Tamsui Oxford Journal of Information and Mathematical Sciences **29(4)** (2013) 427-434
Aletheia University

On Generalized Derivations and Lie Ideals in Prime Rings *

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Received December 13, 2012, Accepted May 28, 2014.

Abstract

Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F, G be two generalized derivations of R with associated derivations d and δ of R respectively. In the present paper, we study the situations (1) $F(u)G(v) - uv \in Z(R)$, (2) $F(u)G(v) + uv \in Z(R)$, (3) $d(u)F(v) - vu \in Z(R)$, (4) $d(u)F(v) + vu \in Z(R)$; for all $u, v \in U$ and show that if $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.

Keywords and Phrases: *Prime ring, Lie ideal, Derivation, Generalized derivation.*

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*2000 *Mathematics Subject Classification.* Primary 16W25, 16R50, 16N60.

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1. Introduction

Let R be a prime ring with center $Z(R)$. For any pair of elements $x, y \in R$, we shall write $[x, y]$ for the commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R, if $[U, R] \subseteq U$. The centralizer of U is denoted by $C_R(U)$ and defined by $C_R(U) = \{x \in R \mid [x, U] = 0\}$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In [5], Brešar introduced the notion of generalized derivations in rings. An additive mapping $F : R \to R$ is said to be a generalized derivation, if there exists a derivation d of R such that for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$ holds (d is called the derivation associated with F). For $a, b \in R$, $F(x) = ax + xb$ for all $x \in R$ is a generalized derivation of R, which is called as inner generalized derivation of R. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [6], [10], [11]). Moreover, several authors studied commutativity in prime and semiprime rings admitting derivations and generalized derivations which satisfy appropriate algebraic conditions on suitable subsets of the rings.

In [3], Ashraf and Rehman established that a prime ring R with a nonzero ideal I must be commutative, if R admits a nonzero derivation d satisfying $d(xy) + xy \in Z(R)$ for all $x, y \in I$ or $d(xy) - xy \in Z(R)$ for all $x, y \in I$. Recently in [1] Ashraf et al. studied the case replacing derivation d with a generalized derivation F in a prime ring R. More precisely, they proved that the prime ring R with a nonzero ideal I must be commutative, if R admits a generalized derivation F associated with a nonzero derivation d satisfying any one of the following situations: (i) $F(xy)-xy \in Z(R)$, (ii) $F(xy)+xy \in Z(R)$, (iii) $F(xy) - yx \in Z(R)$, (iv) $F(xy) + yx \in Z(R)$, (v) $F(x)F(y) - xy \in Z(R)$, (vi) $F(x)F(y) + xy \in Z(R)$; for all $x, y \in I$. Again, in [2] Ashraf et al. considered the situations $d(x)F(y)-xy \in Z(R)$ and $d(x)F(y)+xy \in Z(R)$ for all x, y in some appropriate subset of a prime ring R, where F is a generalized derivation with associated derivation d of R. Further Golbasi and Koc [8] studied all the cases (i) - (vi) in a square closed Lie ideal U in a 2-torsion free prime ring R and obtained that if $d \neq 0$, then $U \subseteq Z(R)$. From these identities, it is natural to consider the situations (1) $F(x)F(y)-yx \in Z(R)$ and (2) $F(x)F(y) + yx \in Z(R)$ for all x, y in some suitable subset of R. Recently in [7] Dhara et al. studied these two identities in a square closed Lie ideal U in a 2-torsion free prime ring R and obtained that if $d \neq 0$, then $U \subseteq Z(R)$.

In view of above results, it is natural to ask, what happen in cases: (1) $F(x)G(y)-xy \in Z(R)$, (2) $F(x)G(y)+xy \in Z(R)$, (3) $d(x)F(y)-yx \in Z(R)$, (4) $d(x)F(y)+yx \in Z(R)$ for all x, y in a square closed Lie ideal U of a prime ring R, where F and G are two generalized derivations of R associated to the derivations d and δ of R respectively.

