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Transmutation Operators Associated with a Bessel Type Operator on The Half Line and Certain of Their Applications *

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Abstract

We consider a second-order singular differential operator L on the half line which generalizes the Bessel operator. We construct a pair transmutation operators between L and the second derivative operator d^2/dx^2. Using these transmutation operators, we firstly establish a Paley-Wiener theorem for the Fourier transform associated to L, and secondly introduce a generalized convolution on $[0,\infty[$ tied to L. Furthermore, a generalization of the classical Sonine integral transform is built.

Keywords and Phrases: *Transmutation operators, Generalized Fourier transform, Generalized convolution, Generalized Sonine integral transform.*

1. Introduction

The german astronomer F.W. Bessel (1784-1846) first achieved fame by computing the orbit of Halley's comet. In addition to many other accomplishments in connection with his studies of planetary motion, he is credited with deriving

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the differential equation bearing his name and carrying out the first systematic study of the general properties of its solutions (now called Bessel functions) in his famous 1824 memoir. Nevertheless, Bessel functions were first discovered in 1732 by D. Bernoulli (1700-1782), who provided a series solution (representing a Bessel function) for the oscillatory displacements of a heavy hanging chain (see [1]). Euler later developed a series similar to that of Bernoulli, which was also a Bessel function, and Bessel's equation appeared in a 1764 article by Euler dealing with the vibrations of a circular drumhead. J. Fourier (1768-1836) also used Bessel functions in his classical treatise on heat in 1822, but it was Bessel who first recognized their special properties. Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane. They also occur in electromagnetic theory and numerous other areas of physics and engineering (see [1]). The more complete reference about Bessel functions is the treatise of Watson [4].

Trimèche [2, 3] has pointed out how the theory of Bessel functions generates an harmonic analysis on the half line tied to the differential operator

$$L_\alpha f(x) = \frac{d^2 f}{dx^2} + \frac{2\alpha+1}{x}\frac{df}{dx}, \quad \alpha > -1/2, \tag{1}$$

which is referred to as the Bessel operator of index α. A summary of this harmonic analysis is provided in Section 2.

Consider the following generalization of the Bessel operator :

$$L_{\alpha,n} f(x) = \frac{d^2 f}{dx^2} + \frac{2\alpha+1}{x}\frac{df}{dx} - \frac{4n(\alpha+n)}{x^2} f(x), \tag{2}$$

with $n = 0, 1, \ldots$. Throughout this paper, several known analytic structures related to the Bessel operator L_α are generalized. More explicitly, we propose the following program.

In Section 3, we construct a pair of transmutation operators $R_{\alpha,n}$ and $W_{\alpha,n}$ between $L_{\alpha,n}$ and the second derivative operator d^2/dx^2. Mainly, we prove that $R_{\alpha,n}$ and $W_{\alpha,n}$ are isomorphism between suitable functional spaces, satisfying the intertwining relations

$$R_{\alpha,n} \circ \frac{d^2}{dx^2} = L_{\alpha,n} \circ R_{\alpha,n}$$

$$\frac{d^2}{dx^2} \circ W_{\alpha,n} = W_{\alpha,n} \circ L_{\alpha,n}$$

In Section 4, we exploit the transmutation operators $R_{\alpha,n}$ and $W_{\alpha,n}$ to build a completely new commutative harmonic analysis on the half line corresponding to the differential operator $L_{\alpha,n}$. More precisely, we define a generalized Fourier transform $\mathcal{F}_{\alpha,n}$ on $[0,\infty[$ associated to $L_{\alpha,n}$ by the formula

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_0^\infty f(x)\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx,$$

with

$$\varphi_{\lambda,\alpha,n}(x) = \Gamma(\alpha+2n+1)\, x^{2n} \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda x/2)^{2n}}{n!\,\Gamma(3n+\alpha+1)}.$$

We establish for the generalized Fourier transform $\mathcal{F}_{\alpha,n}$, a Paley-Wiener theorem, an inversion formula and a Plancherel theorem.

Next, we introduce a generalized convolution product $*_{\alpha,n}$ on $[0,\infty[$ tied to the differential operator $L_{\alpha,n}$, by putting

$$f *_{\alpha,n} g(x) = \int_0^\infty T^x_{\alpha,n} f(y)g(y)y^{2\alpha+1}dy,$$

where $T^x_{\alpha,n}$ stand for the generalized translation operators tied to $L_{\alpha,n}$ given by

$$T^x_{\alpha,n}\, f(y) = \frac{1}{2}\,(R_{\alpha,n})_x\,(R_{\alpha,n})_y\left[R^{-1}_{\alpha,n}f(x+y) + R^{-1}_{\alpha,n}f(x-y)\right].$$

Such a convolution is mapped firstly by the generalized Fourier transform $\mathcal{F}_{\alpha,n}$ into the simple product, and secondly by the transmutation operator $W_{\alpha,n}$ into the ordinary convolution.

