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Mean-periodic Functions Associated with a Singular Differential-difference Operator on The Real Line *

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Abstract

We extend the classical theory of mean-periodic functions to a first-order singular differential-difference operator on the real line which includes as particular cases the one-dimensional Cherednik and Dunkl operators.

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1. Introduction

We consider the first-order singular differential-difference operator on the real line

$$\Lambda f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - \rho f(-x),$$

where

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$, and $\rho \geq 0$.

For $A(x) = |x|^{2\alpha+1}$, $\alpha > -\frac{1}{2}$, we regain the differential-difference operator

$$D_\alpha f = \frac{df}{dx} + (\alpha + \frac{1}{2})\frac{f(x)-f(-x)}{x},$$

which is referred to as the Dunkl operator with parameter $\alpha+\frac{1}{2}$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl in connection with a generalization of the classical theory of spherical harmonics (see [5, 14, 19] and the references therein). Besides its mathematical interest, the Dunkl operator D_α has quantum-mechanical applications; it is naturally involved in the study of one-dimensional harmonic oscillators governed by Wigner's commutation rules [10, 13, 21].

Notice that the differential-difference operator

$$\begin{aligned} D_{\alpha,\beta} f &= \frac{df}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x](\frac{f(x)-f(-x)}{2}) \\ &- (\alpha+\beta+1)f(-x), \end{aligned}$$

which is referred to as the Jacobi-Cherednik operator (see [7]) is of the same type as Λ with

$$\begin{cases} A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}; \quad \alpha \geq \beta > -\frac{1}{2}; \\ \rho = \alpha+\beta+1; \quad \delta = 2. \end{cases}$$

The one-dimensional Cherednik operator (see [3]) is a particular case of $D_{\alpha,\beta}$. Such operators have been used by Heckmann and Opdam to develop a theory generalizing harmonic analysis on symmetric spaces (see [8, 12]. For recent important results in this direction we refer to [15, 20].

Mourou [11] has proved that there exists a unique automorphism of the space $\mathcal{E}(\mathbb{R})$ of C^∞ functions on $\mathbb{R}$, satisfying

$$V\frac{df}{dx} = \Lambda V f \qquad \text{and} \qquad Vf(0) = f(0), \tag{1}$$

for all $f \in \mathcal{E}(\mathbb{R})$. The intertwining operator V has been exploited to initiate a quite new commutative harmonic analysis on the real line related to the differential-difference operator Λ in which several analytic structures on $\mathbb{R}$ were generalized. A summary of this harmonic analysis is provided in Sec. 2. Through this paper, the classical theory of mean-periodic functions on $\mathbb{R}$ is extended to the differential-difference operator Λ. More explicitly, a function f in $\mathcal{E}(\mathbb{R})$ is called Λ-mean-periodic if there exists a non zero compactly supported distribution μ on $\mathbb{R}$, such that

$$\mu \# f(x) = 0, \quad \text{for all } x \in \mathbb{R},$$

$\#$ being the generalized convolution generated by the differential-difference operator Λ. By using the intertwining operator V and the results of Schwartz in the classical setting [16], we express in Sec. 3 the Λ-mean-periodic function f in terms the elementary functions

$$\Psi_{\lambda,l}(x) = V\left(y^l e^{i\lambda y}\right)(x).$$

Namely, f may be expanded formally as

$$f(x) = \sum_{(\lambda,l)} \sum_{0\le s\le l-1} c_{\lambda,s}\Psi_{\lambda,s}(x), \quad c_{\lambda,s} \in \mathbb{C},$$

the summation being extended over the distinct roots λ of $\mathcal{F}_\Lambda(\mu)$ counted with multiplicities l, where $\mathcal{F}_\Lambda(\mu)$ stands for the generalized Fourier transform of μ defined by

$$\mathcal{F}_\Lambda(\mu)(\lambda) = \langle \mu_y, \Psi_{-\lambda}(y)\rangle, \quad \lambda \in \mathbb{C}.$$

Starting from the distribution μ, we construct in Sec. 4 a biorthogonal system which shows that the coefficients $c_{\lambda,s}$ in the series above are uniquely determined by f. In Sec. 5, we show that the series above is actually convergent to f in the topology of $\mathcal{E}(\mathbb{R})$, after a certain Abel summation procedure is performed. Moreover, we introduce a class of distributions μ for which the Abelian summation process can be dispensed.

In the classical setting, the notion of mean-periodicity was first introduced by Delsarte [4], and then analyzed in depth by Schwartz [16], Kahane [9], Berenstein and Taylor [2]. Later, Trimeche [18] extended the theory of mean-periodic functions to a class of singular second-order differential operator on the half-line . It is pointed out that all the results obtained in [1] emerge as easy consequences of those stated in the present article.

2. Preliminaries

In this section we provide some facts about harmonic analysis related to the differential-difference operator Λ. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [11].