2. Preliminaries

Let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Then for all $u, v \in U$, we get $uv + vu = (u+v)^2 - u^2 - v^2 \in U$. Again from definition of Lie ideal, we have $uv - vu \in U$ for all $u, v \in U$. Combining these two we get $2uv \in U$ for all $u, v \in U$.

In all that follows, let R be a prime ring with center $Z(R)$ and char $(R) \neq 2$.

Lemma 2.1. *([4, Lemma 2]) If $U \nsubseteq Z(R)$ is a Lie ideal of R, then $C_R(U) = Z(R)$.*

Lemma 2.2. *([4, Lemma 3]) If U is a Lie ideal of R, then $C_R([U,U]) = C_R(U)$.*

Lemma 2.3. *([4, Lemma 4]) If $U \nsubseteq Z(R)$ is a Lie ideal of R and $aUb = 0$, then either $a = 0$ or $b = 0$.*

Lemma 2.4. *([9, Theorem 5]) Let d be a nonzero derivation of R and U a nonzero Lie ideal of R such that $[u, d(u)] \in Z(R)$ for all $u \in U$. Then $U \subseteq Z(R)$.*

Lemma 2.5. *([7, Lemma 2.5]) Let U be a nonzero Lie ideal of R and $V = \{u \in U \mid d(u) \in U\}$. Then V is also a nonzero Lie ideal of R. Moreover, if U is noncentral, then V is also noncentral.*

3. Main Results

Theorem 3.1. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F, G be two generalized derivations of R associated to the derivations d and δ of R respectively. If $F(u)G(v) - uv \in Z(R)$ for all $u, v \in U$ and if $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

Proof. We assume on the contrary that $U \not\subseteq Z(R)$. By hypothesis we have

$$F(u)G(v) - uv \in Z(R) \qquad \text{for all} \quad u, v \in U. \tag{1}$$

If $F(U) = 0$ or $G(U) = 0$, then we have $uv \in Z(R)$ for all $u, v \in U$. Interchanging v and u, we obtain $vu \in Z(R)$. These two conditions yield that $[u, v] \in Z(R)$ for all $u, v \in U$, that is $[U, U] \subseteq Z(R)$. This implies $C_R([U, U]) = R$ and hence by Lemma 2.2 we get $C_R(U) = R$. This gives by Lemma 2.1 that $U \subseteq Z(R)$, a contradiction.

Next we assume that $F(U) \neq 0$ and $G(U) \neq 0$. Replacing v by $2vw$ in (1) we get $2(F(u)(G(v)w + v\delta(w)) - uvw) \in Z(R)$ for all $u, v, w \in U$. Since char $(R) \neq 2$, this gives $F(u)(G(v)w + v\delta(w)) - uvw \in Z(R)$ i.e.,

$$(F(u)G(v) - uv)w + F(u)v\delta(w) \in Z(R) \qquad \text{for all} \quad u, v, w \in U. \tag{2}$$

Commuting both sides with w, we get

$$[(F(u)G(v) - uv)w, w] + [F(u)v\delta(w), w] = 0 \qquad \text{for all} \quad u, v, w \in U. \tag{3}$$

Since $F(u)G(v) - uv \in Z(R)$ for all $u, v \in U$, above relation reduces to

$$[F(u)v\delta(w), w] = 0 \qquad \text{for all} \quad u, v, w \in U. \tag{4}$$

Now replacing u with $2uw$ in (4) and then using the restriction on characteristic, we obtain

$$[(F(u)w + ud(w))v\delta(w), w] = 0 \qquad \text{for all} \quad u, v, w \in U. \tag{5}$$

