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Starlikeness of a Double Integral Operator *

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Abstract

For $\alpha, \gamma \geq 0$, sufficient conditions are obtained that ensure the normalized analytic functions f satisfying a differential inequality

$$\left|(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-1\right|<\lambda,$$

to be starlike of order β in the open unit disc. As an application, we construct new starlike function f of order β which can be expressed in terms of double integral

$$f(z)=\int_0^1\int_0^1 g(r,s,z)drds,$$

of some suitable analytic function in the open unit disc.

Keywords and Phrases: *Differential subordination, Starlike function, Convex function.*

1. Introduction

Let $\mathcal{H}$ denotes the class of all analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$. For a positive integer n and $a \in \mathcal{C}$(Complex plane), define the classes of functions:

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\},$$

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$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\},$$

with $\mathcal{A}_1 = \mathcal{A}$. Let $\mathcal{B}_n$ denote the class of all analytic functions ω, such that $\omega(0) = 0$ and $|\omega(z)| < |z|^n$. Further, denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E. A function $f \in \mathcal{A}$ is said to be starlike of order β in E iff it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in E, \tag{1.1}$$

for some $\beta (0 \leq \beta < 1)$. We denote by $S^*(\beta)$ the subclass of $\mathcal{A}$ consisting of all functions f which are starlike of order β. Set $S^*(0) = S^*$, where S^* is the well-known class of normalized analytic functions starlike with respect to the origin.

Let the functions f and g be analytic in E. We say that f is subordinate to g (in symbols, $f(z) \prec g(z)$) in E, if there exists a Schwarz function ω analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. If the function g is univalent in E, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

In 2003, Fournier et. al. [2] investigated some differential inequalities which imply starlikeness. The study of such differential inequalities has been a constant theme of geometric function theory. In a recent paper, Miller and Mocanu [4] extended some of the results of Fournier et. al [2] and also investigated starlikeness properties of functions f defined by double integral operators of the form

$$f(z) = \int_0^1 \int_0^1 W(r, s, z)drds.$$

In a very recent paper, R. M. Ali et al [1] discussed the starlikeness of a linear integral transform over functions f in the class $\mathcal{W}_\beta(\alpha, \gamma)$

$$\left\{f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - \beta\right) > 0,\, z \in E\right\}. \tag{1.2}$$

Motivated by the definition of the class $\mathcal{W}_\beta(\alpha, \gamma)$, the aim of the present paper is to present a new differential inequality which generates starlike function of order β. As an application of this inequality, we construct new starlike function of order β which can be expressed in terms of double integrals of some suitable function in the class $\mathcal{H}$.

2. Preliminaries

We follow the notations used in [1]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{2.1}$$

When $\alpha = 1 + 2\gamma$, (2.1) yields $\mu + \nu = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$. In particular, for $\gamma > 0$, choosing $\mu = 1$ gives $\nu = \gamma$.
We shall also need the following lemma to prove our results.

Lemma 2.1. *([3], p.71) Let h be a convex function with $h(0) = a$ and let $\Re(\gamma) > 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad in \quad E$$

then

$$p(z) \prec q(z) \prec h(z) \quad in \quad E,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt.$$

This result is sharp.

3. Main Results

Theorem 3.1. *Let μ, ν satisfy (2.1) such that $\mu > 0$ and $\nu > \frac{2}{1-\beta}$ $(0 \leq \beta < 1)$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|(1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - 1\right| < \frac{(1 + n\mu)(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)}, \tag{3.1}$$

for $z \in E$, then $f \in S^*(\beta)$.

Proof. The differential inequality (3.1) can be written as follows:

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z) \prec 1+\frac{(1 + n\mu)(1 + n\nu)(\nu(1 - \beta) - 2)}{\nu(n + 1 - \beta)}z. \tag{3.2}$$

If we set

$$\begin{aligned} p(z) &= (1-\nu)\frac{f(z)}{z} + \nu f'(z), \\ &= 1 + (1+n\nu)a_{n+1}z^n + \cdots, \end{aligned}$$

then $p \in \mathcal{H}[1, n]$ and the subordination (3.2) becomes

$$p(z) + \mu z p'(z) \prec 1 + \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)} z = h(z) \ (say). \quad (3.3)$$

It can be easily seen that h is convex and $h(0) = p(0)$. So, applying Lemma 2.1 (with $\gamma = 1/\mu$), we obtain

$$p(z) \prec \frac{1}{n\mu z^{1/n\mu}} \int_0^z \zeta^{\frac{1}{n\mu}-1} h(\zeta) d\zeta, \quad z \in E.$$

Equivalently

$$(1-\nu)\frac{f(z)}{z} + \nu f'(z) \prec 1 + \frac{(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)} z, \ z \in E. \quad (3.4)$$