Notation. We denote by

– $\mathcal{E}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, endowed with the topology of compact convergence for all derivatives;

– $\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support;

– $\mathcal{D}_a(\mathbb{R})$, $a > 0$, the space of C^∞ functions on $\mathbb{R}$ supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$;

– $\mathcal{D}(\mathbb{R}) = \cup_{a>0}\mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology;

– $\mathbf{H}_a$, $a > 0$, the space of entire, rapidly decreasing functions of exponential type a; that is, $f \in \mathbf{H}_a$ if and only if , f is entire on $\mathbb{C}$ and for all $m = 0, 1, ...$,

$$p_m(f) = \sup_{\lambda\in\mathbb{C}} \left|(1 + \lambda)^m f(\lambda)e^{-a|Im\lambda|}\right| < \infty,$$

$\mathbf{H}_a$ is equipped with the topology defined by the semi-norms p_m, $m = 0, 1, ...$;

– $\mathbf{H} = \cup_{a>0}\mathbf{H}_a$, equipped with the inductive limit topology;

– $\mathcal{H}_a$, $a > 0$, the space of entire, slowly increasing functions of exponential type a; that is, $f \in \mathcal{H}_a$, if and only if, f is entire on $\mathbb{C}$ and there is $m = 0, 1, ...$ such that,

$$\sup_{\lambda\in\mathbb{C}} \left|(1 + |\lambda|)^{-m} f(\lambda)e^{-a|Im\lambda|}\right| < \infty;$$

– $\mathcal{H} = \cup_{a>0}\mathcal{H}_a$.

Remark 2.1. *Clearly* Λ *is a bounded linear operator from* $\mathcal{E}(\mathbb{R})$ *into itself. If* $\mu \in \mathcal{E}'(\mathbb{R})$ *and* $n \in \mathbb{N}$, *define* $\Lambda^n \mu \in \mathcal{E}'(\mathbb{R})$ *by*

$$\langle \Lambda^n \mu, f \rangle = (-1)^n \langle \mu, \Lambda^n f \rangle, \quad f \in \mathcal{E}(\mathbb{R}).$$

2.1. Intertwining operators

It is shown in [11] that for each $\lambda \in \mathbb{C}$, the differential-difference equation

$$\Lambda u = i\lambda u, \qquad u(0) = 1,$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \dfrac{1}{i\lambda - \rho} \dfrac{d}{dx} \varphi_\lambda(x) & \text{if } \lambda \neq -i\rho, \\[2ex] 1 + \dfrac{2\rho}{A(x)} \displaystyle\int_0^x A(t)dt & \text{if } \lambda = -i\rho, \end{cases} \tag{2}$$

where φ_λ denotes the solution of the differential equation

$$\Delta u = -\left(\lambda^2 + \rho^2\right) u, \quad u(0) = 1, \quad u'(0) = 1,$$

Δ being the second-order singular differential operator defined by

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.$$

Remark 2.2. *(i) If* $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, *then the differential operator* Δ *is just the Bessel operator [17], and*

$$\varphi_\lambda(x) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n \, (\lambda x/2)^{2n}}{n! \, \Gamma(n+\alpha+1)}.$$

(ii) For $A(x) = (\sinh|x|)^{2\alpha+1} (\cosh x)^{2\beta+1}$, $\alpha \geq \beta > -1/2$, *the differential operator* Δ *reduces to the so-called Jacobi operator. The eigenfunction* φ_λ *is given by*

$$\varphi_\lambda(x) = {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}; \alpha+1; -(\sinh x)^2\right)$$

where ${}_2F_1$ *is the Gauss hypergeometric function [17].*

For $x \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}$, the eigenfunction $\Phi_\lambda(x)$ has the Laplace type integral representation

$$\Phi_\lambda(x) = \int_{-|x|}^{|x|} K(x,y) e^{i\lambda y} dy, \tag{3}$$

where $K(x,.)$ is a real-valued function on $\mathbb{R}$, continuous on $]-|x|,|x|[$, and supported in $[-|x|,|x|]$.

As a consequence of this integral representation, we deduce that the intertwining operator V is in fact an integral transform :

$$Vf(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y) f(y) dy & \text{if } x \neq 0, \\ f(0) & \text{if } x = 0. \end{cases}$$

For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, V reads

$$Vf(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t)^{\alpha-\frac{1}{2}} (1+t)^{\alpha+\frac{1}{2}} f(tx) dt,$$

and referred to as the Dunkl intertwining operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$.