Again, putting $v = 2wv$ in (4) we get

$$[F(u)wv\delta(w), w] = 0 \qquad \text{for all} \quad u, v, w \in U. \tag{6}$$

Subtracting (6) from (5), we arrive at

$$[ud(w)v\delta(w), w] = 0 \qquad \text{for all} \quad u, v, w \in U. \tag{7}$$

Replacing u by $2tu$ and using char $(R) \neq 2$, above relation gives

$$\begin{aligned} 0 &= [tud(w)v\delta(w), w] \\ &= t[ud(w)v\delta(w), w] + [t, w]ud(w)v\delta(w) \qquad \text{for all} \quad u, v, w, t \in U. \end{aligned} \tag{8}$$

Using (7), (8) yields

$$0 = [t, w]ud(w)v\delta(w) \qquad \text{for all} \;\; u, v, w, t \in U. \tag{9}$$

By Lemma 2.3, for each $w \in U$, either $[t, w] = 0$ for all $t \in U$ or $d(w) = 0$ or $\delta(w) = 0$. Let $T_1 = \{w \in U | [U, w] = 0\}$, $T_2 = \{w \in U | d(w) = 0\}$ and $T_3 = \{w \in U | \delta(w) = 0\}$. Then T_1, T_2 and T_3 are three additive subgroups of U such that $T_1 \cup T_2 \cup T_3 = U$. Let $w \in T_3$. Then $\delta(w) = 0$. Hence from (2) we have $(F(u)G(v) - uv)w \in Z(R)$ for all $u, v \in U$. This implies

$$[(F(u)G(v) - uv)w, r] = 0 \qquad \text{for all} \;\; u, v \in U \;\; \text{and for all } r \in R. \tag{10}$$

Since $F(u)G(v) - uv \in Z(R)$, we get from above

$$(F(u)G(v) - uv)[w, r] = 0 \qquad \text{for all} \;\; u, v \in U \;\; \text{and for all } r \in R. \tag{11}$$

Since center of prime ring contains no divisor of zero, we have either $F(u)G(v) - uv = 0$ for all $u, v \in U$ or $w \in Z(R)$. In case $F(u)G(v) - uv = 0$ for all $u, v \in U$, replacing v with $2vt$, where $t \in U$ we have that

$$0 = 2(F(u)G(vt) - uvt) = 2(F(u)G(v) - uv)t + 2F(u)v\delta(t) = 2F(u)v\delta(t).$$

Since char $(R) \neq 2$, $F(u)v\delta(t) = 0$ for all $u, v, t \in U$. By Lemma 2.3, either $F(U) = 0$ or $\delta(U) = 0$. Since $F(U) \neq 0$, we have $\delta(U) = 0$. This implies $U \subseteq Z(R)$, by Lemma 2.4, a contradiction. Hence, we have $w \in Z(R)$. Then $[U, w] = 0$, i.e., $w \in T_1$. Thus $T_3 \subseteq T_1$. Therefore, $T_1 \cup T_2 = U$. Since a group can not be union of its two proper subgroups, either $T_1 = U$ or $T_2 = U$. Now $T_1 = U$ implies $U \subseteq Z(R)$ by Lemma 2.1, a contradiction. $T_2 = U$ implies $d(U) = 0$, implying $U \subseteq Z(R)$ by Lemma 2.4, again a contradiction.

Theorem 3.2. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F, G be two generalized derivations of R with associated derivations d and δ of R respectively. If $F(u)G(v) + uv \in Z(R)$ for all $u, v \in U$ and if $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

Proof. We note that $-F$ and $-G$ are two generalized derivations of R with associated derivations $-d$ and $-\delta$ respectively. Hence Replacing F by $-F$ in Theorem 3.1, we have $(-F)(u)G(v) - uv \in Z(R)$ for all $u, v \in U$, that is $F(u)G(v) + uv \in Z(R)$ for all $u, v \in U$ implies $U \subseteq Z(R)$.

Corollary 3.3. *Let R be a prime ring of characteristic not 2 and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If d and δ are two derivations of R such that $d(u)\delta(v) \pm uv \in Z(R)$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

Theorem 3.4. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to the derivation d of R. If $d(u)F(v) - vu \in Z(R)$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

Proof. We assume on the contrary that $U \not\subseteq Z(R)$.

