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Subordination and Superordination for Multivalent Functions Defined by Linear Operators *

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Abstract

Numerous linear operators were introduced in geometric function theory and the properties of functions defined by them were derived using a recurrence relation satisfied by them. All these linear operators are unified in this paper and subordination and superordination

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properties of p-valent analytic functions defined using the general linear operator as well as a related integral transform are investigated. Some applications to univalent functions are also provided.

Keywords and Phrases: *p-valent function, Linear operator, Starlike function, Strongly starlike function.*

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$, which contains functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ and let $\mathcal{A}_1 := \mathcal{A}$. For two functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by $(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k$. For two analytic functions f and g, we say that f is *subordinate* to g or g *superordinate* to f, denoted by $f \prec g$, if there is a Schwarz function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The class $\mathcal{T}(\alpha)$ is defined to be the class of all functions $f \in \mathcal{A}$ satisfying $\operatorname{Re}(f(z)/z) > \alpha$, $0 \leq \alpha < 1$, $z \in \mathbb{D}$ and let $\mathcal{T} := \mathcal{T}(0)$. For an analytic function φ with $\varphi(0) = 1$, let $\mathcal{S}^*(\varphi)$ denote the class of all $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$. Several special choices of φ reduce to well-known classes. For $-1 \leq B < A \leq 1$, $\mathcal{S}^*[A,B] := \mathcal{S}^*((1+Az)/(1+Bz))$ is the Janowski starlike functions [13] (see [28]) and $\mathcal{S}^*[1-2\alpha,-1]$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α and $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike functions. For $0 < \eta \leq 1$, the class $\mathcal{S}^*(((1+z)/(1-z))^\eta)$ is the class $\mathcal{SS}^*(\eta)$ of strongly starlike function of order η. For $\eta > 0$, the class $\mathcal{S}^*((1+z)^\eta)$ is the class $\mathcal{SL}(\eta)$; the class $\mathcal{SL} := \mathcal{SL}(1/2)$ was introduced by Sokół and Stankiewicz [35] and studied recently by Ali *et al.* [1].

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the Dziok-Srivastava operator [11, 36] $H_p^{l,m}[\alpha_1] = H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by

$$H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \ldots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \ldots (\beta_m)_{n-p}} \frac{a_n z^n}{(n-p)!}, \tag{1.1}$$

where $(a)_n$ is the Pochhammer symbol defined by $(a)_n := \Gamma(a+n)/\Gamma(a)$. Several interesting properties of the classes defined by Dziok-Srivastava operator or its various particular cases including the Hohlov operator [12], the Carlson-Shaffer operator (*cf.* [7, 18]), the Ruscheweyh derivatives [31], the generalized Bernardi-Libera-Livingston integral operator (*cf.* [4, 16, 19]) and the Srivastava-Owa fractional derivative operators (*cf.* [25, 26]), rests on the following identity:

$$z(H_p^{l,m}[\alpha_1]f(z))' = \alpha_1 H_p^{l,m}[\alpha_1+1]f(z) - (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z). \qquad (1.2)$$

The multiplier transformation $I_p(r,\lambda)$ on $\mathcal{A}_p$, introduced by Sivaprasad Kumar et al. [33] and investigated in [2, 3, 34], defined by the following infinite series

$$I_p(r,\lambda)f(z) := z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right)^r a_n z^n \quad (\lambda \in \mathbb{C} \setminus \{-1,-2,\ldots\}), \qquad (1.3)$$

satisfies the identity:

$$z(I_p(r,\lambda)f(z))' = (p+\lambda)I_p(r+1,\lambda)f(z) - \lambda I_p(r,\lambda)f(z). \qquad (1.4)$$

The operator $I_p(r,\lambda)$ is closely related to the Sălăgean derivative operators [32]. The operator $I_\lambda^r := I_1(r,\lambda)$ was studied by Cho and Srivastava [9] and Cho and Kim [10]. The operator $I_r := I_1(r,1)$ was studied by Uralegaddi and Somanatha [37]. Several other operators investigated recently also satisfies a relation similar to the relations (1.2) and (1.4). Notable among them are the operators introduced by Al-Kharasani and Al-Areefi [3] which includes the operators defined in [15], [23] and [22] as well as the Jung-Kim-Srivastava operator [14] and its p-valent analogue of Liu [17].

In the following definition, all these operators investigated one by one are unified.

Definition 1.1. *Let $\mathcal{O}_p$ be the class of all linear operators L_p^a defined on $\mathcal{A}_p$ satisfying*

$$z[L_p^a f(z)]' = \alpha_a L_p^{a+1} f(z) - (\alpha_a - p)L_p^a f(z).$$

One can also consider operators satisfying $z[L_p^b f(z)]' = \alpha_b L_p^{b-1} f(z) - (\alpha_b - p)L_p^b f(z)$ but their properties are very similar to the operators in the above definition. In the following sections, several subordination and superordination theorems as well as corresponding sandwich theorems are proved. A related integral transform is also discussed. Further several sufficient conditions for functions to belong to the classes $\mathcal{S}$, $\mathcal{S}^*(\alpha)$, $\mathcal{SS}^*(\eta)$ and $\mathcal{SL}$ are investigated. Our results are motivated by recent results of Miller and Mocanu [21] on second order differential superordinations. Their results were later used extensively by Bulboacă [5, 6] to investigate superordination-preserving integral operators as well as by several others [2, 3, 11, 29, 30, 33, 34, 36].

We need the following:

Definition 1.2. *[21, Definition 2, p.817] Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\overline{\mathbb{D}} - E(f)$, where*

$$E(f) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} - E(f)$.

Lemma 1.1 (cf. Miller and Mocanu[20, Theorem 3.4h, p.132])**.** *Let $\psi(z)$ be univalent in the unit disk $\mathbb{D}$ and let ϑ and φ be analytic in a domain $D \supset \psi(\mathbb{D})$ with $\varphi(w) \neq 0$, when $w \in \psi(\mathbb{D})$. Set*

$$Q(z) := z\psi'(z)\varphi(\psi(z)), \quad h(z) := \vartheta(\psi(z)) + Q(z).$$

Suppose that

1. *$Q(z)$ is starlike univalent in $\mathbb{D}$ and*
2. *$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathbb{D}$.*

If $q(z)$ is analytic in $\mathbb{D}$, with $q(0) = \psi(0)$, $q(\mathbb{D}) \subset D$ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)), \tag{1.5}$$

then $q(z) \prec \psi(z)$ and $\psi(z)$ is the best dominant.

