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On Some Classes of Submanifolds Satisfying Chen's Equality in An Euclidean Space *

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Abstract

The object of the present paper is to study submanifolds satisfying Chen's equality in a Euclidean space. Also we study submanifolds satisfying the condition $\tilde{C}.\tilde{C} = 0$ and $\tilde{C}.S = 0$, where $\tilde{C}$ denotes the concircular curvature tensor.

Keywords and Phrases: *Chen invariant, Chen's equality, Totally geodesic submanifold, Minimal submanifold.*

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1. Introduction

One of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. In [1] and [4], B.Y. Chen established inequalities in this respect, called Chen inequalities. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants, namely the scalar curvature and the Ricci curvature; and the well known modern curvature invariant, namely Chen invariant [2]. In 1993, Chen obtained an interesting basic inequality for submanifolds in a real space form involving the squared mean curvature and the Chen invariant and found several of its applications. This inequality is now well known as Chen's inequality; and in the equality case it is known as Chen's equality.

In [7], Dillen, Petrovic and Verstraelen studied Einstein, conformally flat and semisymmetric submanifolds satisfying Chen's equality in Euclidean spaces. The hypersurfaces in $\mathbb{E}^{n+1}$ satisfying Chen's equality have been studied in [5] and others. Recently *Özgür* and De [6] studied projectively semisymmetric submanifolds satisfying Chen's equality in a Euclidean space and the submanifold satisfying the condition $P.P = 0$, where P is the projective curvature tensor. Motivated by the above studies in this paper, we study submanifolds satisfying Chen's equality and the conditions $\tilde{C}.\tilde{C} = 0$ and $\tilde{C}.S = 0$ in a Euclidean space.

The paper is organized as follows:

In Section 2, we give some idea about Riemannian submanifolds and Chen's inequality. Section 3 deals with some priliminaries about Chen's equality which will be used in the next Sections. Section 4 is devoted to study submanifolds satisfying the condition $\tilde{C}.\tilde{C} = 0$. Finally, we study submanifolds satisfying the condition $\tilde{C}.S = 0$.

2. Chen's ineqality

Let M be an n-dimensional submanifold of an $(n+m)$-dimensional Euclidean space $\mathbb{E}^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\nabla_X N = -A_N X + \nabla^{\perp}_X N$

for all $X, Y \in T(M)$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle TM of M respectively, and σ is the second fundamental

form related to the shape operator A by $\langle \sigma(X,Y), N\rangle = \langle A_N X, Y\rangle$. The equation of Gauss is given by

$$R(X,Y,Z,W) = \langle \sigma(X,W), \sigma(Y,Z)\rangle - \langle \sigma(X,Z), \sigma(Y,W)\rangle \tag{2.1}$$

for all X, Y, Z, $W \in TM$, where R is the curvature tensor of M.
The mean curvature vector H is given by $H = \frac{1}{n} trace(\sigma)$. The submanifold M is totally geodesic in E^{m+n} if $\sigma = 0$, and minimal if $H = 0$ [3]. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal tangent frame field on M. For the plane section $e_i \wedge e_j$ of the tangent bundle TM spanned by the vectors e_i and e_j $(e_i \neq e_j)$ the scalar curvature of M is defined by $\kappa = \sum_{i,j=1}^{n} K(e_i \wedge e_j)$, where K denotes the sectional curvature of M. Consider the real function $\inf K$ on M^n defined for every $x \in M$ by

$$(\inf K)(x) := \inf\{\ K(\pi)|\ \pi\ \textit{is plane in}\ T_x M^n\}.$$

Note that since the set of planes at a certain point is compact, this infimum is actually a minimum.

Lemma 2.1. [1] *Let M, $n \geq 2$, be any submanifold of E^{m+n}. Then*

$$\inf K \geq \frac{1}{2}\{\kappa - \frac{n^2(n-2)}{n-1}|H|^2\}. \tag{2.2}$$

Equality holds in (2.2) *at a point x if and only if with respect to suitable local orthonormal frames e_1,e_2,...,$e_n \in T_x M^n$, the Weingarten maps A_t with respect to the normal sections $\xi_t = e_{n+t}$, $t = 1, 2, ..., p$ are given by*

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{bmatrix}$$

$$A_t = \begin{bmatrix} c_t & d_t & 0 & \cdots & 0 \\ d_t & -c_t & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$(t > 1)$, *where* $\mu = a + b$ *for any such frame,* $\inf K(x)$ *is attained by the plane* $e_1 \wedge e_2$.
The inequality (2.2) is well known as Chen's inequality. In case of equality, it is known as Chen's equality. For dimension $n = 2$, the Chen's equality is always true.