Section 5 is devoted to the study of the following integral transform

$$S^{m,n}_{\alpha,\beta}(f)(x) = \frac{2\,\Gamma(\beta+2m+1)}{\Gamma(\alpha+2n+1)\Gamma(\beta-\alpha+2(m-n))}\, x^{2(m-n)} \times$$

$$\times \int_0^1 f(tx)\,(1-t^2)^{\beta-\alpha+2(m-n)-1}\, t^{2\alpha+2n+1}\, dt,$$

where $\beta > \alpha > -1/2$ and m, n two non-negative integers such that $m \geq n$. For $m = n = 0$, $S^{m,n}_{\alpha,\beta}$ reduces to the classical Sonine integral transform of order (α,β) (see [3]). Essentially, it is shown that $S^{m,n}_{\alpha,\beta}$ and its dual ${}^tS^{m,n}_{\alpha,\beta}$ are isomorphism between appropriate functional spaces, satisfying the intertwining relations

$$S^{m,n}_{\alpha,\beta} \circ L_{\alpha,n} = L_{\beta,m} \circ S^{m,n}_{\alpha,\beta}$$

$$L_{\alpha,n} \circ {}^tS^{m,n}_{\alpha,\beta} = {}^tS^{m,n}_{\alpha,\beta} \circ L_{\beta,m}$$

Thanks to $S^{m,n}_{\alpha,\beta}$ and ${}^tS^{m,n}_{\alpha,\beta}$, all harmonic analysis tools related to $L_{\beta,m}$ may be expressed in terms of their analogous related to $L_{\alpha,n}$.

2. Preliminaries

Notation. Throughout this section assume $\alpha > -1/2$. We denote by $\mathcal{E}(\mathbb{R})$ the space of C^∞ even functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. For $a > 0$, $\mathcal{D}_a(\mathbb{R})$ designates the space of C^∞ even functions on $\mathbb{R}$, which are supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$. Put

$$\mathcal{D}(\mathbb{R}) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R}),$$

endowed with the inductive limit topology. We denote by $\mathbf{H}_a$, $a > 0$, the space of entire even and rapidly decreasing functions of exponential type a; that is, $f \in \mathbf{H}_a$ if and only if, f is entire, even on $\mathbb{C}$ and for all $m = 0, 1, ...,$

$$p_m(f) = \sup_{\lambda\in\mathbb{C}} |(1+\lambda)^m f(\lambda) e^{-a|Im\lambda|}| < \infty.$$

$\mathbf{H}_a$ is equipped with the topology defined by the semi-norms p_m, $m = 0, 1, ...$ Put

$$\mathbf{H} = \bigcup_{a>0} \mathbf{H}_a$$

equipped with the inductive limit topology.

In this section we recall some facts about harmonic analysis related to the Bessel operator L_α. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [2, 3].

The normalized spherical Bessel function of index α is defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} \quad (z\in\mathbb{C}) \tag{3}$$

The function j_α possesses the Laplace type integral representation

$$j_\alpha(z) = a_\alpha\int_0^1 cos(zt)(1-t^2)^{\alpha-1/2}\,dt, \tag{4}$$

where

$$a_\alpha = \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}. \tag{5}$$

The function j_α is the unique solution of the differential equation

$$L_\alpha u + u = 0, \quad u(0) = 1, \quad u'(0) = 0. \tag{6}$$

The Riemann-Liouville integral transform R_α is defined on $\mathcal{E}(\mathbb{R})$ by

$$R_\alpha(f)(x) = a_\alpha\int_0^1 f(tx)(1-t^2)^{\alpha-1/2}\,dt, \quad x\in\mathbb{R}. \tag{7}$$

From [2] it is known that R_α is the unique automorphism of $\mathcal{E}(\mathbb{R})$ satisfying

$$R_\alpha\circ\frac{d^2}{dx^2}(f) = L_\alpha\circ R_\alpha(f) \qquad \text{and} \qquad R_\alpha(f)(0) = f(0), \tag{8}$$

for all $f\in\mathcal{E}(\mathbb{R})$.

