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Three Classes of The Stirling Formula for The q-factorial Function *

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Abstract

In this paper, q-analogues of the Stirling formula for the q-factorial function are derived and expressed as infinite integral, infinite series and double infinite series.

Keywords and Phrases: *q-Factorial function; q-Stirling formula.*

1. Introduction

The Stirling formula and its generalizations have a large class of applications in science as in statistical physics or probability theory. In consequence, it has been deeply studied by a large number of authors, due to its practical importance. For details see [1-3] and the references given therein.

Many of the classical facts for the gamma function have been extended to the q-gamma function which is defined as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad |q|<1,\ z\neq 0,-1,-2,\cdots \tag{1.1}$$

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where $(a;q)_\infty$ is the q-shifted factorial defined by the limit

$$(a;q)_\infty = \lim_{m\to\infty}(a;q)_m = \lim_{m\to\infty}\prod_{k=0}^{m-1}(1-aq^k), \qquad |q|<1 \tag{1.2}$$

(see [4-10] and the references therein). An important fact for the gamma function in applied mathematics as well as in probability is the Stirling formula that gives a pretty accurate idea about the size of the gamma function. With the Euler-Maclaurin formula, Moak [6] obtained the following q-analogue of Stirling formula

$$\begin{aligned}\log\Gamma_q(x) \sim &\left(x-\frac{1}{2}\right)\log[x]_q + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + C_{\hat q} + \frac{1}{2}H(q-1)\log q \\ &+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log\hat q}{\hat q^x-1}\right)^{2k-1}\hat q^x p_{2k-3}(\hat q^x), \qquad x\to\infty\end{aligned} \tag{1.3}$$

where $H(\cdot)$ denotes the Heaviside step function,

$$\hat q = \begin{cases} q & \text{if} \quad 0<q\le 1 \\ q^{-1} & \text{if} \quad q\ge 1\end{cases},$$

$[x]_q = (1-q^x)/(1-q)$, $\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as [11]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt; \qquad z\notin(1,\infty), \tag{1.4}$$

p_k is a polynomial of degree k satisfying

$$p_k(z) = (z-z^2)p'_{k-1}(z) + (kz+1)p_{k-1}(z), \qquad p_0 = p_{-1} = 1, \quad k=1,2,\cdots \tag{1.5}$$

and

$$\begin{aligned}C_q =&\frac{1}{2}\log(2\pi) + \frac{1}{2}\log\left(\frac{q-1}{\log q}\right) - \frac{1}{24}\log q \\ &+\log\left(\sum_{m=-\infty}^{\infty}\left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right)\right),\end{aligned} \tag{1.6}$$

where $r = \exp(4\pi^2/\log q)$. It is easy to see that

$$\lim_{q\to 1} C_q = C_1 = \frac{1}{2}\log(2\pi), \quad \lim_{q\to 1} \frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x \quad \text{and} \quad P_k(1) = (k+1)! \tag{1.7}$$

and so (1.3) when letting $q \to 1$, tends to the ordinary Stirling formula [11]

$$\log\Gamma(x) \sim \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \frac{1}{x^{2k-1}}, \qquad x \to \infty. \tag{1.8}$$

Mansour [12] derived an asymptotic expansion of the q-gamma function $\Gamma_q(x)$ as

$$\Gamma_q(x) = [2]_q^{\frac{1}{2}} \Gamma_{q^2}(\frac{1}{2})(1-q)^{1/2-x} e^{\frac{\theta q^x}{1-q-q^x}}, \qquad 0 < \theta < 1,\ 0 < q < 1. \tag{1.9}$$

In this paper, the Euler-Maclaurin formula is exploited to provide an expression for the q-factorial function as an infinite integral. This integral representation for the q-factorial function is used to express it as infinite series and double infinite series.