Now, if we set

$$q(z) = \frac{f(z)}{z} = 1 + a_{n+1}z^n + \cdots,$$

then $q \in \mathcal{H}[1, n]$ and the subordination (3.4) leads to

$$q(z) + \nu z q'(z) \prec 1 + \frac{(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)} z = h_1(z) \ \text{(say)}.$$

The function h_1 satisfies the conditions of Lemma 2.1. Thus, we obtain

$$\frac{f(z)}{z} = q(z) \prec \frac{1}{n\nu z^{1/n\nu}} \int_0^z t^{\frac{1}{n\nu}-1} h_1(t) dt = 1 + \frac{(\nu(1-\beta)-2)}{\nu(n+1-\beta)} z. \quad (3.5)$$

It follows from the subordination (3.4) that

$$\begin{aligned} \left|(1-\nu)\frac{f(z)}{z} + \nu f'(z)\right| &< 1 + \frac{(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)} \\ &= \frac{(n\nu+2)(\nu(1-\beta)-1)}{\nu(n+1-\beta)}, \ z \in E, \quad (3.6) \end{aligned}$$

while from the subordination (3.5), we have

$$\left|\frac{f(z)}{z}\right| > 1 - \frac{(\nu(1-\beta)-2)}{\nu(n+1-\beta)} = \frac{n\nu+2}{\nu(n+1-\beta)}, \quad z \in E. \tag{3.7}$$

Combining these last two inequalities, we see that

$$\begin{aligned}\frac{n\nu+2}{\nu(n+1-\beta)}\left|\frac{zf'(z)}{f(z)} - \left(1-\frac{1}{\nu}\right)\right| &< \frac{1}{\nu}\left|\nu f'(z) + (1-\nu)\frac{f(z)}{z}\right| \\ &< \frac{1}{\nu}\left[\frac{(n\nu+2)(\nu(1-\beta)-1)}{\nu(n+1-\beta)}\right],\end{aligned}$$

which simplifies to

$$\left|\frac{zf'(z)}{f(z)} - \left(1-\frac{1}{\nu}\right)\right| < \left(1-\frac{1}{\nu}-\beta\right). \tag{3.8}$$

Thus,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left(1-\frac{1}{\nu}\right) - \left(1-\frac{1}{\nu}-\beta\right) = \beta, \tag{3.9}$$

which implies that f is starlike of order β in E. □

Taking $\alpha = 1 + 2\gamma\,(\mu = 1, \nu = \gamma)$ in Theorem 3.1 leads to the following result:

Corollary 3.1. *Let $f \in \mathcal{A}_n$ and ν be a real number such that $\nu > \frac{2}{1-\beta}$ $(0 \le \beta < 1)$. If f satisfies*

$$|f'(z) + \nu z f''(z) - 1| < \frac{(n+1)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}, \tag{3.10}$$

for $z \in E$, then $f \in S^*(\beta)$.

If we make $\nu \longrightarrow \infty$ in Corollary 3.1, we obtain the following criterion for starlikeness :

Corollary 3.2. *Let $f \in \mathcal{A}_n$ and $0 \le \beta < 1$. If f satisfies*

$$|zf''(z)| < \frac{n(n+1)(1-\beta)}{(n+1-\beta)}, \tag{3.11}$$

for $z \in E$, then $f \in S^*(\beta)$.

Remark 3.1. *We note that $\beta = 0$ and $n = 1$ in Corollary 3.2 leads us to the well known result of Obradovic [5].*

Substituting $n = 1$ and $\beta = 0$ in Corollary 3.1 yields the following interesting criterion for starlikeness.

Corollary 3.3. *Let $f \in \mathcal{A}$ and ν be a real number such that $\nu > 2$. If f satisfies*

$$\left| zf''(z) + \frac{1}{\nu}\left(f'(z) - 1\right)\right| < \frac{(1+\nu)(\nu-2)}{\nu^2}, \tag{3.12}$$

for $z \in E$, then

$$\left| \frac{zf'(z)}{f(z)} - \left(1 - \frac{1}{\nu}\right)\right| < \left(1 - \frac{1}{\nu}\right),$$

i.e $f \in S^*$.

Remark 3.2. *Corollary 3.3 is a particular case of the Theorem 1.7 in [6] with $\alpha = \frac{1}{\nu} \in \mathbb{R}$.*

We now present the following example in support of Theorem 3.1.