The Weyl integral transform W_α is defined on $\mathcal{D}(\mathbb{R})$ by

$$W_\alpha(f)(y) = a_\alpha\int_{|y|}^{\infty} f(x)\,(x^2-y^2)^{\alpha-1/2}\,x\,dx, \quad y\in\mathbb{R}. \tag{9}$$

The transforms R_α and W_α are dual in the sense of the relationship

$$\int_0^\infty R_\alpha(f)(x)g(x)x^{2\alpha+1}dx = \int_0^\infty f(y)W_\alpha(g)(y)dy, \tag{10}$$

which is valid for any $f\in\mathcal{E}(\mathbb{R})$ and $g\in\mathcal{D}(\mathbb{R})$.

By [2] we know that W_α is an automorphism of $\mathcal{D}(\mathbb{R})$ satisfying the intertwining relation

$$W_\alpha \circ L_\alpha(f) = \frac{d^2}{dx^2} \circ W_\alpha(f), \qquad f \in \mathcal{D}(\mathbb{R}). \tag{11}$$

The Fourier-Bessel transform of a function $f \in \mathcal{D}(\mathbb{R})$ is defined by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx, \quad \lambda \in \mathbb{C}. \tag{12}$$

Theorem 2.1. *(i) The Fourier-Bessel transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{D}(\mathbb{R})$ onto $\mathbf{H}$. More precisely, $f \in \mathcal{D}_a(\mathbb{R})$ if, and only if, $\mathcal{F}_\alpha(f) \in \mathbf{H}_a$.*

(ii) For every $f \in \mathcal{D}(\mathbb{R})$, we have

$$f(x) = m_\alpha \int_0^\infty \mathcal{F}_\alpha(f)(\lambda) j_\alpha(\lambda x) \lambda^{2\alpha+1} d\lambda,$$

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx = m_\alpha \int_0^\infty |\mathcal{F}_\alpha(f)(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

where

$$m_\alpha = \frac{1}{4^\alpha (\Gamma(\alpha+1))^2}. \tag{13}$$

The Bessel translation operators T_α^x, $x \in \mathbb{R}$, are defined by

$$T_\alpha^x(f)(y) = a_\alpha \int_0^\pi f(\sqrt{x^2 + y^2 + 2\,xy\,cos\theta})(sin\theta)^{2\alpha} d\theta, \quad y \in \mathbb{R}.$$

Proposition 2.1. *(i) For every $f \in \mathcal{E}(\mathbb{R})$,*

$$T_\alpha^x(f)(y) = (R_\alpha)_x\,(R_\alpha)_y \left[\sigma_x\, R_\alpha^{-1}(f)(y)\right], \tag{14}$$

where

$$\sigma_x\, f(y) = \frac{f(x+y) + f(x-y)}{2}.$$

(ii) For all $x \in \mathbb{R}$, T_α^x is a linear bounded operator from $\mathcal{E}(\mathbb{R})$ into itself; the function $x \mapsto T_\alpha^x$ is C^∞.

(iii) We have

$$T_\alpha^0 = identity, \quad T_\alpha^x T_\alpha^y = T_\alpha^y T_\alpha^x, \quad L_\alpha T_\alpha^x = T_\alpha^x L_\alpha.$$

(iv) For all $f \in \mathcal{E}(\mathbb{R})$,

$$T_\alpha^x f(y) = T_\alpha^y f(x).$$

(v) For each $\lambda \in \mathbb{C}$, *we have the product formula*

$$T_\alpha^x(j_\alpha(\lambda\cdot))(y) = j_\alpha(\lambda x) j_\alpha(\lambda y).$$

(vi) Let f *be in* $\mathcal{D}_a(\mathbb{R})$. *Then for all* $x \in \mathbb{R}$, $T_\alpha^x f$ *is an element of* $\mathcal{D}_{a+|x|}(\mathbb{R})$ *and*

$$\mathcal{F}_\alpha(T_\alpha^x f)(\lambda) = j_\alpha(\lambda x)\, \mathcal{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{C}. \tag{15}$$

(vii) For all $f \in \mathcal{E}(\mathbb{R})$ *and* $g \in \mathcal{D}(\mathbb{R})$,

$$\int_0^\infty T_\alpha^x(f)(y) g(y) y^{2\alpha+1} dy = \int_0^\infty f(y)\, T_\alpha^x(g)(y) y^{2\alpha+1} dy. \tag{16}$$