The dual operator ${}^tV$ of V, is defined on $\mathcal{E}'(\mathbb{R})$ by

$$\left\langle {}^tV\mu, f \right\rangle = \langle \mu, Vf \rangle, \quad f \in \mathcal{E}(\mathbb{R}).$$

If $f \in \mathcal{D}(\mathbb{R})$, then the distribution ${}^tV(Af)$ is given by the function

$${}^tVf(y) = \int_{|x|\geq|y|} K(x,y) f(x) A(x) dx, \quad y \in \mathbb{R}.$$

Theorem 2.1. *(i) The dual transform* ${}^tV$ *of* V*, is a bijection from* $\mathcal{E}'(\mathbb{R})$ *onto itself. More precisely,* $\operatorname{supp}\mu \subset [-a,a]$ *if, and only if,*
$\operatorname{supp}{}^tV\mu \subset [-a,a]$*. Moreover,*

$$\frac{d}{dx}\,{}^tV\mu = {}^tV\Lambda\mu, \quad \text{for all } \mu \in \mathcal{E}'(\mathbb{R}).$$

(ii) The integral transform ${}^tV$ *is a topological automorphism of* $\mathcal{D}(\mathbb{R})$ *satisfying the intertwining relation*

$$\frac{d}{dx}\,{}^tVf = {}^tV\,\widetilde{\Lambda}\,f, \quad f \in \mathcal{D}(\mathbb{R}),$$

$\widetilde{\Lambda}$ *being the dual operator of* Λ *defined by*

$$\widetilde{\Lambda}f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right) + 2\rho f(-x).$$

2.2. Generalized Fourier transform

The generalized Fourier transform of a distribution $\mu \in \mathcal{E}'(\mathbb{R})$ is defined by

$$\mathcal{F}_\Lambda(\mu)(\lambda) = \langle \mu, \Phi_{-\lambda} \rangle, \quad \lambda \in \mathbb{C}.$$

The generalized Fourier transform of a function $f \in \mathcal{D}(\mathbb{R})$ is defined by

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-\lambda}(x)A(x)dx, \quad \lambda \in \mathbb{C}.$$

Recall the following identities :

$$\begin{aligned} \mathcal{F}_\Lambda(\mu) &= \mathcal{F}_u\left({}^tV\mu\right), \quad \mu \in \mathcal{E}'(\mathbb{R}), \qquad (4)\\ \mathcal{F}_\Lambda(f) &= \mathcal{F}_u\left({}^tVf\right), \quad f \in \mathcal{D}(\mathbb{R}),\\ \mathcal{F}_\Lambda(\Lambda\mu)(\lambda) &= i\lambda\,\mathcal{F}_\Lambda(\mu)(\lambda), \quad \mu \in \mathcal{E}'(\mathbb{R}),\\ \mathcal{F}_\Lambda\left(\widetilde{\Lambda}f\right)(\lambda) &= i\lambda\,\mathcal{F}_\Lambda(f)(\lambda), \quad f \in \mathcal{D}(\mathbb{R}), \end{aligned}$$

$\mathcal{F}_u$ being the usual Fourier transform on $\mathbb{R}$ given by

$$\mathcal{F}_u(\mu)(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x}\,d\mu(x), \quad \mu \in \mathcal{E}'(\mathbb{R}).$$

An outstanding result about the generalized Fourier transform $\mathcal{F}_\Lambda$ is as follows.

Theorem 2.2. ***(Paley-Wiener)***

(i) The generalized Fourier transform $\mathcal{F}_\Lambda$ *is a bijection from* $\mathcal{E}'(\mathbb{R})$ *onto* $\mathcal{H}$. *More precisely,* μ *has its support in* $[-a, a]$ *if, and only if,* $\mathcal{F}_\Lambda(\mu) \in \mathcal{H}_a$.

(ii) The generalized Fourier transform $\mathcal{F}_\Lambda$ *is a topological isomorphism from* $\mathcal{D}(\mathbb{R})$ *onto* $\boldsymbol{H}$. *More precisely,* $f \in \mathcal{D}_a(\mathbb{R})$ *if, and only if,* $\mathcal{F}_\Lambda(f) \in \boldsymbol{H}_a$.

2.3. Generalized convolution

The generalized translation operators T^x, $x \in \mathbb{R}$, tied to Λ are defined on $\mathcal{E}(\mathbb{R})$ by

$$T^x f(y) = V_x V_y \left[V^{-1} f(x+y)\right], \quad y \in \mathbb{R}.$$

The T^x, $x \in \mathbb{R}$, are linear bounded operator from $\mathcal{E}(\mathbb{R})$ into itself, and possess the following fundamental properties :

$$T^0 = \text{identity}, \quad T^x T^y = T^y T^x, \quad T^x f(y) = T^y f(x),$$

$$\Lambda T^x = T^x \Lambda \quad \text{and} \quad T^x(\Phi_\lambda)(y) = \Phi_\lambda(x)\Phi_\lambda(y).$$

The generalized convolution product of two distributions $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$, is the distribution $\mu \# \nu \in \mathcal{E}^{'}(\mathbb{R})$ given by

$$\langle \mu \# \nu, f \rangle = \langle \mu_x, \langle \nu_y, T^x f(y) \rangle \rangle, \quad f \in \mathcal{E}(\mathbb{R}).$$

The generalized convolution of $\mu \in \mathcal{E}^{'}(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$, is the function $\mu \# f \in \mathcal{E}(\mathbb{R})$ given by

$$\mu \# f(x) = \left\langle \mu_y, T^{-x} f^{-}(y) \right\rangle, \quad x \in \mathbb{R},$$

with $f^{-}(y) = f(-y)$.