Lemma 1.2. *[6, Corollary 3.2, p.289] Let $\psi(z)$ be univalent in the unit disk $\mathbb{D}$ and ϑ and φ be analytic in a domain D containing $\psi(\mathbb{D})$. Suppose that*

1. *$\operatorname{Re}[\vartheta'(\psi(z))/\varphi(\psi(z))] > 0$ for $z \in \mathbb{D}$,*
2. *$Q(z) := z\psi'(z)\varphi(\psi(z))$ is starlike univalent in $\mathbb{D}$.*

If $q(z) \in \mathcal{H}[\psi(0), 1] \cap \mathcal{Q}$, with $q(\mathbb{D}) \subseteq D$, and $\vartheta(q(z)) + zq'(z)\varphi(q(z))$ is univalent in $\mathbb{D}$, then

$$\vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.6}$$

implies $\psi(z) \prec q(z)$ and $\psi(z)$ is the best subordinant.

2. Subordination, Superordination and Sandwich Results

For functions $f, F \in \mathcal{A}_p$, let

$$\Omega^a_{L,\mu,\nu}(f(z)) = \left(\frac{L_p^{a+1}f(z)}{z^p}\right)^\mu \left(\frac{z^p}{L_p^a f(z)}\right)^\nu, \quad \Omega^a_{L,\mu,\nu}(f(z),F(z)) := \frac{\Omega^a_{L,\mu,\nu}(f(z))}{\Omega^a_{L,\mu,\nu}(F(z))}$$

where the powers are principal one, μ and ν are real numbers such that they do not assume the value zero simultaneously.

Theorem 2.1. *Let ψ be convex univalent in $\mathbb{D}$ with $\psi(0) = 1$ and $f \in \mathcal{A}_p$. Let $\alpha_{a+1} \neq 0$, $\operatorname{Re}[\alpha_{a+1}\mu - \alpha_a\nu] \geq 0$. Assume that χ and Φ are respectively defined by*

$$\chi(z) := \frac{1}{\alpha_{a+1}}[(\alpha_{a+1}\mu - \alpha_a\nu)\psi(z) + z\psi'(z)] \tag{2.1}$$

and

$$\Phi(z) := \Omega^a_{L,\mu,\nu}(f(z))\Upsilon_L(z), \tag{2.2}$$

where

$$\Upsilon_L(z) := \mu\Omega^{a+1}_{L,1,1}(f(z)) - \frac{\alpha_a\nu}{\alpha_{a+1}}\Omega^a_{L,1,1}(f(z)).$$

1. *If $\Phi(z) \prec \chi(z)$, then*

   $$\Omega^a_{L,\mu,\nu}(f(z)) \prec \psi(z)$$

   and $\psi(z)$ is the best dominant.

2. *If $\chi(z) \prec \Phi(z)$,*

$$0 \neq \Omega^a_{L,\mu,\nu}(f(z)) \in \mathcal{H}[1,1] \cap \mathcal{Q} \text{ and } \Phi(z) \text{ is univalent in } \mathbb{D}, \tag{2.3}$$

then

$$\psi(z) \prec \Omega^a_{L,\mu,\nu}(f(z))$$

and $\psi(z)$ is the best subordinant.

Proof. Define the function q by

$$q(z) := \Omega^a_{L,\mu,\nu}(f(z)), \tag{2.4}$$

where the branch of $q(z)$ is so chosen such that $q(0)=1$. Then $q(z)$ is analytic in $\mathbb{D}$. By a simple computation, we find from (2.4) that

$$\begin{aligned}\frac{zq'(z)}{q(z)} &= \frac{z[\Omega^a_{L,\mu,\nu}(f(z))]'}{\Omega^a_{L,\mu,\nu}(f(z))}\\ &= \mu\frac{z(L_p^{a+1}f(z))'}{L_p^{a+1}f(z)} - \nu\frac{z(L_p^a f(z))'}{L_p^a f(z)} + p(\nu-\mu).\end{aligned} \tag{2.5}$$

By making use of the identity

$$z(L_p^a f(z))' = \alpha_a L_p^{a+1}f(z) - (\alpha_a - p)L_p^a f(z), \tag{2.6}$$

in (2.5), we have

$$\begin{aligned}\Omega^a_{L,\mu,\nu}(f(z))&\left(\mu\Omega^{a+1}_{L,1,1}(f(z)) - \frac{\alpha_a\nu}{\alpha_{a+1}}\Omega^a_{L,1,1}(f(z))\right)\\ &= \frac{1}{\alpha_{a+1}}[(\alpha_{a+1}\mu - \alpha_a\nu)q(z) + zq'(z)].\end{aligned} \tag{2.7}$$

In view of (2.7), the subordination $\Phi(z) \prec \chi(z)$ becomes

$$(\alpha_{a+1}\mu - \alpha_a\nu)q(z) + zq'(z) \prec (\alpha_{a+1}\mu - \alpha_a\nu)\psi(z) + z\psi'(z)$$

and this can be written as (1.5), by defining

$$\vartheta(w) := (\alpha_{a+1}\mu - \alpha_a\nu)w \text{ and } \varphi(w) := 1.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w),\ \varphi(w)$ are analytic in $\mathbb{C} - \{0\}$. Set

$$\begin{aligned}Q(z) &:= z\psi'(z)\\ h(z) &:= \vartheta(\psi(z)) + Q(z) = (\alpha_{a+1}\mu - \alpha_a\nu)\psi(z) + z\psi'(z).\end{aligned}$$

In light of the hypothesis of our Theorem 2.1, we see that $Q(z)$ is starlike and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\alpha_{a+1}\mu - \alpha_a\nu + 1 + \frac{z\psi''(z)}{\psi'(z)}\right) > 0.$$

By an application of Lemma 1.1, we obtain that $q(z) \prec \psi(z)$ or

$$\Omega^a_{L,\mu,\nu}(f(z)) \prec \psi(z).$$

The second half of Theorem 2.1 follows by a similar application of Lemma 1.2. □

Using Theorem 2.1, we obtain the following "sandwich result".