The Bessel convolution product of $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$ is defined by

$$f *_\alpha g(x) = \int_0^\infty T_\alpha^x f(y) g(y) y^{2\alpha+1} dy, \quad x \in \mathbb{R}. \tag{17}$$

Proposition 2.2. *(i) Let* $f \in \mathcal{D}_a(\mathbb{R})$ *and* $g \in \mathcal{D}_b(\mathbb{R})$. *Then* $f *_\alpha g \in \mathcal{D}_{a+b}(\mathbb{R})$ *and*

$$\mathcal{F}_\alpha(f *_\alpha g)(\lambda) = \mathcal{F}_\alpha(f)(\lambda)\mathcal{F}_\alpha(g)(\lambda), \quad \lambda \in \mathbb{C}.$$

(ii) For all $f, g \in \mathcal{D}(\mathbb{R})$,

$$W_\alpha(f *_\alpha g) = W_\alpha(f) * W_\alpha(g),$$

where $*$ *is the symmetric convolution product on* $\mathbb{R}$ *given by*

$$f * g(x) = \int_0^\infty \sigma_x f(y) g(y) dy. \tag{18}$$

3. Transmutation operators

Notation. Throughout this section assume $\alpha > -1/2$ and n a non-negative integer. Let $\mathcal{E}_n(\mathbb{R})$ (resp. $\mathcal{D}_n(\mathbb{R})$) stand for the subspace of $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{D}(\mathbb{R})$) consisting of functions f such that

$$f(0) = \cdots = f^{(2n-1)}(0) = 0.$$

For $a > 0$, put

$$\mathcal{D}_{a,n}(\mathbb{R}) = \mathcal{D}_a(\mathbb{R}) \cap \mathcal{E}_n(\mathbb{R}).$$

The following technical lemma will be useful.

Lemma 3.1. *(i) The map*

$$\mathcal{M}_n(f)(x) = x^{2n} f(x) \tag{19}$$

is an isomorphism

– *from $\mathcal{E}(\mathbb{R})$ onto $\mathcal{E}_n(\mathbb{R})$;*

– *from $\mathcal{D}(\mathbb{R})$ onto $\mathcal{D}_n(\mathbb{R})$.*

(ii) For all $f \in \mathcal{E}(\mathbb{R})$,

$$L_{\alpha,n} \circ \mathcal{M}_n(f) = \mathcal{M}_n \circ L_{\alpha+2n}(f), \tag{20}$$

where $L_{\alpha+2n}$ is the Bessel operator of order $\alpha + 2n$ given by (1).

(iii) The differential operator $L_{\alpha,n}$ is self-adjoint ,i.e,

$$\int_0^\infty L_{\alpha,n} f(x) g(x) x^{2\alpha+1} dx = \int_0^\infty f(x) L_{\alpha,n} g(x)\, x^{2\alpha+1} dx, \tag{21}$$

for all $f \in \mathcal{E}_n(\mathbb{R})$ and $g \in \mathcal{D}_n(\mathbb{R})$.

Proof. Assertion (i) is easily checked. For all $f \in \mathcal{E}(\mathbb{R})$ we have

$$\begin{aligned} L_{\alpha,n}\left(x^{2n} f\right)(x) &= \left(x^{2n} f\right)'' + \frac{2\alpha+1}{x}\left(x^{2n} f\right)' - 4n(\alpha+n) x^{2n-2} f(x) \\ &= x^{2n} f''(x) + (2\alpha + 4n + 1) x^{2n-1} f'(x) \\ &= x^{2n}\left(f''(x) + \frac{2\alpha+4n+1}{x} f'(x)\right) \\ &= x^{2n} L_{\alpha+2n} f(x), \end{aligned}$$

which gives (ii). Let us verify (iii). If $f \in \mathcal{E}_n(\mathbb{R})$ and $g \in \mathcal{D}_n(\mathbb{R})$, then by (2),

$$\begin{aligned}\int_0^\infty L_{\alpha,n} f(x) g(x) x^{2\alpha+1} dx &= \int_0^\infty \Big(L_\alpha f(x) - \frac{4n(\alpha+n)}{x^2} f(x) \Big) g(x) x^{2\alpha+1} dx \\ &= \int_0^\infty L_\alpha f(x) g(x) x^{2\alpha+1} dx \\ &- \int_0^\infty \frac{4n(\alpha+n)}{x^2} f(x) g(x) x^{2\alpha+1} dx.\end{aligned}$$