Proposition 2.1. *(i) Let $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$ and $f \in \mathcal{D}(\mathbb{R})$. Then*

$$\mathcal{F}_\Lambda(\mu \# \nu) = \mathcal{F}_\Lambda(\mu)\mathcal{F}_\Lambda(\nu), \tag{5}$$

$$\mathcal{F}_\Lambda(\mu \# f) = \mathcal{F}_\Lambda(\mu)\mathcal{F}_\Lambda(f). \tag{6}$$

(ii) For $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$ we have

$$\mu \# (\nu \# f) = (\mu \# \nu) \# f.$$

(iii) If $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$ then

$$V\left({}^tV\mu * f\right) = \mu \# Vf, \tag{7}$$

$${}^tV(\mu \# \nu) = {}^tV\mu * {}^tV\nu,$$

where $*$ denotes the classical convolution on $\mathbb{R}$.

3. Λ-mean-periodic functions

According to Schwartz [16], a function f in $\mathcal{E}(\mathbb{R})$ is called mean-periodic relatively to a distribution μ in $\mathcal{E}'(\mathbb{R})$, if it is a solution of the convolution equation

$$\mu * f(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

In this section we extend the notion of mean-periodicity to the differential-difference operator Λ, by replacing in the equation above the ordinary convolution * by the generalized convolution #.

Definition 3.1. We say that a function $f \in \mathcal{E}(\mathbb{R})$ is Λ-mean-periodic, if there exists $0 \neq \mu \in \mathcal{E}'(\mathbb{R})$ such that

$$\mu \# f(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

If we want to emphasize the equation satisfied by f we will say that f is mean-periodic with respect to μ or μ-Λ-mean-periodic.

Notation. For $f \in \mathcal{E}(\mathbb{R})$, write $\tau(f)$ for the closure of the subspace of $\mathcal{E}(\mathbb{R})$ spanned by $T^{-x}f^{-}$, $x \in \mathbb{R}$.

Remark 3.1. *(i) Notice that*

$$\mu \# f = 0 \Leftrightarrow \mu = 0 \text{ on } \tau(f) \Leftrightarrow \mu \in (\tau(f))^{\perp}$$

(ii) According to the Hahn-Banach theorem, Definition 3.1 is equivalent to $\tau(f) \neq \mathcal{E}(\mathbb{R})$.

Examples. (i) Let a be a nonzero real number. Each function $f \in \mathcal{E}(\mathbb{R})$ satisfying

$$T^{-x}f^{-}(a) = f(x), \quad \text{for all } x \in \mathbb{R},$$

is Λ-mean-periodic with respect to $\mu = \delta_a - \delta_0$, where δ_a denotes the Dirac measure at the point a.

(ii) By virtue of (6) and Theorem 2.2, every $0 \neq f \in \mathcal{D}(\mathbb{R})$ is not Λ-mean-periodic.

Proposition 3.1. *For* $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ *and* $l \in \mathbb{N}$, *put*

$$\psi_{\lambda,l}(x) = x^l e^{i\lambda x} \quad \text{and} \quad \Psi_{\lambda,l}(x) = V(\psi_{\lambda,l})(x) \tag{8}$$

Then

(i) $\Psi_{\lambda,l}(x) = (-i)^l \dfrac{\partial^l}{\partial\lambda^l}\Phi_\lambda(x)$.

(ii) *For all* $\mu \in \mathcal{E}^{'}(\mathbb{R})$, *we have*

$$\left(\mathcal{F}_\Lambda(\mu)\right)^{(l)}(\lambda) = (-i)^l \langle\mu, \Psi_{-\lambda,l}\rangle, \tag{9}$$

$$\mu\#\Psi_{\lambda,l}(x) = \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} \Psi_{\lambda,l-s}(x)(-i)^s \left(\mathcal{F}_\Lambda(\mu)\right)^{(s)}(\lambda). \tag{10}$$

(iii) *The function* $x \to \Psi_{\lambda,l}(x)$ *is* Λ*-mean-periodic.*

Proof. Assertion (i) follows by using (3) and differentiation under the integral sign. Formula (9) follows also by using differentiation under the integral sign. Let us check (10). By (7) and (8),

$$\mu\#\Psi_{\lambda,l} = V\left({}^tV\mu * \psi_{\lambda,l}\right). \tag{11}$$

But an easy computation shows that

$$\nu * \psi_{\lambda,l}(x) = \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} \psi_{\lambda,l-s}(x)(-i)^s \left(\mathcal{F}_u(\nu)\right)^{(s)}(\lambda),$$

for all $\nu \in \mathcal{E}^{'}(\mathbb{R})$. So

$${}^tV\mu * \psi_{\lambda,l}(x) = \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} \psi_{\lambda,l-s}(x)(-i)^s \left(\mathcal{F}_\Lambda(\nu)\right)^{(s)}(\lambda), \tag{12}$$

by virtue of (4). Identity (10) follows now by combining (8), (11) and (12). Finally, to have $\mu\#\Psi_{\lambda,l} \equiv 0$, it is sufficient in view of (10), to choose $0 \neq \mu \in \mathcal{E}^{'}(\mathbb{R})$ such that λ is a zero of order at least l of $\mathcal{F}_\Lambda(\mu)$. This completes the proof. □