2. Main Results

The Euler-Maclaurin formula provides a powerful connection between integrals and sums. It can be used to approximate integrals by finite sums, or conversely to evaluate finite sums and infinite series using integrals and the machinery of calculus and defined as

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx + \frac{1}{2}(f(n) + f(1)) + \int_1^n P_1(x) f'(x)dx \tag{2.1}$$

where $P_n(x) = B_n(\{x\})$ is the periodic Bernoulli functions and $\{x\} = x - [x]$ and $[x]$ denotes the largest integer less than or equal to x. The q-analogue of the factorial function is defined for positive integer n as

$$[n]_q! = \prod_{k=1}^{n} [k]_q = \frac{(q;q)_n}{(1-q)^n} \quad \text{with} \quad [0]_q! = 1, \quad q \neq 1. \tag{2.2}$$

From now on, we will fix $q \in (0, 1)$.

Theorem 2.1. *For a positive integer n, the q-factorial function (2.2) can be expressed via the infinite integral as*

$$[n]_q! = \sqrt{1-q^n}\, e^{\frac{-\pi^2}{6\log q}}(q;q)_\infty [n]_q^n \exp\left(\frac{Li_2(1-q^n)}{\log q} + \int_n^\infty \frac{q^x \log q\left(\{x\}-\frac{1}{2}\right)}{1-q^x}dx\right) \tag{2.3}$$

$$= \sqrt{2\pi[n]_q}\, S_q [n]_q^n \exp\left(\frac{Li_2(1-q^n)}{\log q} + \int_n^\infty \frac{q^x \log q\left(\{x\}-\frac{1}{2}\right)}{1-q^x}dx\right) \tag{2.4}$$

where

$$S_q = q^{\frac{-1}{24}}\sqrt{\frac{q-1}{\log q}} \sum_{m=-\infty}^{\infty} \left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right), \quad r = \exp(4\pi^2/\log q). \tag{2.5}$$

Proof. Let $f(x) = \log(1-q^x)$ and substitute into Euler-Maclaurin formula (2.1), we get

$$\sum_{k=1}^{n} \log(1-q^k) = \int_1^n \log(1-q^x)dx + \frac{1}{2}\left(\log(1-q^n) + \log(1-q)\right) - \int_1^n \frac{q^x \log q}{1-q^x} P_1(x)dx. \tag{2.6}$$

It is easy to show that the first integral can be computed as

$$\int_1^n \log(1-q^x)dx = \int_q^{q^n} \frac{\log(1-t)}{t\log q}dt = \frac{\mathrm{Li}_2(q) - \mathrm{Li}_2(q^n)}{\log q}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function defined as in (1.4) and it has the identity

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) + \frac{\pi^2}{6} - \log z \log(1-z), \qquad 0 < z \le 1$$

which reveals that

$$\int_1^n \log(1-q^x)dx = \frac{\mathrm{Li}_2(1-q^n) - \mathrm{Li}_2(1-q)}{\log q} + n\log(1-q^n) - \log(1-q). \tag{2.7}$$

Substituting $P_1(x) = \{x\} - \frac{1}{2}$ and (2.7) into (2.6) to obtain

$$\log[n]_q! = \frac{\mathrm{Li}_2(1-q^n) - \mathrm{Li}_2(1-q)}{\log q} + \left(n + \frac{1}{2}\right)\log[n]_q - \int_1^n \frac{q^x \log q\left(\{x\} - \frac{1}{2}\right)}{1-q^x}dx$$

which can be rewritten as

$$[n]_q! = [n]_q^n \sqrt{[n]_q} \exp\left(\frac{\mathrm{Li}_2(1-q^n)}{\log q} + \delta_n\right), \tag{2.8}$$

where

$$\delta_n = -\frac{\mathrm{Li}_2(1-q)}{\log q} - \int_1^n \frac{q^x \log q\left(\{x\} - \frac{1}{2}\right)}{1-q^x}dx.$$

The Dirichlet test for convergence of infinite integral shows that

$$\int_1^\infty \frac{q^x\left(\{x\} - \frac{1}{2}\right)}{1-q^x}dx$$

is convergent for $0 < q < 1$. Therefore, we can define

$$\lim_{n\to\infty} \delta_n = -\frac{\mathrm{Li}_2(1-q)}{\log q} - \int_1^\infty \frac{q^x \log q\left(\{x\} - \frac{1}{2}\right)}{1-q^x}dx = \delta.$$