But by [2],

$$\int_0^\infty L_\alpha f(x) g(x) x^{2\alpha+1} dx = \int_0^\infty f(x) L_\alpha g(x) x^{2\alpha+1} dx.$$

So

$$\begin{aligned}\int_0^\infty L_{\alpha,n} f(x) g(x) x^{2\alpha+1} dx &= \int_0^\infty f(x) L_\alpha g(x) x^{2\alpha+1} dx \\ &- \int_0^\infty \frac{4n(\alpha+n)}{x^2} f(x) g(x) x^{2\alpha+1} dx \\ &= \int_0^\infty f(x) \Big(L_\alpha g(x) - \frac{4n(\alpha+n)}{x^2} g(x) \Big) x^{2\alpha+1} dx \\ &= \int_0^\infty f(x) L_{\alpha,n} g(x) x^{2\alpha+1} dx.\end{aligned}$$

This ends the proof. □

Remark 3.1. *In view of Lemma 3.1, $L_{\alpha,n}$ is a bounded linear operator from $\mathcal{E}_n(\mathbb{R})$ (resp. $\mathcal{D}_n(\mathbb{R})$) into itself.*

Theorem 3.1. *The integral transform*

$$R_{\alpha,n}(f)(x) = a_{\alpha+2n}\, x^{2n} \int_0^1 f(tx)(1-t^2)^{\alpha+2n-1/2}\, dt, \tag{22}$$

where $a_{\alpha+2n}$ is given by (5), is an isomorphism from $\mathcal{E}(\mathbb{R})$ onto $\mathcal{E}_n(\mathbb{R})$ satisfying the intertwining relation

$$R_{\alpha,n} \circ \frac{d^2}{dx^2}(f) = L_{\alpha,n} \circ R_{\alpha,n}(f), \quad f \in \mathcal{E}(\mathbb{R}).$$

Proof. By (7) and (22) observe that

$$R_{\alpha,n} = \mathcal{M}_n \circ R_{\alpha+2n}, \tag{23}$$

where $\mathcal{M}_n$ is given by (19). As $R_{\alpha+2n}$ is an automorphism of $\mathcal{E}(\mathbb{R})$, we deduce from (23) and Lemma 3.1(i) that $R_{\alpha,n}$ is an isomorphism from $\mathcal{E}(\mathbb{R})$ onto $\mathcal{E}_n(\mathbb{R})$. Moreover, by (8), (20) and (23),

$$\begin{aligned} R_{\alpha,n} \circ \frac{d^2}{dx^2}(f) &= \mathcal{M}_n \circ R_{\alpha+2n} \circ \frac{d^2}{dx^2}(f) \\ &= \mathcal{M}_n \circ L_{\alpha+2n} \circ R_{\alpha+2n}(f) \\ &= L_{\alpha,n} \circ \mathcal{M}_n \circ R_{\alpha+2n}(f) \\ &= L_{\alpha,n} \circ R_{\alpha,n}(f), \end{aligned}$$

which completes the proof. □

Remark 3.2. *According to Theorem 3.1, $R_{\alpha,n}$ is a transmutation operator between $L_{\alpha,n}$ and d^2/dx^2.*

Define the dual transform of $R_{\alpha,n}$ on $\mathcal{D}_n(\mathbb{R})$ by

$$W_{\alpha,n}(f)(y) = a_{\alpha+2n} \int_{|y|}^{\infty} f(x)\,(x^2 - y^2)^{\alpha+2n-1/2}\,\frac{dx}{x^{2n-1}}, \quad y \in \mathbb{R}, \tag{24}$$

where $a_{\alpha+2n}$ is given by (5).

Remark 3.3. *By (9) and (24) notice that*

$$W_{\alpha,n} = W_{\alpha+2n} \circ \mathcal{M}_n^{-1}. \tag{25}$$

Proposition 3.1. *We have the duality relation*

$$\int_0^{\infty} R_{\alpha,n}(f)(x)g(x)x^{2\alpha+1}dx = \int_0^{\infty} f(y)W_{\alpha,n}(g)(y)dy, \tag{26}$$

valid for any $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}_n(\mathbb{R})$.