Proposition 3.2. *Let* $f \in \mathcal{E}(\mathbb{R})$ *be* Λ*-mean-periodic. Then* $\Psi_{\lambda,l} \in \tau(f)$ *if and only if, for all* $\mu \in (\tau(f))^{\perp}$, *we have*

$$\left(\mathcal{F}_\Lambda(\mu)\right)^{(l)}(-\lambda) = 0.$$

Proof. The result follows by using (9) and the Hahn-Banach theorem. □

Definition 3.2. We call spectrum of a Λ-mean-periodic function $f \in \mathcal{E}(\mathbb{R})$, denoted by $sp(f)$, the set of pairs (λ, l), $\lambda \in \mathbb{C}$, $l \in \mathbb{N}$, such that the functions $\Psi_{\lambda,s}$ belong to $\tau(f)$ for $0 \leq s \leq l-1$ and not for $s = l$.

Remark 3.2. *According to Proposition 3.2,* $(\lambda, l) \in sp(f)$ *if and only if,* $-\lambda$ *is a common zero of order* l *of the generalized Fourier transforms of elements of* $(\tau(f))^{\perp}$.

The next statement clarifies the relationship between Λ-mean-periodic functions and classical mean-periodic functions.

Proposition 3.3. *A function* $f \in \mathcal{E}(\mathbb{R})$ *is* Λ*-mean-periodic with respect to a distribution* $\mu \in \mathcal{E}'(\mathbb{R})$ *if, and only if,* $V^{-1}f$ *is a classical mean-periodic function with respect to* ${}^tV\mu$.

Proof. The result is a direct consequence of (7). □

From the work of Schwartz [16] and the proposition above, we deduce the following characterization of Λ-mean-periodic functions.

Theorem 3.1. *Let* $f \in \mathcal{E}(\mathbb{R})$ *be* Λ*-mean-periodic. Then* f *can be approximated in the topology of* $\mathcal{E}(\mathbb{R})$ *by finite linear combinations of functions of the type* $\Psi_{\lambda,l}$, $(\lambda, l) \in sp(f)$.

4. Biorthogonal system

Notation. Throughout this section fix $0 \neq \mu \in \mathcal{E}'(\mathbb{R})$. Put

$$\mathcal{Z}_\Lambda(\mu) = \{(\lambda_k, l_k), k \in \mathbb{N}, l_k \in \mathbb{N}\},$$

where λ_k is a zero of order l_k of the entire function $\mathcal{F}_\Lambda(\mu)$.

Starting from the distribution μ, we construct in this section a biorthogonal system in $\mathcal{E}'(\mathbb{R})$, that is, a family of distributions $\mu_{k,m} \in \mathcal{E}'(\mathbb{R})$, satisfying

$$\langle \mu_{k,m}, \Psi_{\lambda_s,j} \rangle = \delta_{k,s}\, \delta_{m,j} \tag{13}$$

for $0 \leq m \leq l_k - 1$ and $0 \leq j \leq l_s - 1$. Given a μ-Λ-mean-periodic function $f \in \mathcal{E}(\mathbb{R})$, formula (13) will allow us to compute the coefficients $c_{k,l}$ in a possible development of f with respect to the functions $\Psi_{\lambda_k,l}$, $k \in \mathbb{N}$, $0 \leq l \leq l_k - 1$. We adopt here the arguments used by Delsarte [4] and Schwartz [16].

Notation. For $f \in \mathcal{E}(\mathbb{R})$, put

$$I_k(f)(x) = \int_0^x f(t)e^{i\lambda_k(x-t)}dt, \quad x \in \mathbb{R}.$$

Lemma 4.1. *Let $f \in \mathcal{E}(\mathbb{R})$. Then*

(i) The general solution of the equation

$$\left(\frac{d}{dx} - i\lambda_k\right)^{l_k} g = f,$$

is given by

$$g(x) = \sum_{s=0}^{l_k-1} \beta_s\, \psi_{\lambda_k,s}(x) + \overbrace{I_k \circ \cdots \circ I_k}^{l_k\, times}(f)(x), \quad \beta_s \in \mathbb{C}.$$