3. Preliminaries

Let M be an n-dimensional $(n \geq 3)$ submanifold of a Euclidean space E^{n+m} satisfying Chen's equality. Then, from Lemma 2.1 we immediately have the following

$$K_{12} = ab - \sum_{r=1}^{m}(c_r^2 + d_r^2), \tag{3.1}$$

$$K_{1j} = a\mu, \tag{3.2}$$

$$K_{2j} = b\mu, \tag{3.3}$$

$$K_{ij} = \mu^2, \tag{3.4}$$

$$S(e_1, e_1) = K_{12} + (n-2)a\mu, \tag{3.5}$$

$$S(e_2, e_2) = K_{12} + (n-2)b\mu, \tag{3.6}$$

$$S(e_i, e_i) = (n-2)\mu^2, \tag{3.7}$$

$$r = 2K_{12} + (n-1)(n-2)\mu^2, \tag{3.8}$$

where $i, j > 2$. Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different [7].
A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular

transformation ([9], [11]). A concircular tranformation is always a conformal transformation [9]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism . An interesing invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$. It is defned by ([10], [11])

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.9}$$

where X, Y, $Z \in T(M)$ and r is the scalar curvature. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.
Using (3.1)-(3.7) in (3.8), we have the following:

Proposition 3.1. *Let M be an n-dimensional $(n \geq 3)$ submanifold in a Euclidean space satisfying Chen's equality, then*

$$\tilde{C}_{122} = \{K_{12} - \frac{r}{n(n-1)}\}e_1, \tag{3.10}$$

$$\tilde{C}_{133} = \{a\mu - \frac{r}{n(n-1)}\}e_1, \tag{3.11}$$

$$\tilde{C}_{131} = \{-a\mu + \frac{r}{n(n-1)}\}e_3, \tag{3.12}$$

$$\tilde{C}_{233} = \{b\mu - \frac{r}{n(n-1)}\}e_2, \tag{3.13}$$

$$\tilde{C}_{211} = \{K_{12} - \frac{r}{n(n-1)}\}e_2, \tag{3.14}$$

$$\tilde{C}_{232} = \{-b\mu + \frac{r}{n(n-1)}\}e_3, \tag{3.15}$$

and

$$\tilde{C}ijk = 0 \tag{3.16}$$

if i, j, k are mutually different.

In [7] Dillen, Petrovic and Verstraelen proved the following:

Theorem 3.1. [7]*Let M be an n-dimensional $(n \geq 3)$ submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. Then M is semisymmetric if and only if M is minimal submanifold (in which case M is $(n-2)$-ruled), or M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$.*

Remark 3.1. *For a concircular curvature tensor $\tilde{C}$ an important observation is that $R.R = R.\tilde{C}$ in Riemannian manifolds. Therefore the above Theorem* (3.1) *is true for concircularly semisymmetric ($R.\tilde{C} = 0$) submanifolds of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality.*

4. $\tilde{C}.\tilde{C} = 0$ in submanifolds of A Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality

Since the condition $\tilde{C}.\tilde{C} = 0$ holds on M, we have

$$\begin{aligned}(\tilde{C}(e_1,e_3).\tilde{C})(e_2,e_3)e_1 &= \tilde{C}(e_1,e_3)\tilde{C}(e_2,e_3)e_1 - \tilde{C}(\tilde{C}(e_1,e_3)e_2,e_3)e_1 \\ &- \tilde{C}(e_2,\tilde{C}(e_1,e_3)e_3)e_1 - \tilde{C}(e_2,e_3)\tilde{C}(e_1,e_3)e_1 = 0\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}(\tilde{C}(e_2,e_3).\tilde{C})(e_1,e_3)e_2 &= \tilde{C}(e_2,e_3)\tilde{C}(e_1,e_3)e_2 - \tilde{C}(\tilde{C}(e_2,e_3)e_1,e_3)e_2 \\ &- \tilde{C}(e_1,\tilde{C}(e_2,e_3)e_3)e_2 - \tilde{C}(e_1,e_3)\tilde{C}(e_2,e_3)e_2 = 0.\end{aligned} \tag{4.2}$$

Using (3.1)-(3.8) and (3.10)-(3.16) we have

$$\{a\mu - \frac{r}{n(n-1)}\}(K_{12} - b\mu) = 0 \tag{4.3}$$

and

$$\{b\mu - \frac{r}{n(n-1)}\}(K_{12} - a\mu) = 0. \tag{4.4}$$

Case 1 $(K_{12} - a\mu) = 0$, $(K_{12} - b\mu) = 0$.
By subtracting we have either $a = b$, or $\mu = 0$.
Sub Case 1 If $\mu = 0$, then M is minimal.
Sub Case 2 If $a = b$, then $\mu = 2a = 2b$,
then $K_{12} - a\mu = 0$ and (3.1) yields

$$a^2 + \sum_{r=1}^{m} c_r^2 + d_r^2 = 0.$$

Hence $a = 0$, $c_r = d_r = 0$. Since $a = b$, we get $b = 0$ also. Thus in this case $a = 0$, $b = 0$, $\mu = 0$ and $c_r = d_r = 0$. So M is totally geodesic.
Case 2 $(a\mu - \frac{r}{n(n-1)}) = 0$ and $K_{12} - a\mu = 0$.
Here, $K_{12} - a\mu = 0$ implies that