Proof. Using (10), (23) and (25), we get

$$\begin{aligned} \int_0^{\infty} R_{\alpha,n}(f)(x)g(x)x^{2\alpha+1}dx &= \int_0^{\infty} R_{\alpha+2n}(f)(x)\,\mathcal{M}_n^{-1}g(x)\,x^{2\alpha+4n+1}dx \\ &= \int_0^{\infty} f(y)\,W_{\alpha+2n}\big(\mathcal{M}_n^{-1}g\big)(y)\,dy \\ &= \int_0^{\infty} f(y)W_{\alpha,n}(g)(y)dy. \quad \square \end{aligned}$$

Theorem 3.2. *The integral transform* $W_{\alpha,n}$ *is an isomorphism from* $\mathcal{D}_n(\mathbb{R})$ *onto* $\mathcal{D}(\mathbb{R})$ *satisfying the intertwining relation*

$$\frac{d^2}{dx^2} \circ W_{\alpha,n}(f) = W_{\alpha,n} \circ L_{\alpha,n}(f), \quad f \in \mathcal{D}_n(\mathbb{R}).$$

Proof. As $W_{\alpha+2n}$ is an automorphism of $\mathcal{D}(\mathbb{R})$, it follows from (25) and Lemma 3.1(i) that $W_{\alpha,n}$ is an isomorphism from $\mathcal{D}_n(\mathbb{R})$ onto $\mathcal{D}(\mathbb{R})$. Furthermore, by (11), (20) and (25), we have

$$\begin{aligned}
\frac{d^2}{dx^2} \circ W_{\alpha,n}(f) &= \frac{d^2}{dx^2} \circ W_{\alpha+2n} \circ \mathcal{M}_n^{-1}(f) \\
&= W_{\alpha+2n} \circ L_{\alpha+2n} \circ \mathcal{M}_n^{-1}(f) \\
&= W_{\alpha+2n} \circ \mathcal{M}_n^{-1} \circ L_{\alpha,n}(f) \\
&= W_{\alpha,n} \circ L_{\alpha,n}(f),
\end{aligned}$$

which achieves the proof. □

Remark 3.4. *From Theorem 3.2 we deduce that* $W_{\alpha,n}$ *is a transmutation operator between* $L_{\alpha,n}$ *and* d^2/dx^2.

4. Generalized Fourier transform – Generalized convolution product

Throughout this section assume $\alpha > -1/2$ and n a non-negative integer.

4.1 The Fourier transform associated with $L_{\alpha,n}$

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\varphi_{\lambda,\alpha,n}(x) = x^{2n} j_{\alpha+2n}(\lambda x), \tag{27}$$

where $j_{\alpha+2n}$ is the normalized Bessel function of index $\alpha + 2n$ given by (3).

Proposition 4.1. *(i) The kernel* $\varphi_{\lambda,\alpha,n}$ *possesses the Laplace type integral representation*

$$\varphi_{\lambda,\alpha,n}(x) = a_{\alpha+2n}\, x^{2n} \int_0^1 cos(\lambda t x)(1-t^2)^{\alpha+2n-1/2}\, dt, \tag{28}$$

where $a_{\alpha+2n}$ is given by (5).

(ii) $\varphi_{\lambda,\alpha,n}$ satisfies the differential equation

$$L_{\alpha,n}\,\varphi_{\lambda,\alpha,n} = -\lambda^2\,\varphi_{\lambda,\alpha,n}. \tag{29}$$

Proof. Statement (i) follows directly from (4) and (27). Set

$$\psi_{\lambda,\alpha,n}(x) = j_{\alpha+2n}(\lambda x).$$

Notice by (27) that

$$\varphi_{\lambda,\alpha,n} = \mathcal{M}_n(\psi_{\lambda,\alpha,n}).$$

Moreover, it is easily seen from (6) that

$$L_{\alpha+2n}\,\psi_{\lambda,\alpha,n} = -\lambda^2\,\psi_{\lambda,\alpha,n}.$$

So using (20), we obtain

$$\begin{aligned} L_{\alpha,n}\,\varphi_{\lambda,\alpha,n} &= L_{\alpha,n}\circ\mathcal{M}_n(\psi_{\lambda,\alpha,n}) \\ &= \mathcal{M}_n\circ L_{\alpha+2n}(\psi_{\lambda,\alpha,n}) \\ &= -\lambda^2\,\mathcal{M}_n(\psi_{\lambda,\alpha,n}) \\ &= -\lambda^2\,\varphi_{\lambda,\alpha,n}, \end{aligned}$$

which proves (ii). □

Remark 4.1. *By (22) and (28) notice that*

$$\varphi_{\lambda,\alpha,n}(x) = R_{\alpha,n}(cos(\lambda\cdot))(x). \tag{30}$$

Definition 4.1. *The generalized Fourier transform of a function $f \in \mathcal{D}_n(\mathbb{R})$ is defined by*