(ii) The general solution of the equation

$$(\Lambda - i\lambda_k)^{l_k}\, g = f, \tag{14}$$

is given by

$$g(x) = \sum_{s=0}^{l_k-1} \beta_s\, \Psi_{\lambda_k,s}(x) + V \circ \overbrace{I_k \circ \cdots \circ I_k}^{l_k\, times} \circ V^{-1}(f)(x), \quad \beta_s \in \mathbb{C}.$$

Proof. Assertion (i) is easily checked. By virtue of (1), equation (14) is equivalent to

$$\left(\frac{d}{dx} - i\lambda_k\right)^{l_k} \left(V^{-1}g\right) = V^{-1}f.$$

Assertion (ii) follows then from (i). □

Lemma 4.2. *There is a unique distribution $\mu_- \in \mathcal{E}'(\mathbb{R})$ such that*

$$\mathcal{F}_\Lambda(\mu_-)(\lambda) = \mathcal{F}_\Lambda(\mu)(-\lambda), \quad \text{for all } \lambda \in \mathbb{C}.$$

Moreover, if $\operatorname{supp} \mu \subset [-a, a]$, *then* $\operatorname{supp}(\mu_-) \subset [-a, a]$.

Proof. The result follows readily from Theorem 2.2(i). □

Remark 4.1. *Define* $\mu^- \in \mathcal{E}'(\mathbb{R})$ *by*

$$\int_{\mathbb{R}} f(x) d\mu^-(x) = \int_{\mathbb{R}} f(-x) d\mu(x), \quad f \in \mathcal{E}(\mathbb{R}).$$

Then according to (2) and Theorem 2.2(i), $\mu_- = \mu^-$ *if and only if* $\rho = 0$.

Notation. If G is a meromorphic function, having γ as a pole, we denote by $[G(\lambda)]_\gamma$ the singular part of $G(\lambda)$ in a neighborhood of γ, hence $G(\lambda) - [G(\lambda)]_\gamma$ is holomorphic in a neighborhood of γ.

Lemma 4.3. *(i) The distribution* $q_k \in \mathcal{E}'(\mathbb{R})$ *defined by*

$$\mathcal{F}_\Lambda(q_k)(\lambda) = (\lambda + \lambda_k)^{l_k} \left[\frac{1}{\mathcal{F}_\Lambda(\mu)(-\lambda)} \right]_{-\lambda_k}$$

has a support concentrated at the origin.

(ii) The distribution $\mu_{k,0} \in \mathcal{E}'(\mathbb{R})$ *defined by*

$$\mathcal{F}_\Lambda(\mu_{k,0})(\lambda) = \begin{cases} \mathcal{F}_\Lambda(\mu)(-\lambda) \left[\frac{1}{\mathcal{F}_\Lambda(\mu)(-\lambda)} \right]_{-\lambda_k} & \text{if } \lambda \neq -\lambda_k, \\ 1 & \text{if } \lambda = -\lambda_k, \end{cases} \tag{15}$$

satisfies

$$\langle \mu_{k,0}, f \rangle = (-i)^{l_k} \left\langle q_k \# \mu_-, V \circ \overbrace{I_k \circ \cdots \circ I_k}^{l_k \, times} \circ V^{-1}(f) \right\rangle,$$

for all $f \in \mathcal{E}(\mathbb{R})$.

Proof. (i) As the function $(\lambda + \lambda_k)^{l_k} [1/\mathcal{F}_\Lambda(\mu)(-\lambda)]_{-\lambda_k}$ is a polynomial $P_k(\lambda)$, it follows by (4) that ${}^tV q_k = P_k(d/dx)(\delta_0)$. Then using Theorem 2.1(i), we deuce that q_k has a support concentrated at the origin.

(ii) As

$$(\lambda + \lambda_k)^{l_k} \mathcal{F}_\Lambda(\mu_{k,0})(\lambda) = \mathcal{F}_\Lambda(q_k)(\lambda) \mathcal{F}_\Lambda(\mu)(-\lambda),$$

it follows from (5) that

$$(-i)^{l_k} (\Lambda + i\lambda_k)^{l_k} \mu_{k,0} = q_k \# \mu_-.$$

So for all g in $\mathcal{E}(\mathbb{R})$,

$$\langle q_k \# \mu_-, g \rangle = (-i)^{l_k} \left\langle (\Lambda + i\lambda_k)^{l_k} \mu_{k,0}, g \right\rangle = i^{\,l_k} \left\langle \mu_{k,0}, (\Lambda - i\lambda_k)^{l_k} g \right\rangle .$$

The result is now a direct consequence of (9) and Lemma 4.1(ii). □

Remark 4.2. *If the zeros* λ_k *of* $\mathcal{F}_\Lambda(\mu)$ *are simple, then*

$$\left[\frac{1}{\mathcal{F}_\Lambda(\mu)(-\lambda)}\right]_{-\lambda_k} = \frac{-1}{(\lambda + \lambda_k)\,(\mathcal{F}_\Lambda(\mu))'(-\lambda_k)},$$

that is,

$$q_k = \frac{-\,\delta_0}{(\mathcal{F}_\Lambda(\mu))'(-\lambda_k)}$$

and

$$\langle \mu_{k,0}, f \rangle = \frac{i}{(\mathcal{F}_\Lambda(\mu))'(-\lambda_k)} \left\langle \mu_-, V \circ I_k \circ V^{-1}(f) \right\rangle$$

for all $f \in \mathcal{E}(\mathbb{R})$.