Example 3.1. *Consider the function*

$$f(z) = z + ((\nu(1-\beta) - 2)/\nu(n+1-\beta))z^{n+1}, \;\; 0 \leq \beta < 1.$$

Now,

$$\begin{aligned}
& \left|(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - 1\right| \\
= \;& \left|((1-\alpha+2\gamma) + (n+1)(\alpha-2\gamma) + n(n+1)\gamma)\,\frac{(\nu(1-\beta)-2)}{\nu(n+1-\beta)}z^n\right| \\
= \;& \left|(1+n\mu)(1+n\nu)\frac{(\nu(1-\beta)-2)}{\nu(n+1-\beta)}\right| |z|^n \\
< \;& (1+n\mu)(1+n\nu)\frac{(\nu(1-\beta)-2)}{\nu(n+1-\beta)}.
\end{aligned}$$

Thus, f satisfies the criterion of Theorem 3.1. Further, for $\mu > 0$ and $\nu > \frac{2}{1-\beta}$

as defined in (2.1)*, we have*

$$\begin{aligned}\Re\{zf'(z)f(z)\} &= \Re\left\{\frac{1+((n+1)(\nu(1-\beta)-2)z^n/\nu(n+1-\beta))}{1+((\nu(1-\beta)-2)z^n/\nu(n+1-\beta))}\right\}\\ &> \left\{\frac{1-((n+1)(\nu(1-\beta)-2)/\nu(n+1-\beta))}{1-((\nu(1-\beta)-2)/\nu(n+1-\beta))}\right\}\\ &= \left\{\frac{n\nu\beta+2n+2}{n\nu+2}\right\}\\ &> \beta.\end{aligned}$$

Theorem 3.2. *Let a function $g\in\mathcal{H}$ satisfy*

$$|g(z)|\le\frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)},\tag{3.13}$$

for some $\mu>0$ and $\nu>\frac{2}{1-\beta}$ as defined in (2.1) *and $0\le\beta<1$. Then the function f given by*

$$f(z)=z+\frac{z^{n+1}}{\mu\nu}\int_0^1\int_0^1 g(rsz)r^{n+1/\mu-1}s^{n+1/\nu-1}drds\tag{3.14}$$

is starlike of order β in E.

Proof. We first consider the function $f\in\mathcal{A}_n$ satisfying the differential equation

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-1=z^ng(z).\tag{3.15}$$

In view of (3.13), we have

$$\begin{aligned}&\left|(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-1\right|=|z|^n|g(z)|\\ &<\frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)},\quad z\in E.\end{aligned}$$

By Theorem 3.1, we see that the solution of differential equation (3.15) must be a starlike function of order β. Further, to obtain the solution, let $\varphi(z)=(1-\nu)\frac{f(z)}{z}+\nu f'(z)\in\mathcal{H}[1,n]$, then the equation (3.15) simplifies to

$$\mu z\varphi'(z)+\varphi(z)-1=z^ng(z).$$

On integration, we get

$$\varphi(z) = 1 + \frac{1/\mu}{z^{1/\mu}} \int_0^z g(\zeta)\zeta^{n+1/\mu-1} d\zeta = 1 + \frac{1}{\mu} z^n \int_0^1 g(rz) r^{n+1/\mu-1} dr.$$

Thus,

$$(1-\nu)\frac{f(z)}{z} + \nu f'(z) = 1 + \frac{1}{\mu} z^n \int_0^1 g(rz) r^{n+1/\mu-1} dr. \tag{3.16}$$

Further, setting $\psi(z) = \frac{f(z)}{z} \in \mathcal{H}[1,n]$, the differential equation (3.16) reduces to

$$\nu z\psi'(z) + \psi(z) = 1 + \frac{1}{\mu} z^n \int_0^1 g(rz) r^{n+1/\mu-1} dr.$$

A simple calculation gives

$$\psi(z) = 1 + \frac{1/\mu\nu}{z^{1/\nu}} \int_0^z \left(\int_0^1 g(r\zeta) r^{n+1/\mu-1} dr \right) \zeta^{n+1/\nu-1} d\zeta.$$

Since, $\psi(z) = f(z)/z$ and a change of variable yields that

$$f(z) = z + \frac{z^{n+1}}{\mu\nu} \int_0^1 \int_0^1 g(rsz) r^{n+1/\mu-1} s^{n+1/\nu-1} dr ds.$$

This completes the proof of the theorem. □

As an illustration of Theorem 3.2, we give the following example.

Example 3.2. *The function* $g(z) = \dfrac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}$ $(0 \leq \beta < 1)$*, satisfies the criteria of the Theorem 3.2. An Application of the theorem yields*

$$f(z) = z + \frac{(\nu(1-\beta)-2)}{\nu(n+1-\beta)} z^{n+1},$$

which is starlike of order β*, as shown in the Example 3.1.*

Further, taking $n = 1$ and $\alpha = 1 + 2\gamma\,(\mu = 1, \nu = \gamma)$ in Theorem 3.2, we have the following :

Corollary 3.4. *Let a function $g \in \mathcal{H}$ satisfy*

$$|g(z)| \leq \frac{2(1+\nu)(\nu(1-\beta)-2)}{\nu(2-\beta)}, \tag{3.17}$$

for some $\nu > \frac{2}{1-\beta}$ and $0 \leq \beta < 1$. Then the function f given by

$$f(z) = z + z^2 \left(\frac{1}{\nu} \int_0^1 \int_0^1 g(rsz) r s^{1/\nu} dr ds \right) \tag{3.18}$$

is starlike of order β in E.

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