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_0^\infty f(x)\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx, \quad \lambda\in\mathbb{C}. \tag{31}$$

Proposition 4.2. *We have*

$$\mathcal{F}_{\alpha,n} = \mathcal{F}_{\alpha+2n}\circ\mathcal{M}_n^{-1}. \tag{32}$$

Proof. Let $f \in \mathcal{D}_n(\mathbb{R})$. From (12), (27) and (31) we have

$$\begin{aligned}\mathcal{F}_{\alpha,n}(f)(\lambda) &= \int_0^\infty f(x)\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx\\ &= \int_0^\infty \mathcal{M}_n^{-1}f(x)\, j_{\alpha+2n}(\lambda x)x^{2\alpha+4n+1}dx\\ &= \mathcal{F}_{\alpha+2n}\big(\mathcal{M}_n^{-1}f\big)(\lambda)\ . \quad \square\end{aligned}$$

Proposition 4.3. *Let $f \in \mathcal{D}_n(\mathbb{R})$. Then*

$$\mathcal{F}_{\alpha,n}(L_{\alpha,n}f)(\lambda) = -\lambda^2\, \mathcal{F}_{\alpha,n}(f)(\lambda).$$

Proof. From (21) and (29) we have

$$\begin{aligned}\mathcal{F}_{\alpha,n}(L_{\alpha,n}f)(\lambda) &= \int_0^\infty L_{\alpha,n}f(x)\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx\\ &= \int_0^\infty f(x)\, L_{\alpha,n}\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx\\ &= -\lambda^2\int_0^\infty f(x)\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx\\ &= -\lambda^2\, \mathcal{F}_{\alpha,n}(f)(\lambda). \quad \square\end{aligned}$$

Proposition 4.4. *We have*

$$\mathcal{F}_{\alpha,n} = \mathcal{F}_c \circ W_{\alpha,n} \tag{33}$$

where $\mathcal{F}_c$ is the cosine transform given by

$$\mathcal{F}_c(f)(\lambda) = \int_0^\infty f(x)cos(\lambda x)dx.$$

Proof. From (26) and (30) it follows that

$$\begin{aligned}\mathcal{F}_{\alpha,n}(f)(\lambda) &= \int_0^\infty f(x)\varphi_{\lambda,\alpha,n}(x)x^{2\alpha+1}dx\\ &= \int_0^\infty f(x)R_{\alpha,n}(cos(\lambda\cdot))(x)x^{2\alpha+1}dx\\ &= \int_0^\infty W_{\alpha,n}(f)(y)cos(\lambda y)dy\\ &= \mathcal{F}_c \circ W_{\alpha,n}(f)(\lambda). \quad \square\end{aligned}$$

Theorem 4.1. ***(Paley-Wiener)*** *The generalized Fourier transform $\mathcal{F}_{\alpha,n}$ is an isomorphism from $\mathcal{D}_n(\mathbb{R})$ onto $\mathbf{H}$. More precisely, $f \in \mathcal{D}_{a,n}(\mathbb{R})$ if, and only if, $\mathcal{F}_{\alpha,n}(f) \in \mathbf{H}_a$.*

Proof. The result follows directly from (32), Lemma 3.1(i) and Theorem 2.1(i). □

By combining (27), (32) and Theorem 2.1 we get the following two standard results

Theorem 4.2. ***(inversion formula)*** *For all $f \in \mathcal{D}_n(\mathbb{R})$,*

$$f(x) = m_{\alpha+2n} \int_0^\infty \mathcal{F}_{\alpha,n}(f)(\lambda)\varphi_{\lambda,\alpha,n}(x)\lambda^{2\alpha+4n+1}d\lambda,$$

where $m_{\alpha+2n}$ is given by (13).

Theorem 4.3. ***(Plancherel)*** *(i) For every $f \in \mathcal{D}_n(\mathbb{R})$, we have the Plancherel formula*

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1}dx = m_{\alpha+2n} \int_0^\infty |\mathcal{F}_{\alpha,n}(f)(\lambda)|^2 \lambda^{2\alpha+4n+1}d\lambda.$$

(ii) The generalized Fourier transform $\mathcal{F}_{\alpha,n}$ extends uniquely to an isometric isomorphism from $L^2([0,\infty[, x^{2\alpha+1}dx)$ onto $L^2([0,\infty[, m_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda)$.