$$a^2 + \sum_{r=1}^{m} c_r^2 + d_r^2 = 0.$$

Therefore $a = 0$, $c_r = d_r = 0$ and hence from (3.1) $K_{12} = 0$. Thus from $(a\mu - \frac{r}{n(n-1)}) = 0$ we obtain $r = 0$. Also we know that $r = 2K_{12} + (n-1)(n-2)\mu^2$, therefore we have $\mu = 0$ and hence M is totally geodesic.
Case 3 $(b\mu - \frac{r}{n(n-1)}) = 0$ and $K_{12} - b\mu = 0$.
Similar to Case 2
Case 4 $a\mu = \frac{r}{n(n-1)}$ and $b\mu = \frac{r}{n(n-1)}$. In this case we obtain $\mu = 0$, or $a = b$. For $\mu = 0$, M is minimal. In view of the above cases we can state the following:

Theorem 4.1. *Let M be an n-dimensional $(n \geq 3)$ submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. If the condition $\tilde{C}.\tilde{C} = 0$ holds on M, then*
(i) M is minimal, or
(ii) M is totally geodesic, or
(iii) $\inf K = 0$, *or*
(iv) $a = b$.

5. $\tilde{C}.S = 0$ in submanifolds of A Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality

Since the condition $\tilde{C}.S = 0$ holds on M, we have

$$(\tilde{C}(e_i, e_j).S)(e_i, e_j) = S(\tilde{C}(e_i, e_j)e_i, e_j) + S(e_i, \tilde{C}(e_i, \tilde{C}(e_i, e_j)e_j) = 0, \quad (5.1)$$

for all $i \neq j$ $(1 \leq i, j \leq n)$. Using (3.9) in (5.1) we obtain

$$S(R(e_i, e_j)e_i, e_j) + S(e_i, R(e_i, e_j)e_j) = 0. \quad (5.2)$$

Thus, we have the following three equations:

$$S(R(e_1, e_2)e_1, e_2) + S(e_1, R(e_1, e_2)e_2) = 0, \quad (5.3)$$

$$S(R(e_1, e_3)e_1, e_3) + S(e_1, R(e_1, e_3)e_3) = 0, \quad (5.4)$$

$$S(R(e_2, e_3)e_2, e_3) + S(e_2, R(e_2, e_3)e_3) = 0. \quad (5.5)$$

Again using (3.10) − (3.16) in (3.9), we get

$$\begin{aligned} R(e_1, e_2)e_1 &= -K_{12}e_2, & R(e_1, e_2)e_2 &= K_{12}e_1 \\ R(e_1, e_3)e_1 &= -a\mu e_3, & R(e_1, e_3)e_3 &= a\mu e_1 \\ R(e_2, e_3)e_2 &= -b\mu e_3, & R(e_2, e_3)e_3 &= b\mu e_2 \end{aligned} \quad (5.6)$$

Therefore substituting (5.6) in (5.3), (5.4) and (5.5) respectively we have the following equations:

$$\mu(n-2)(a-b)K_{12} = 0, \quad (5.7)$$

$$a\mu[K_{12} + (n-2)a\mu - (n-2)\mu^2] = 0, \quad (5.8)$$

$$b\mu[K_{12} + (n-2)b\mu - (n-2)\mu^2] = 0. \quad (5.9)$$

Case 1 If $\mu = 0$, then M is minimal.
Case 2 $\mu \neq 0$, $a \neq 0$, $b \neq 0$.
Therefore, from (5.6) we have $K_{12} = 0$, i. e., $infK = 0$. Also from (5.7) and (5.8), we get

$$(a-b)^2(n-2)\mu^2 = 0.$$

Since $\mu \neq 0$ and $n \geq 3$, we have $a = b$.
Case 3 $\mu \neq 0$ and $a \neq 0$.

Therefore, from (5.6) we have either $K_{12} = 0$ or, $a = b$.
Subcase 1 If $K_{12} = 0$ $(i.e., a \neq b)$,
then, from (5.7) we have $a = \mu = a + b$. This implies $b = 0$. Again $K_{12} = 0$ yields $ab - \sum_{r=1}^{m}(c_r^2 + d_r^2) = 0$ which gives us $c_r = d_r = 0$ (as $b = 0$). So, by [8], M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$.
Case 4 $\mu \neq 0$ and $b \neq 0$.

Similar to Case 3.
In view of the above cases we can state the following:

Theorem 5.1. *Let M be an n-dimensional $(n \geq 3)$ submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. If the condition $\tilde{C}.S = 0$ holds on M, then*
(i) M is minimal, or
(ii) M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$, or
(iii) $\inf K = 0$, *or*
(iv) $a = b$.

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