Corollary 2.1. *Let* ψ_j $(j = 1, 2)$ *be convex univalent in* $\mathbb{D}$ *with* $\psi_j(0) = 1$. *Assume that* $\operatorname{Re}[\alpha_{a+1}\mu - \alpha_a\nu] \geq 0$ *and* Φ *be as defined in (2.2). Further assume that*

$$\chi_j(z) := \frac{1}{\alpha_{a+1}}\left[(\alpha_{a+1}\mu - \alpha_a\nu)\psi_j(z) + z\psi_j'(z)\right].$$

If (2.3) holds and $\chi_1(z) \prec \Phi(z) \prec \chi_2(z)$, *then*

$$\psi_1(z) \prec \Omega^a_{L,\mu,\nu}(f(z)) \prec \psi_2(z).$$

Theorem 2.2. *Let* ψ *be convex univalent in* $\mathbb{D}$ *with* $\psi(0) = 1$ *and* α_a *be a complex number. Assume that* $\operatorname{Re}(\mu\alpha_{a+1} - \nu\alpha_a) \geq 0$ *and* $f \in \mathcal{A}_p$. *Define the functions* F, χ *and* Ψ *respectively by*

$$F(z) := \frac{\alpha_a}{z^{\alpha_a - p}}\int_0^z t^{\alpha_a - p - 1} f(t)dt, \tag{2.8}$$

$$\chi(z) := (\mu\alpha_{a+1} - \nu\alpha_a)\psi(z) + z\psi'(z) \tag{2.9}$$

and

$$\Psi(z) := \Omega^a_{L,\mu,\nu}(F(z))\left[\mu\alpha_{a+1}\Omega^a_{L,1,0}(f(z), F(z)) - \nu\alpha_a\Omega^a_{L,0,-1}(f(z), F(z))\right]. \tag{2.10}$$

1. *If* $\Psi(z) \prec \chi(z)$, *then*
$$\Omega^a_{L,\mu,\nu}(F(z)) \prec \psi(z)$$
and $\psi(z)$ *is the best dominant.*

2. *If* $\chi(z) \prec \Psi(z)$,
$$0 \neq \Omega^a_{L,\mu,\nu}(F(z)) \in \mathcal{H}[1, 1] \cap \mathcal{Q} \text{ and } \Psi(z) \text{ is univalent in } \mathbb{D}, \tag{2.11}$$
then
$$\psi(z) \prec \Omega^a_{L,\mu,\nu}(F(z))$$

and $\psi(z)$ *is the best subordinant.*

Proof. From the definition of F, we obtain that

$$\alpha_a f(z) = (\alpha_a - p)F(z) + zF'(z). \tag{2.12}$$

By convoluting (2.12) with $\mathscr{L}_a(z)$, where

$$L_p^a(f(z)) = \mathscr{L}_a(z) * f(z)$$

and using the fact that $z(f*g)'(z) = f(z) * zg'(z)$, we obtain

$$\alpha_a L_p^a(f(z)) = (\alpha_a - p)L_p^a(F(z)) + z(L_p^a(F(z)))'. \tag{2.13}$$

Define the function q by

$$q(z) := \Omega^a_{L,\mu,\nu}(F(z)), \tag{2.14}$$

where the branch of $q(z)$ is so chosen such that $q(0) = 1$. Clearly $q(z)$ is analytic in $\mathbb{D}$. Using (2.13) and (2.14), we have

$$\begin{aligned}\Omega^a_{L,\mu,\nu}(F(z))\left(\mu\alpha_{a+1}\Omega^a_{L,1,0}(f(z),F(z)) - \nu\alpha_a\Omega^a_{L,0,-1}(f(z),F(z))\right)&\\ = (\mu\alpha_{a+1} - \nu\alpha_a)q(z) + zq'(z).&\end{aligned} \tag{2.15}$$

Using(2.15), the subordination $\Psi(z) \prec \chi(z)$ becomes

$$(\mu\alpha_{a+1} - \nu\alpha_a)q(z) + zq'(z) \prec (\mu\alpha_{a+1} - \nu\alpha_a)\psi(z) + z\psi'(z)$$

and this can be written as (1.5), by defining

$$\vartheta(w) := (\mu\alpha_{a+1} - \nu\alpha_a)\psi(z) \text{ and } \varphi(w) := 1.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w),\ \varphi(w)$ are analytic in $\mathbb{C} - \{0\}$. Set

$$\begin{aligned} Q(z) &:= z\psi'(z) \\ h(z) &:= \vartheta(\psi(z)) + Q(z) = (\mu\alpha_{a+1} - \nu\alpha_a)\psi(z) + z\psi'(z). \end{aligned}$$

In light of the assumption of our Theorem 2.2, we see that $Q(z)$ is starlike and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\mu\alpha_{a+1} - \nu\alpha_a + 1 + \frac{z\psi''(z)}{\psi'(z)}\right) > 0.$$

An application of Lemma 1.1, gives $q(z) \prec \psi(z)$ or

$$\Omega^a_{L,\mu,\nu}(F(z)) \prec \psi(z).$$

By an application of Lemma 1.2 the proof of the second half of Theorem 2.2 follows at once. □

As a consequence of Theorem 2.2, we obtain the following "sandwich result".