From (2.8), we have

$$e^{2\delta_n - \delta_{2n}} = \frac{([n]_q!)^2 [2n]_q^{2n}\sqrt{[2n]_q}}{[2n]_q! [n]_q^{2n+1}} \exp\left(\frac{\mathrm{Li}_2(1-q^{2n}) - 2\mathrm{Li}_2(1-q^n)}{\log q}\right).$$

When letting $n \to \infty$, this leads to

$$\begin{aligned} e^\delta &= \sqrt{1-q}\, e^{\frac{-\pi^2}{6\log q}} \lim_{n\to\infty} \frac{(q;q)_n^2\sqrt{1-q^{2n}}}{(q;q)_{2n}(1-q^n)} \lim_{n\to\infty}\left(\frac{1-q^{2n}}{1-q^n}\right)^{2n} \\ &= \sqrt{1-q}\, e^{\frac{-\pi^2}{6\log q}}(q;q)_\infty. \end{aligned}$$

These conclude that

$$\begin{aligned} [n]_q! &= \sqrt{1-q^n}\, e^{\frac{-\pi^2}{6\log q}}(q;q)_\infty [n]_q^n \exp\left(\frac{\mathrm{Li}_2(1-q^n)}{\log q} + \delta_n - \delta\right) \\ &= \sqrt{1-q^n}\, e^{\frac{-\pi^2}{6\log q}}(q;q)_\infty [n]_q^n \exp\left(\frac{\mathrm{Li}_2(1-q^n)}{\log q} + \int_n^\infty \frac{q^x \log q\left(\{x\} - \frac{1}{2}\right)}{1-q^x}dx\right). \end{aligned}$$

In order to prove (2.4), inserting the formula

$$(q;q)_\infty = \left(\frac{r}{q}\right)^{\frac{1}{24}} \sqrt{\frac{2\pi}{-\log q}} \sum_{m=-\infty}^{\infty} \left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right)$$

which was proven by Moak [6], into (2.3) to obtain the desired result (2.4).

Corollary 2.2. *For a positive integer n, the q-factorial function (2.2) can be expressed via the double infinite series as*

$$[n]_q! = \sqrt{2\pi[n]_q} S_q [n]_q^n e^{\frac{Li_2(1-q^n)}{\log q}} \times \exp\left(\sum_{k=n}^{\infty}\sum_{i=1}^{\infty} \frac{i(-1)^{i-1}(k+\frac{1}{2})q^{ik}\log q + (1-q^k)^i([k+1]_q^i - [k]_q^i)}{i^2[k]_q^i \log q}\right). \tag{2.9}$$

Proof. Since the infinite integral in the previous theorem is convergent, then we have

$$\begin{aligned}
&\int_n^\infty \frac{q^x \log q\left(\{x\} - \frac{1}{2}\right)}{1-q^x} dx \\
&= \sum_{k=n}^{\infty} \int_k^{k+1} \frac{q^x \log q\left(\{x\} - \frac{1}{2}\right)}{1-q^x} dx \\
&= \sum_{k=n}^{\infty} \int_k^{k+1} \frac{q^x \log q\left(x - k - \frac{1}{2}\right)}{1-q^x} dx \\
&= \sum_{k=n}^{\infty} \left[\left(k+\frac{1}{2}\right)\log\left(\frac{1-q^{k+1}}{1-q^k}\right) + \frac{\mathrm{Li}_2(1-q^{k+1}) - \mathrm{Li}_2(1-q^k)}{\log q}\right] \\
&= \sum_{k=n}^{\infty} \left[\left(k+\frac{1}{2}\right)\log\left(1+\frac{q^k}{[k]_q}\right) + \frac{\mathrm{Li}_2(1-q^{k+1}) - \mathrm{Li}_2(1-q^k)}{\log q}\right] \\
&= \sum_{k=n}^{\infty}\sum_{i=1}^{\infty} \frac{i(-1)^{i-1}(k+\frac{1}{2})q^{ik}\log q + (1-q^k)^i([k+1]_q^i - [k]_q^i)}{i^2[k]_q^i \log q}.
\end{aligned}$$

This completes the proof.