Proposition 4.1. *Define* $\mu_{k,m} \in \mathcal{E}'(\mathbb{R})$, $0 \le m \le l_k - 1$, *by*

$$\mu_{k,m} = \frac{(-1)^m}{m!}\,(\Lambda + i\lambda_k)^m\,\mu_{k,0} + \tau_{k,m}\,\#\mu_-, \tag{16}$$

where

– $\mu_{k,0} \in \mathcal{E}'(\mathbb{R})$ *is defined in Lemma 3.2.*

– $\tau_{k,m} \in \mathcal{E}'(\mathbb{R})$ *with support concentrated at the origin, whose the generalized Fourier transform is given by*

$$\mathcal{F}_\Lambda(\tau_{k,m})(\lambda) = \frac{(-i)^m}{m!}\,R_{k,m}(\lambda) \tag{17}$$

with

$$R_{k,m}(\lambda) = \left[\frac{(\lambda + \lambda_k)^m}{\mathcal{F}_\Lambda(\mu)(-\lambda)}\right]_{-\lambda_k} - (\lambda + \lambda_k)^m \left[\frac{1}{\mathcal{F}_\Lambda(\mu)(-\lambda)}\right]_{-\lambda_k}$$

Then the family $(\mu_{k,m})$ *satisfies (13).*

Proof. Notice that $R_{k,m}(\lambda)$ is a polynomial, so the support of $\tau_{k,m}$ is concentrated at the origin. A combination of (15), (16) and (17) yields

$$\mathcal{F}_\Lambda(\mu_{k,m})(\lambda) = \frac{(-i)^m}{m!}\,\mathcal{F}_\Lambda(\mu)(-\lambda)\left[\frac{(\lambda+\lambda_k)^m}{\mathcal{F}_\Lambda(\mu)(-\lambda)}\right]_{-\lambda_k}. \tag{18}$$

According to (9) and (18), $\langle \mu_{k,m}, \Psi_{\lambda_s,j}\rangle = 0$ for $s \neq k$. A straightforward calculation shows that

$$\mathcal{F}_\Lambda(\mu_{k,m})(\lambda) = (-i)^m\,\frac{(\lambda+\lambda_k)^m}{m!} + O\left((\lambda+\lambda_k)^{l_k+1}\right),$$

in a neighborhood of $-\lambda_k$. We conclude, in view of (9), that $\langle \mu_{k,m}, \Psi_{\lambda_k,j}\rangle = 0$ for $j \neq m$, and $\langle \mu_{k,m}, \Psi_{\lambda_k,m}\rangle = 1$. This achieves the proof. $\square$

Corollary 4.1. *Let $f \in \mathcal{E}(\mathbb{R})$. Assume that there are disjoint finite subsets $\mathcal{Z}_j$ (groupings) such that $\mathcal{Z}_\Lambda(\mu) = \bigcup_1^\infty \mathcal{Z}_j$ and*

$$\sum_{j=1}^{\infty}\left(\sum_{(\lambda_k,l_k)\in\mathcal{Z}_j}\ \sum_{0\le l\le l_k-1} c_{k,l}\,\Psi_{\lambda_k,l}\right) \tag{19}$$

is convergent in $\mathcal{E}(\mathbb{R})$ to f, with a suitable mode of convergence. Then f is μ-Λ-mean-periodic and the coefficients $c_{k,l}$ can be computed by the formula

$$c_{k,l} = \langle \mu_{k,l}, f\rangle. \tag{20}$$

Proof. The function f is μ-Λ-mean-periodic because that is true for each term in (19). Identity (20) follows immediately from Proposition 4.1. $\square$

5. Series expansion with respect to the functions Ψ_{λ_k,l_k}

Like in the classical setting, the series (19) is not actually convergent in $\mathcal{E}(\mathbb{R})$, without a certain abelian summation procedure is performed :

Theorem 5.1. *Let $f \in \mathcal{E}(\mathbb{R})$ be Λ-mean-periodic with respect to $\mu \in \mathcal{E}'(\mathbb{R})$. Then there are disjoint finite subsets $\mathcal{Z}_j$ (groupings) such that $\mathcal{Z}_\Lambda(\mu) = \bigcup_1^\infty \mathcal{Z}_j$ and for every $\varepsilon > 0$ the series*

$$\sum_{j=1}^{\infty}\left(\sum_{(\lambda_k,l_k)\in\mathcal{Z}_j}\ \sum_{0\le l\le l_k-1} c_{k,l}\,\Psi_{\lambda_k,l}\,e^{-\varepsilon|\lambda_k|}\right)$$

converges in $\mathcal{E}(\mathbb{R})$ to a function f_ε satisfying :

$$\lim_{\varepsilon \longrightarrow 0} f_\varepsilon = f, \quad \text{in } \mathcal{E}(\mathbb{R}).$$

The coefficients $c_{k,l}$ being determined by (20).