Corollary 2.2. *Let* ψ_j $(j=1,2)$ *be convex univalent in* $\mathbb{D}$ *with* $\psi_j(0)=1$ *and* α_a *be a complex number. Further assume that* $\mathrm{Re}(\mu\alpha_{a+1}-\nu\alpha_a)\geq 0$ *and* Ψ *be as defined in (2.10). If (2.11) holds and* $\chi_1(z)\prec\Psi(z)\prec\chi_2(z)$*, then*

$$\psi_1(z)\prec\Omega^a_{L,\mu,\nu}(F(z))\prec\psi_2(z),$$

where

$$\chi_j(z):=(\mu\alpha_{a+1}-\nu\alpha_a)\psi_j(z)+z\psi_j'(z)\quad(j=1,2)$$

and F *is defined by (2.8).*

Theorem 2.3. *Let* ϕ *be analytic in* $\mathbb{D}$ *with* $\phi(0)=1$ *and* α_a *is independent of* a*. If* $f\in\mathcal{A}_p$*, then*

$$\Omega^a_{L,\mu,\nu}(f(z))\prec\phi(z)\Leftrightarrow\Omega^{a+1}_{L,\mu,\nu}(F(z))\prec\phi(z).$$

Further

$$\phi(z)\prec\Omega^a_{L,\mu,\nu}(f(z))\Leftrightarrow\phi(z)\prec\Omega^{a+1}_{L,\mu,\nu}(F(z)),$$

where F *is defined by (2.8).*

Proof. Using the following identity

$$z[L^a_p(f(z))]'=\alpha_aL^{a+1}_p(f(z))-(\alpha_a-p)L^a_p(f(z))$$

in (2.13), we get

$$L^a_p(f(z))=L^{a+1}_p(F(z)). \tag{2.16}$$

Since α_a is independent of a, $\alpha_{a+1}=\alpha_a$, we have

$$\begin{aligned}\alpha_aL^{a+1}_p(f(z)) &= z(L^a_p(f(z)))'+(\alpha_a-p)L^a_p(f(z))\\ &= z(L^{a+1}_p(F(z)))'+(\alpha_a-p)L^{a+1}_p(F(z))\\ &= \alpha_{a+1}L^{a+2}_p(F(z)). \end{aligned}\tag{2.17}$$

Therefore, from (2.16) and (2.17), we have

$$\Omega^{a+1}_{L,\mu,\nu}(F(z))=\Omega^a_{L,\mu,\nu}(f(z))$$

and hence the result follows at once. □

Now we will use Theorem 2.3 to state the following "sandwich result".

Corollary 2.3. *Let $f \in \mathcal{A}_p$ and α_a is independent of a. Let ϕ_i $(i = 1, 2)$ be analytic in $\mathbb{D}$ with $\phi_i(0) = 1$ and F is defined by (2.8). Then*

$$\phi_1(z) \prec \Omega^{a}_{L,\mu,\nu}(f(z)) \prec \phi_2(z)$$

if and only if

$$\phi_1(z) \prec \Omega^{a+1}_{L,\mu,\nu}(F(z)) \prec \phi_2(z).$$

3. Applications

We begin with some interesting applications of subordination part of Theorem 2.1 for the case when $L = H$, the Dziok Srivastava Operator. Note that the subordination part of Theorem 2.1 holds even if we assume

$$\operatorname{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > \max\{0, \operatorname{Re}[\alpha_1(\nu - \mu) - \mu]\}$$

instead of "$\psi(z)$ is convex and $\operatorname{Re}[\alpha_1(\mu - \nu) + \mu] \geq 0$" and leads to the following corollary to the first part of Theorem 2.1 by taking $\psi(z) = (1 + Az)/(1 + Bz)$.

Corollary 3.1. *Let $-1 < B < A \leq 1$ and $\operatorname{Re}(u - vB) \geq |v - \bar{u}B|$ where $u = \alpha_1(\mu - \nu) + \mu + 1$ and $v = [\alpha_1(\mu - \nu) + \mu - 1]B$. If $f \in \mathcal{A}_p$ satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \left(\mu\Omega^{\alpha_1+1}_{H,1,1}(f(z)) - \frac{\alpha_1 \nu}{\alpha_1 + 1}\Omega^{\alpha_1}_{H,1,1}(f(z))\right)$$
$$\prec \frac{1}{\alpha_1 + 1}\left([\alpha_1(\mu - \nu) + \mu]\frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2}\right) \quad (\alpha_1 \neq -1),$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \prec \frac{1 + Az}{1 + Bz}$$

and $(1 + Az)/(1 + Bz)$ is the best dominant.

Proof. Let

$$\psi(z)=\frac{1+Az}{1+Bz}\quad(-1<B<A\leq 1), \tag{3.1}$$

then clearly $\psi(z)$ is univalent and $\psi(0)=1$. Upon logarithmic differentiation of ψ given by (3.1), we obtain that

$$z\psi'(z)=\frac{(A-B)z}{(1+Bz)^2}. \tag{3.2}$$

Another differentiation of (3.2), yields

$$1+\frac{z\psi''(z)}{\psi'(z)}=\frac{1-Bz}{1+Bz}.$$

If $z=re^{i\theta}$, $0\leq r<1$, then we have

$$\mathrm{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)=\frac{1-B^2r^2}{1+B^2r^2+2Br\cos\theta}\geq 0.$$

Hence $\psi(z)$ is convex in $\mathbb{D}$. Also it follows that

$$\begin{aligned}[\alpha_1(\mu-\nu)+\mu]+1+\frac{z\psi''(z)}{\psi'(z)} &= \frac{[\alpha_1(\mu-\nu)+\mu+1]+[\alpha_1(\mu-\nu)+\mu-1]Bz}{1+Bz}\\ &= \frac{u+vz}{1+Bz},\end{aligned}$$

where $u=\alpha_1(\mu-\nu)+\mu+1$ and $v=[\alpha_1(\mu-\nu)+\mu-1]\,B$. The function $w(z)=\frac{u+vz}{1+Bz}$ maps $\mathbb{D}$ into the disk

$$\left|w-\frac{\bar{u}-\bar{v}B}{1-B^2}\right|\leq\frac{|v-\bar{u}B|}{1-B^2}.$$

Which implies that

$$\mathrm{Re}\left([\alpha_1(\mu-\nu)+\mu]+1+\frac{z\psi''(z)}{\psi'(z)}\right)\geq\frac{\mathrm{Re}(\bar{u}-\bar{v}B)-|v-\bar{u}B|}{1-B^2}\geq 0$$

provided

$$\mathrm{Re}(\bar{u}-\bar{v}B)\geq|v-\bar{u}B|$$

or

$$\mathrm{Re}(u-vB)\geq|v-\bar{u}B|.$$

Thus the result follows at once by an application of the first part of Theorem 2.1. □