Remark 2.3. It is not difficult to see that the results obtained in (2.4) and (2.9) when letting $q \to 1$ tend to the same results for the factorial function (see [3]).

Corollary 2.4. *For a positive integer n, the q-factorial function (2.2) can be expressed via the infinite series as*

$$[n]_q! = \sqrt{2\pi[n]_q}S_q(1-q)^{-n}\exp\left(\frac{\pi^2}{6\log q}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{q^{in}(1+q^i)}{i(1-q^i)}\right) \tag{2.10}$$

and the q-shifted factorial can also be expressed as

$$(q;q)_n = \sqrt{2\pi[n]_q}S_q\exp\left(\frac{\pi^2}{6\log q}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{q^{in}(1+q^i)}{i(1-q^i)}\right). \tag{2.11}$$

Proof. If $\lim_{N\to\infty} f(N)$ exists, then we get

$$\sum_{k=n}^{\infty}(f(k+1)-f(k)) = \lim_{N\to\infty}\sum_{k=n}^{N}(f(k+1)-f(k)) = \lim_{N\to\infty} f(N+1)-f(n).$$

Applying the above rule with using the results obtained in the proof of Corollary 2.2 would yield

$$\begin{aligned}
&\int_n^{\infty}\frac{q^x\log q\left(\{x\}-\frac{1}{2}\right)}{1-q^x}dx\\
&=\sum_{k=n}^{\infty}\left[\left(k+\frac{1}{2}\right)\log\left(\frac{1-q^{k+1}}{1-q^k}\right)+\frac{\mathrm{Li}_2(1-q^{k+1})-\mathrm{Li}_2(1-q^k)}{\log q}\right]\\
&=\lim_{N\to\infty}\sum_{k=n}^{N}\left[(k+1)\log(1-q^{k+1})-k\log(1-q^k)\right]\\
&+\lim_{N\to\infty}\sum_{k=n}^{N}\left[\frac{\mathrm{Li}_2(1-q^{k+1})-\mathrm{Li}_2(1-q^k)}{\log q}\right]-\frac{1}{2}\sum_{k=n}^{\infty}\left[\log(1-q^{k+1})+\log(1-q^k)\right]\\
&=-n\log(1-q^n)+\frac{\pi^2}{6\log q}-\frac{\mathrm{Li}_2(1-q^n)}{\log q}+\frac{1}{2}\sum_{k=n}^{\infty}\sum_{i=1}^{\infty}\frac{q^{ik}(1+q^i)}{i}\\
&=-n\log(1-q^n)+\frac{\pi^2}{6\log q}-\frac{\mathrm{Li}_2(1-q^n)}{\log q}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{q^{in}(1+q^i)}{i(1-q^i)}.
\end{aligned}$$

Substituting into (2.4) to end the proof.

Remark 2.5. In view of the value of the term $\frac{\pi^2}{6\log q}$ when letting $q \to 1$, we get $\lim_{q\to1} \frac{\pi^2}{6\log q} = \infty$ but $\lim_{q\to1} \frac{-\pi^2}{6\log q^{-1}} = -\infty$ which occurs confused. However, we can realize this by changing the form of exponent in (2.10) and (2.11) to be

$$f(q) = \frac{1}{1-q}\left(\frac{\pi^2}{6}\frac{1-q}{\log q} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{q^{in}(1+q^i)}{i[i]_q}\right) \tag{2.12}$$

which reveals that $\lim_{q\to1}(1-q)f(q) = 0$. By using the l'Hospital rule, we deduce that $\lim_{q\to1} f(q) = -\infty$ and thus the right hand side of (2.11) approaches to zero when letting $q \to 1$ which equals precisely the left hand side in the same equation.

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