Proof. By Proposition 3.3, $V^{-1}f$ is a classical mean-periodic function with respect to the distribution ${}^tV\mu$. So using (4) and the results of Schwartz [16], we can find :

– finite subsets $\mathcal{Z}_j$ such that $\mathcal{Z}_\Lambda(\mu) = \bigcup_1^\infty \mathcal{Z}_j$

– a sequence of complex numbers $\tilde{c}_{k,l}$

such that for every $\varepsilon > 0$ the series

$$\sum_{j=1}^{\infty} \left(\sum_{(\lambda_k, l_k) \in \mathcal{Z}_j} \ \sum_{0 \le l \le l_k - 1} \tilde{c}_{k,l} \, \psi_{\lambda_k, l} \, e^{-\varepsilon |\lambda_k|} \right)$$

converges in $\mathcal{E}(\mathbb{R})$ to a function f_ε satisfying :

$$\lim_{\varepsilon \longrightarrow 0} f_\varepsilon = V^{-1}f, \quad \text{in } \mathcal{E}(\mathbb{R}).$$

As the intertwining operator V is an automorphism of $\mathcal{E}(\mathbb{R})$, it follows by (4) that

$$V(f_\varepsilon) = \sum_{j=1}^{\infty} \left(\sum_{(\lambda_k, l_k) \in \mathcal{Z}_j} \ \sum_{0 \le l \le l_k - 1} \tilde{c}_{k,l} \, \Psi_{\lambda_k, l} \, e^{-\varepsilon |\lambda_k|} \right)$$

and

$$\lim_{\varepsilon \to 0} V(f_\varepsilon) = f,$$

where both the series and the limit are meaningful in the topology of $\mathcal{E}(\mathbb{R})$. Finally, we deduce from Corollary 4.1 that

$$\tilde{c}_{k,l} = c_{k,l}, \qquad 0 \le l \le l_k - 1, \quad k \in \mathbb{N}.$$

This ends the proof. □

Following Ehrenpreis [6], we introduce a class of distributions for which the Abel summation process is not necessary.

Definition 5.1. A distribution $\mu \in \mathcal{E}'(\mathbb{R})$ is called Λ-slowly-decreasing, if there are positive constants c, d such that for any $x \in \mathbb{R}$,

$$\max\left\{|\mathcal{F}_\Lambda(\mu)(y)|,\, y \in \mathbb{R},\, |x-y| \le d\, \log\left(1+|x|^2\right)\right\} \ge c\,(1+|x|)^{-1/c}.$$

Using the results of [6] and Proposition 3.3, it is not hard to establish the following theorem.

Theorem 5.2. *Let $f \in \mathcal{E}(\mathbb{R})$ be Λ-mean-periodic with respect to a Λ-slowly-decreasing distribution $\mu \in \mathcal{E}'(\mathbb{R})$. Then there exist finite groupings $\mathcal{Z}_j$ of $\mathcal{Z}_\Lambda(\mu)$ such that the series*

$$\sum_{j=1}^{\infty}\left(\sum_{(\lambda_k,l_k)\in\mathcal{Z}_j}\ \sum_{0\le l\le l_k-1} c_{k,l}\,\Psi_{\lambda_k,l}\right) \tag{21}$$

converges to f in $\mathcal{E}(\mathbb{R})$. The coefficients $c_{k,l}$ being determined by (20).

The next statement characterizes the Λ-slowly-decreasing distributions $\mu \in \mathcal{E}'(\mathbb{R})$ for which every grouping $\mathcal{Z}_j$ in (21) can be taken to contain a single point of $\mathcal{Z}_\Lambda(\mu)$.

Theorem 5.3. *Let $f \in \mathcal{E}(\mathbb{R})$ be Λ-mean-periodic with respect to a Λ-slowly-decreasing distribution $\mu \in \mathcal{E}'(\mathbb{R})$. A necessary and sufficient condition that the series (21) converges to f in $\mathcal{E}(\mathbb{R})$ without groupings (i.e.,* $\operatorname{card}(\mathcal{Z}_j) = 1$ *for all j) is that for some c, $d > 0$ we have*

$$\left|\frac{d^l}{d\lambda^l}\mathcal{F}_\Lambda(\mu)(\lambda)\right| \ge d\,\frac{\exp(-c\,|\mathcal{I}m\lambda|)}{(1+|\lambda|)^c}$$

for all $(\lambda, l) \in \mathcal{Z}_\Lambda(\mu)$.

Proof. The result follows easily by combining the results of [2] and Proposition 3.3. $\square$

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