Corollary 3.2. *Let* $0 \leq \alpha < 1$ *and* $\operatorname{Re}(\alpha_1(\mu-\nu)+\mu) \geq 0$. *If*

$$\Omega_{H,\mu,\nu}^{\alpha_1}(f(z))\left(\mu\Omega_{H,1,1}^{\alpha_1+1}(f(z)) - \frac{\alpha_1\nu}{\alpha_1+1}\Omega_{H,1,1}^{\alpha_1}(f(z))\right) \prec \frac{1}{\alpha_1+1}\left((\alpha_1(\mu-\nu)+\mu)\frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)z}{(1-z)^2}\right) \quad (\alpha_1 \neq -1),$$

then

$$\operatorname{Re}\Omega_{H,\mu,\nu}^{\alpha_1}(f(z)) > \alpha.$$

Proof. Let

$$\psi(z) = \frac{1+(1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1),$$

then obviously $\psi(z)$ is univalent and $\psi(0) = 1$. By a simple calculation, we have

$$1 + \frac{z\psi''(z)}{\psi'(z)} = \frac{1+z}{1-z},$$

which clearly indicates that $\psi(z)$ is convex. If we assume $\beta = \alpha_1(\mu-\nu)+\mu$ then by hypothesis we have $\operatorname{Re}\beta \geq 0$. So if we take

$$w(z) = \beta + \frac{1+z}{1-z} = \frac{(1+\beta)+(1-\beta)z}{1-z},$$

then $w(z)$ maps the unit disc $\mathbb{D}$ on to $\operatorname{Re} w > \operatorname{Re}\beta \geq 0$. The result now follows by an application of the subordination part of Theorem 2.1. □

Note that if $p = 1, l = m+1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ..., m)$, then $H_1[1]f(z) = f(z), H_1[2]f(z) = zf'(z)$ and $H_1[3]f(z) = \frac{1}{2}z^2f''(z) + zf'(z)$. Putting $\alpha = 1$, $p = 1, l = m+1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ..., m)$ in Corollary 3.2, we obtain the following.

Corollary 3.3. *Let* $0 \leq \alpha < 1$ *and* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\operatorname{Re}\left((f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right) - \nu\frac{zf'(z)}{f(z)}\right)\right) > \frac{2(2\mu-\nu)\alpha-(1-\alpha)}{2},$$

then

$$\operatorname{Re}\left((f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right) > \alpha.$$

Proof. From Corollary 3.2, we see that

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right)$$
$$\prec (2\mu-\nu)\frac{1+(1-2\alpha)z}{1-z}+\frac{2(1-\alpha)z}{(1-z)^2} =: h(z).$$

We now investigate the image of $h(\mathbb{D})$. Assuming $a = 1-2\alpha$ and $b = 2\mu-\nu$, we have

$$h(z) = \frac{b+(1+a-b+ab)z-abz^2}{(1-z)^2},$$

where $h(0) = b$ and $h(-1) = [2b(1-a)-(1+a)]/4$. The boundary curve of the image of $h(\mathbb{D})$ is given by $h(e^{i\theta}) = u(\theta)+iv(\theta)$, $-\pi < \theta < \pi$, where

$$u(\theta) = \frac{(1+a-b+ab)+(1-a)b\cos\theta}{2(\cos\theta-1)} \quad \text{and} \quad v(\theta) = \frac{(1+a)b\sin\theta}{2(1-\cos\theta)}.$$

By eliminating θ, we obtain the equation of the boundary curve as

$$v^2 = -b^2(1+a)\left(u-\frac{2b(1-a)-(a+1)}{4}\right). \tag{3.3}$$

Obviously (3.3) represents a parabola opening towards the left, with the vertex at the point $\left(\frac{2b(1-a)-(a+1)}{4},0\right)$ and negative real axis as its axis. Hence $h(\mathbb{D})$ is the exterior of the parabola (3.3) which includes the right half plane

$$u > \frac{2b(1-a)-(a+1)}{4}.$$

Hence the result follows at once. □

Setting $\mu = 0$ and $\nu = -1$ in Corollary 3.3, we obtain the following result.

Example 3.1. *Let* $0 \le \alpha < 1$. *If* $f \in \mathcal{A}$ *and* $\operatorname{Re} f'(z) > \frac{3\alpha-1}{2}$, *then* $f \in \mathcal{T}(\alpha)$.

Remark 3.1. *The above Example 3.1 reduces to [24, Theorem 2] when* $\alpha = 1/3$.

If we take $\psi(z) = ((1+z)/(1-z))^{\eta}$ with $0 < \eta \le 1$ in Theorem 2.1 for the case $L = H$, the Dziok Srivastava operator, then clearly $\psi(z)$ is convex in $\mathbb{D}$ and consequently corresponding to the subordination part of the Theorem 2.1, we have the following.

Corollary 3.4. *Let* $0 < \eta \leq 1$, $\alpha_1 \neq -1$ *and* $\operatorname{Re}(\alpha_1(\mu-\nu)+\mu) \geq 0$. *If* $f \in \mathcal{A}_p$ *and satisfies*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z))\left(\mu\Omega^{\alpha_1+1}_{H,1,1}(f(z)) - \frac{\alpha_1\nu}{\alpha_1+1}\Omega^{\alpha_1}_{H,1,1}(f(z))\right) \prec \frac{1}{\alpha_1+1}\left((\alpha_1(\mu-\nu)+\mu) + \frac{2\eta z}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\eta},$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \prec \left(\frac{1+z}{1-z}\right)^{\eta}$$

and $((1+z)/(1-z))^{\eta}$ *is the best dominant.*

By taking $p = 1, l = m+1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$, in the above Corollary 3.4, we have the following:

Corollary 3.5. *Let* $0 < \eta \leq 1$ *and* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right) - \nu\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\delta\pi}{2},$$

then

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}$$

where

$$\delta = \eta + 1 - \frac{2}{\pi}\arctan\frac{2\mu-\nu}{\eta}.$$

Proof. In view of the Corollary 3.4, we have

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right) - \nu\frac{zf'(z)}{f(z)}\right) \prec \left((2\mu-\nu) + \frac{2\eta z}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\eta} =: h(z)$$

implies

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu} \prec \left(\frac{1+z}{1-z}\right)^{\eta}$$

or

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2} \quad (z \in \mathbb{D}).$$

Now we need to find the minimum value of $\arg h(\mathbb{D})$. Let $z = e^{i\theta}$. Since $h(\mathbb{D})$ is symmetrical about the real axis, we shall restrict ourself to $0 < \theta \leq \pi$. Setting $t = \cot\theta/2$, we have $t \geq 0$ and for $z = \frac{it-1}{it+1}$, we arrive at

$$\begin{aligned} h(e^{i\theta}) &= (it)^{\eta-1}\left[(2\mu-\nu)it - \frac{\eta(1+t^2)}{2}\right] \\ &= (it)^{\eta-1}G(t), \end{aligned}$$

where

$$G(t) = \left[(2\mu-\nu)it - \frac{\eta(1+t^2)}{2}\right].$$

Let $G(t) = U(t) + iV(t)$, where $U(t) = -\frac{\eta(1+t^2)}{2}$ and $V(t) = (2\mu-\nu)t$, there arises two cases namely $2\mu > \nu$ and $2\mu = \nu$. If $2\mu > \nu$, then a calculation shows that $\min_{t\geq 0}\arg G(t)$ occurs at $t = 1$ and

$$\min_{t\geq 0}\arg G(t) = \pi - \arctan\frac{2\mu-\nu}{\eta}.$$

Thus

$$\min_{|z|<1}\arg h(z) = \frac{(\eta+1)\pi}{2} - \arctan\frac{2\mu-\nu}{\eta}.$$

If $2\mu = \nu$, then $\arg G(t) = \pi$ and $\min_{|z|<1}\arg h(z) = (\eta+1)\pi/2$. Thus for $2\mu \geq \nu$, we have

$$\begin{aligned} \min_{|z|<1}\arg h(z) &= \min\left\{\frac{(\eta+1)\pi}{2}, \frac{(\eta+1)\pi}{2} - \arctan\frac{2\mu-\nu}{\eta}\right\} \\ &= \frac{(\eta+1)\pi}{2} - \arctan\frac{2\mu-\nu}{\eta}. \end{aligned}$$

This completes the proof of the corollary. □

We now enlist a few applications of Theorem 2.1 for the operator $L = H$, the Dziok Srivastava operator, by taking $\psi(z) = \sqrt{1+z}$ as dominant. Obviously $\psi(z)$ is a convex function in the open unit disk $\mathbb{D}$ with $\psi(0) = 1$. The subordination part of Theorem 2.1, leads to the following result.

Corollary 3.6. *Let $\alpha_1 \neq -1$ and $\operatorname{Re}[\alpha_1(\mu-\nu)+\mu] \geq 0$. If $f \in \mathcal{A}_p$ and satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \left(\mu\Omega^{\alpha_1+1}_{H,1,1}(f(z)) - \frac{\alpha_1\nu}{\alpha_1+1}\Omega^{\alpha_1}_{H,1,1}(f(z))\right) \prec \frac{1}{\alpha_1+1}\left([\alpha_1(\mu-\nu)+\mu]\sqrt{1+z} + \frac{z}{2\sqrt{1+z}}\right),$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ is the best dominant.

By taking $p = 1, l = m+1$, $\alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$ in Corollary 3.6, we obtain the following result.

Corollary 3.7. *Let $2\mu \geq \nu$. If $f \in \mathcal{A}$ and satisfies the subordination*

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right) - \nu\frac{zf'(z)}{f(z)}\right) \prec (2\mu-\nu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu} \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ is the best dominant.

We obtain the following example from Corollary 3.7.

Example 3.2. *If $f \in \mathcal{A}$ and satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \sqrt{1.22} \approx 1.10,$$

then $f \in \mathcal{SL}$.

Proof. Putting $\mu = \nu = 1$ in Corollary 3.7, we have

$$\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} + \frac{z}{2\sqrt{1+z}} =: h(z),$$

implies

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}.$$

The dominant $h(z)$ can be written as

$$h(z) = \frac{3z+2}{2\sqrt{1+z}}.$$

Writing $h(e^{i\theta}) = u(\theta) + iv(\theta), -\pi < \theta < \pi$, we have

$$u(\theta) = \frac{3\cos(3\theta/4) + 2\cos(\theta/4)}{2\sqrt{2\cos(\theta/2)}}$$

and

$$v(\theta) = \frac{3\sin(3\theta/4) - 2\sin(\theta/4)}{2\sqrt{2\cos(\theta/2)}}.$$

A simple calculation gives

$$u^2(\theta) + v^2(\theta) = \frac{13 + 12\cos\theta}{8\cos(\theta/2)} =: k(\theta).$$

A computation shows that $k(\theta)$ has minimum at $\theta = \arccos(\sqrt{1/24})$ and $k(\theta) \geq \sqrt{3/2} \approx 1.22$. Since $h(0) = 1$ and $h(-1) = -\infty$, by a computation we come to know that the image of $h(\mathbb{D})$ is the interior of the domain bounded by parabola opening towards left which contains the interior of the circle $u^2+v^2 = 1.22$. Hence the result follows at once. $\square$

We now give some interesting applications of Theorem 2.2 for the case $L = H$. Note that if we replace the statement "$\psi(z)$ is convex in the open unit disc $\mathbb{D}$ and $\mathrm{Re}\,[(\mu-\nu)\alpha_1 + \mu] \geq 0$" by

$$\mathrm{Re}\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > \max\{0, \mathrm{Re}\,[(\nu-\mu)\alpha_1 - \mu]\}$$

in the hypothesis of Theorem 2.2 still the subordination part of the result holds so we obtain the following corollary as a straight forward consequence to the first part of Theorem 2.2 by taking $\psi(z) = (1+(1-2\alpha)z)/(1-z), 0 \leq \alpha < 1$.