If $d = 0$ or $F = 0$, then $uv \in Z(R)$ for all $u, v \in U$. Then by same argument of Theorem 3.1, we obtain $U \subseteq Z(R)$, a contradiction.

Let $d \neq 0$ and $F \neq 0$. We have

$$d(u)F(v) - vu \in Z(R) \quad \text{for all} \ \ u, v \in U. \tag{12}$$

Replacing v with $2vu$ in (12) we get $2(d(u)(F(v)u + vd(u)) - vu^2) \in Z(R)$ for all $u, v \in U$. Since char $(R) \neq 2$, this gives $d(u)(F(v)u + vd(u)) - vu^2 \in Z(R)$ i.e. $(d(u)F(v) - vu)u + d(u)vd(u) \in Z(R)$ for all $u, v \in U$. Commuting both sides with u, we get $[(d(u)F(v) - vu)u, u] + [d(u)vd(u), u] = 0$ for all $u, v \in U$. Since $d(u)F(v) - vu \in Z(R)$ for all $u, v \in U$, above relation reduces to

$$[d(u)vd(u), u] = 0 \quad \text{for all} \ \ u, v \in U. \tag{13}$$

Set $V = \{u \in U / d(u) \in U\}$. Then by Lemma 2.5, V is a noncentral nonzero Lie ideal of R. Since $V \subseteq U$, it follows from (13) that $[d(u)vd(u), u] = 0$ i.e.

$$d(u)vd(u)u - ud(u)vd(u) = 0 \quad \text{for all} \ \ u \in V \text{ and } v \in U. \tag{14}$$

Thus $2d(u)v \in U$ for all $v \in U$ and $u \in V$ and hence $4wd(u)v \in U$ for all $w, v \in U$ and $u \in V$. Now we substitute $4vd(u)w$ for v in (14), where $w \in U$ and $u \in V$, and then obtain by using char $(R) \neq 2$ that

$$d(u)vd(u)wd(u)u - ud(u)vd(u)wd(u) = 0 \quad \text{for all} \ \ v, \ w \in U \text{ and } u \in V. \tag{15}$$

Using (14), this can be written as $d(u)vud(u)wd(u) - d(u)vd(u)uwd(u) = 0$ i.e. $d(u)v[d(u), u]wd(u) = 0$ for all $v, w \in U$ and $u \in V$. By Lemma 2.3, above relation yields either $d(u) = 0$ or $[d(u), u] = 0$. In any case, it follows that $[d(u), u] = 0$ for all $u \in V$. By Lemma 2.4, we have $V \subseteq Z(R)$. By Lemma 2.5, it leads a contradiction.

By using the same technique of the proof of Theorem 3.4, we have the following:

Theorem 3.5. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F be a generalized derivation of R associated to the derivation d of R. If $d(u)F(v) + vu \in Z(R)$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

Corollary 3.6. *Let R be a prime ring of characteristic not 2 and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If d is a derivation of R such that $d(u)d(v) \pm vu \in Z(R)$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

The following example illustrates that R to be prime and char $(R) \neq 2$ in the hypothesis of the above theorems are not superfluous.

Example: Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$ and $U = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\}$. Then U is a nonzero square closed Lie ideal of R. Note that R is not prime for $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$. Define map $F : R \longrightarrow R$ by $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation with associated derivation d given by $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ satisfying $F(u)F(v) \pm uv \in Z(R)$ and $d(u)F(v) \pm vu \in Z(R)$ for all $u, v \in U$, but U is not central.

4. Conjecture

Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F, G be two generalized derivations of R associated to the nonzero derivations d and δ of R respectively.

(1) Does $F(u)G(v) - vu \in Z(R)$ for all $u, v \in U$ implies $U \subseteq Z(R)$;

(2) Does $F(u)G(v) + vu \in Z(R)$ for all $u, v \in U$ implies $U \subseteq Z(R)$.

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