always greater than zero. Hence the spacetime under consideration cannot contain pure matter.

Now we determine the sign of pressure in such a spacetime without pure matter. Hence for $\rho = 0$, (7.14) implies that

$$p = -\frac{r}{3k}. \tag{7.21}$$

Again for $\rho = 0$, (7.5) yields $r = 6\alpha^2$. Therefore (7.21) reduces to

$$p = -\frac{2\alpha^2}{k}.$$

Thus we can state the following:

Theorem 7.2. *If a perfect fluid Lorentzian para-Sasakian type spacetime obeying Einstein's equation without cosmological constant and the square of the length of the Ricci operator is $\frac{1}{3}r^2$, then the spacetime can not contain pure matter. Moreover in such a spacetime without pure matter the pressure of the fluid is always negative.*

Next, we consider a conformally flat perfect fluid Lorentzian para-Sasakian type spacetime obeying Einstein equation without cosmological constant. Hence using (7.5) and (7.6) in (7.13) we have

$$S(X,Y) = (\frac{r}{3} - \alpha^2)g(X,Y) + (\frac{r}{3} - 4\alpha^2)\eta(X)\eta(Y). \tag{7.22}$$

which implies

$$QX = (\frac{r}{3} - \alpha^2)X + (\frac{r}{3} - 4\alpha^2)\eta(X)\xi, \tag{7.23}$$

where Q is the symmetric endomorphism given by $S(X,Y) = g(QX,Y)$. Since the spacetime is assumed to be conformally flat, we have [14]

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{7.24}$$

Using (7.22) and (7.23) in (7.24), we have

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{6}-\alpha^2)[g(Y,Z)X - g(X,Z)Y] \\ &+ \frac{1}{2}(\frac{r}{3}-4\alpha^2)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &+ \frac{1}{2}(\frac{r}{3}-4\alpha^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi. \end{aligned} \tag{7.25}$$

Let $\xi^{\perp}$ denote the $3-$dimensional distribution in Lorentzian para-Sasakian type spacetime orthogonal to ξ, then

$$R(X,Y)Z = (\frac{r}{6}-\alpha^2)[g(Y,Z)X - g(X,Z)Y] \qquad \text{for all } X,Y,Z \in \xi^{\perp} \tag{7.26}$$

and

$$R(X,\xi)\xi = -(\frac{r}{6}-\alpha^2)X \qquad \text{for every } X \in \xi^{\perp}. \tag{7.27}$$

Let $X, Y \in \xi^{\perp}$, K_1 denote sectional curvature of the plane determined by X, Y and K_2 denote the sectional curvature determined by X, ξ . Then

$$\begin{aligned} K_1 &= \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - \{g(X,Y)\}^2} \\ &= (\frac{r}{6}-\alpha^2). \end{aligned}$$

Again

$$\begin{aligned} K_2 &= \frac{g(R(X,\xi)\xi,X)}{g(X,X)g(\xi,\xi) - \{g(X,\xi)\}^2} \\ &= -(\frac{r}{6}-\alpha^2). \end{aligned}$$

Summing up we can state the following theorem:

Theorem 7.3. *A conformally flat perfect fluid Lorentzian para-Sasakian type spacetime obeying the Einstein equation without cosmological constant and having the characteristic vector field ξ as the velocity vector field has the following property:*
All planes perpendicular to ξ have sectional curvature $(\frac{r}{6}-\alpha^2)$ and all planes containing ξ have sectional curvature $-(\frac{r}{6}-\alpha^2)$.

By Karcher [7] a Lorentz manifold is called infinitesimally spatially isotropic relative to a timelike unit vector field ξ if its Riemannian curvature R satisfies the relations

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y] \;\; forX, Y, Z \;\in\; \xi^{\perp}$$

and

$$R(X,\xi)\xi = mX \quad forX \;\in\; \xi^{\perp}$$

where l,m are real valued functions on the manifold. By virtue of (7.26) and (7.27) we can state the following:

Theorem 7.4. *A conformally flat perfect fluid Lorentzian para-Sasakian type spacetime obeying the Einstein equation without cosmological constant and having the characteristic vector field as the velocity vector field of the fluid is infinitesimally spatially isotropic relative to the velocity vector field.*

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