Corollary 3.8. *Let* $0 \leq \alpha < 1$ *and* $\mathrm{Re}[(\mu-\nu)\alpha_1+\mu] \geq 0$. *If* $f \in \mathcal{A}_p$, F *as defined in (2.8) and*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z))(\mu(\alpha_1+1)\Omega^{\alpha_1}_{H,1,0}(f(z),F(z)) - \nu\alpha_1\Omega^{\alpha_1}_{H,0,-1}(f(z),F(z))) \prec ((\mu-\nu)\alpha_1+\mu)\frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)z}{(1-z)^2}$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z)) \prec \frac{1+(1-2\alpha)z}{1-z}$$

and $(1+(1-2\alpha)z)/(1-z)$ *is the best dominant.*

Putting $p=1, l=m+1, \alpha_1=1$ and $\alpha_{i+1}=\beta_i$ $(i=1,2,...m)$ in Corollary 3.8, we obtain the following result:

Corollary 3.9. *Let* $0 \leq \alpha < 1$ *and* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$, F *as defined in (2.8) and*

$$\mathrm{Re}\left\{(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right)\right\} < \frac{2(2\mu-\nu)\alpha-(1-\alpha)}{2},$$

then

$$\mathrm{Re}\left[(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\right] > \alpha.$$

Proof. From Corollary 3.8, we see that

$$(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right) \prec (2\mu-\nu)\frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)z}{(1-z)^2} =: h(z) \quad (3.4)$$

implies

$$\mathrm{Re}\left[(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\right] > \alpha.$$

Let $z = e^{i\theta}, -\pi \leq \theta \leq \pi$. Then

$$\begin{aligned}\mathrm{Re}(h(e^{i\theta})) &= \mathrm{Re}\left\{(2\mu-\nu)\frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}} + \frac{2(1-\alpha)e^{i\theta}}{(1-e^{i\theta})^2}\right\} \\ &= (2\mu-\nu)\alpha - \frac{(1-\alpha)}{2}\left(\frac{1}{\sin^2(\theta/2)}\right) =: k(\theta).\end{aligned}$$

A calculation shows that $k(\theta)$ attains its maximum at $\theta = \pi$ and

$$\max_{|\theta|\le\pi} k(\theta) = \frac{2(2\mu-\nu)\alpha-(1-\alpha)}{2}.$$

Hence the result follows at once. □

By taking $\psi(z) = ((1+z)/(1-z))^\eta$ in the subordination part of Theorem 2.2 for the case $L = H$, the Dzoik Srivastava operator, we have the following result.

Corollary 3.10. *Let* $0 < \eta \le 1$ *and* $\mathrm{Re}[(\mu-\nu)\alpha_1 + \mu] \ge 0$. *If* $f \in \mathcal{A}_p$, F *as defined in (2.8) and satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z))\left((\alpha_1+1)\mu\Omega^{\alpha_1}_{H,1,0}(f(z),F(z)) - \nu\alpha_1\Omega^{\alpha_1}_{H,0,-1}(f(z),F(z))\right) \prec \left(((\mu-\nu)\alpha_1+\mu) + \frac{2\eta z}{(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^\eta,$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z)) \prec \left(\frac{1+z}{1-z}\right)^\eta$$

and $((1+z)/(1-z))^\eta$ *is the best dominant.*

By putting $p = 1, l = m+1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$ in the above Corollary 3.10, we obtain the following result.

Corollary 3.11. *Let* $0 < \eta \le 1$ *and* $2\mu \ge \nu$. *If* $f \in \mathcal{A}$, F *as defined in (2.8) and*

$$\left|\arg\left\{(F'(z))^\mu\left(\frac{z}{F(z)}\right)^\nu\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right)\right\}\right| < \frac{(\eta+1)\pi}{2} - \arctan\frac{(2\mu-\nu)}{\eta},$$

then

$$\left|\arg\left\{(F'(z))^\mu\left(\frac{z}{F(z)}\right)^\nu\right\}\right| < \frac{\eta\pi}{2}.$$

Proof. The proof of the above Corollary 3.11 is similar to that of the Corollary 3.5 hence it is skipped here. □

Taking the dominant $\psi(z) = \sqrt{1+z}$, which is a convex function in the open unit disc $\mathbb{D}$, in the subordination part of Theorem 2.2, we have the following corollary for the Dzoik Srivastava operator $H = L$.

Corollary 3.12. *Let* $0 < \eta \leq 1$ *and* $\mathrm{Re}[(\alpha_1(\mu-\nu)+\mu] \geq 0$. *If* $f \in \mathcal{A}_p$, F *as defined in (2.8) and*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z))\left((\alpha_1+1)\mu\Omega^{\alpha_1}_{H,1,0}(f(z),F(z)) - \alpha_1\nu\Omega^{\alpha_1}_{H,0,-1}(f(z),F(z))\right) \prec (\alpha_1(\mu-\nu)+\mu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z)) \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ *is the best dominant.*

Putting $p=1, l=m+1, \alpha_1=1$ and $\alpha_{i+1}=\beta_i$ $(i=1,2,...m)$ in Corollary 3.12, we obtain the following result.

Corollary 3.13. *Let* $0 < \eta \leq 1$ *and* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$, F *as defined in (2.8) and*

$$(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right) \prec (2\mu-\nu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then

$$(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu} \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ *is the best dominant.*

Putting $\mu=\nu=1$ in the above Corollary 3.13, we have the following example.

Example 3.3. *Let* $0 < \eta \leq 1$. *If* $f \in \mathcal{A}$, F *as defined in (2.8) and*

$$\left|\frac{zF'(z)}{F(z)}\left(2\frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)}\right)\right| < \sqrt{1.22} \approx 1.10,$$

then $F \in \mathcal{SL}$.

Proof. The above result can be proved using the technique adopted in the proof of Example 3.2 and hence it is omitted here. □

Next we discuss some applications of Theorem 2.1 when $L = I$, the multiplier transformation. In Theorem 2.1, the subordination part yields the following corollary by taking $\psi(z) = (1+(1+2\alpha)z)/(1-z), 0 \leq \alpha < 1$ and

$$\mathrm{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right) > \max\{0, \mathrm{Re}[(\nu-\mu)(\lambda+p)]\}$$

in place of "ψ is convex and $\mathrm{Re}[(\mu-\nu)(\lambda+p)] \geq 0$".

Corollary 3.14. *Let* $0 \leq \alpha < 1$, $\lambda \neq -p$ *be a complex number and* $\mathrm{Re}[(\mu-\nu)(\lambda+p)] \geq 0$. *If* $f \in \mathcal{A}_p$ *and*

$$\Omega^r_{I,\mu,\nu}(f(z))\left(\mu\Omega^{r+1}_{I,1,1}(f(z)) - \nu\Omega^r_{I,1,1}(f(z))\right) \prec (\mu-\nu)\frac{1+(1-2\alpha)z}{1-z} + \frac{1}{\lambda+p}\frac{2(1-\alpha)z}{(1-z)^2},$$

then

$$\Omega^r_{I,\mu,\nu}(f(z)) \prec \frac{1+(1-2\alpha)z}{1-z}$$

and $(1+(1-2\alpha)z)/(1-z)$ *is the best dominant.*

Note that for $p=1, \lambda=0$ and $r=0$, we have $I_1(0,0)f(z) = f(z), I_1(1,0)f(z) = zf'(z), I_1(2,0)f(z) = z(zf''(z)+f'(z))$. Putting these values in Corollary 3.14, we have the following result.

Corollary 3.15. *Let* $0 \leq \alpha < 1$ *and* $\mu \geq \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\mathrm{Re}\left[(f'(z))^\mu\left(\frac{z}{f(z)}\right)^\nu\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right)\right] > \frac{2(\mu-\nu)\alpha-(1-\alpha)}{2},$$

then

$$\mathrm{Re}\left[(f'(z))^\mu\left(\frac{z}{f(z)}\right)^\nu\right] > \alpha.$$

Proof. The proof is similar to that of the Corollary 3.3 and hence omitted here. □

Setting $\mu = \nu = 1$ in Corollary 3.15, we have the following result:

Example 3.4. *Let* $0 \le \alpha < 1$. *If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)\right] > \frac{\alpha-1}{2},$$

then $f \in S^*(\alpha)$.

Remark 3.2. *For* $\alpha = 0$, *the above asserted Example 3.4 reduces to a result obtained by Owa and Obradović [27, Corollary 2].*

Putting $\mu = 1$ and $\nu = 0$ in Corollary 3.15, we have the following result:

Example 3.5. *Let* $0 \le \alpha < 1$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\operatorname{Re}[f'(z) + zf''(z)] > \frac{3\alpha - 1}{2},$$

then $\operatorname{Re} f'(z) > \alpha$.

Remark 3.3. *The above Example 3.5 extends the result [8, Theorem 5] due to Chichra. Further corollary 3.15 reduces to [24, Theorem 2] when* $\mu = 0, \nu = -1$ *and* $\alpha = 1/3$.

If we take $\psi(z) = ((1+z)/(1-z))^\eta$ *with* $0 < \eta \le 1$, *for the case* $L = I$, *then clearly* $\psi(z)$ *is convex in the open unit disc* $\mathbb{D}$ *and we have the following corollary from the subordination part of Theorem 2.1.*

Corollary 3.16. *Let* $0 < \eta \le 1$, $\lambda \ne -p$ *be a complex number and* $\operatorname{Re}[(\mu - \nu)(\lambda + p)] \ge 0$. *If* $f \in \mathcal{A}_p$, *and satisfies the subordination*

$$\Omega^r_{I,\mu,\nu}(f(z))\left(\mu\Omega^{r+1}_{I,1,1}(f(z)) - \nu\Omega^r_{I,1,1}(f(z))\right) \prec \left((\mu-\nu) + \frac{2\eta z}{(\lambda+p)(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^\eta,$$

then

$$\Omega^r_{I,\mu,\nu}(f(z)) \prec \left(\frac{1+z}{1-z}\right)^\eta$$

and $\left(\frac{1+z}{1-z}\right)^\eta$ *is the best dominant.*

Putting $p = 1, \lambda = 0$ and $r = 0$ in Corollary 3.16, we obtain the following corollary.

Corollary 3.17. *Let $0 < \eta \leq 1$ and $\mu \geq \nu$. If $f \in \mathcal{A}$ and satisfies*

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\delta\pi}{2},$$

where

$$\delta = \eta + 1 - \frac{2}{\pi}\arctan\frac{\mu-\nu}{\eta},$$

then

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}.$$

Proof. The proof is much akin to the proof of Corollary 3.5 hence it is left here. □

Taking $\psi(z) = \sqrt{1+z}$, convex function in the open unit disc $\mathbb{D}$, as dominant in the subordination part of the Theorem 2.1, we obtain the following corollary.

Corollary 3.18. *Let $\lambda \neq -p$ be a complex number and* $\mathrm{Re}[(\mu-\nu)(\lambda+p)] \geq 0$. *If $f \in \mathcal{A}_p$, and satisfies the subordination*

$$\Omega^{r}_{I,\mu,\nu}(f(z))\left(\mu\Omega^{r+1}_{I,1,1}(f(z)) - \nu\Omega^{r}_{I,1,1}(f(z))\right) \prec (\mu-\nu)\sqrt{1+z} + \frac{z}{2(\lambda+p)\sqrt{1+z}},$$

then

$$\Omega^{r}_{I,\mu,\nu}(f(z)) \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ is the best dominant.

Putting $p = 1, \lambda = 0$ and $r = 0$ in Corollary 3.18, we have the following corollary.

Corollary 3.19. *Let $\mu \geq \nu$. If $f \in \mathcal{A}$ and satisfies the subordination*

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right) \prec (\mu-\nu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu} \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ is the best dominant.

Example 3.6. *If $f \in \mathcal{A}$ and satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{1}{2\sqrt{2}} \approx 0.35,$$

then $f \in \mathcal{SL}$.

Proof. Putting $\mu = \nu = 1$ in Corollary 3.19 and using the technique used in the proof of Example 3.2, we get the required result. □

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