4.2 The convolution product associated with $L_{\alpha,n}$

Definition 4.2. The generalized translation operators $T^x_{\alpha,n}$, $x \in \mathbb{R}$, associated with $L_{\alpha,n}$ are defined on $\mathcal{E}_n(\mathbb{R})$ by

$$T^x_{\alpha,n}\, f(y) = (R_{\alpha,n})_x\, (R_{\alpha,n})_y \left[\sigma_x\, R^{-1}_{\alpha,n} f(y)\right], \quad y \in \mathbb{R}. \tag{34}$$

Proposition 4.5. *We have*

$$T^x_{\alpha,n} = x^{2n}\, \mathcal{M}_n \circ T^x_{\alpha+2n} \circ \mathcal{M}_n^{-1}. \tag{35}$$

Proof. From (14), (23) and (34) we deduce that

$$\begin{aligned} T^x_{\alpha,n}\, f(y) &= (R_{\alpha,n})_x\, (R_{\alpha,n})_y \left[\sigma_x\, R^{-1}_{\alpha,n}\, f(y)\right] \\ &= (\mathcal{M}_n \circ R_{\alpha+2n})_x\, (\mathcal{M}_n \circ R_{\alpha+2n})_y \left[\sigma_x\, R^{-1}_{\alpha+2n}\mathcal{M}_n^{-1} f(y)\right] \\ &= x^{2n}\, y^{2n} (R_{\alpha+2n})_x\, (R_{\alpha+2n})_y \left[\sigma_x\, R^{-1}_{\alpha+2n}\mathcal{M}_n^{-1} f(y)\right] \\ &= x^{2n}\, y^{2n}\, T^x_{\alpha+2n}\big(\mathcal{M}_n^{-1} f\big)(y). \quad \square \end{aligned}$$

A combination of (20), (27), (35) and Proposition 2.1 yields the next statement which contains the essential properties of the generalized translation operators.

Theorem 4.4. *(i) For all $x \in \mathbb{R}$, $T^x_{\alpha,n}$ is a linear bounded operator from $\mathcal{E}_n(\mathbb{R})$ into itself; the function $x \mapsto T^x_{\alpha,n}$ is C^∞.*

(ii) We have

$$T^x_{\alpha,n} T^y_{\alpha,n} = T^y_{\alpha,n} T^x_{\alpha,n}, \quad L_{\alpha,n} T^x_{\alpha,n} = T^x_{\alpha,n} L_{\alpha,n}.$$

(iii) For all $f \in \mathcal{E}_n(\mathbb{R})$,

$$T^x_{\alpha,n} f(y) = T^y_{\alpha,n} f(x).$$

(iv) For each $\lambda \in \mathbb{C}$, $\varphi_{\lambda,\alpha,n}$ satisfies the product formula

$$T^x_{\alpha,n}(\varphi_{\lambda,\alpha,n})(y) = \varphi_{\lambda,\alpha,n}(x)\varphi_{\lambda,\alpha,n}(y).$$

Theorem 4.5. *(i) Let f be in $\mathcal{D}_{a,n}(\mathbb{R})$. Then for all $x \in \mathbb{R}$, $T^x_{\alpha,n} f$ is an element of $\mathcal{D}_{a+|x|,n}$, and*

$$\mathcal{F}_{\alpha,n}\big(T^x_{\alpha,n} f\big)(\lambda) = \varphi_{\lambda,\alpha,n}(x)\, \mathcal{F}_{\alpha,n}(f)(\lambda), \quad \lambda \in \mathbb{C}.$$

(ii) For all $f \in \mathcal{E}_n(\mathbb{R})$ and $g \in \mathcal{D}_n(\mathbb{R})$,

$$\int_0^\infty T^x_{\alpha,n}\, f(y)\, g(y) y^{2\alpha+1} dy = \int_0^\infty f(y)\, T^x_{\alpha,n}\, g(y) y^{2\alpha+1} dy.$$

Proof. (i) By (15), (27), (32) and (35) we have

$$\begin{aligned}
\mathcal{F}_{\alpha,n}\big(T^x_{\alpha,n} f\big)(\lambda) &= x^{2n}\, \mathcal{F}_{\alpha+2n}\big(T^x_{\alpha+2n}\, \mathcal{M}_n^{-1} f\big)(\lambda) \\
&= x^{2n}\, j_{\alpha+2n}(\lambda x)\, \mathcal{F}_{\alpha+2n}\big(\mathcal{M}_n^{-1} f\big)(\lambda) \\
&= \varphi_{\lambda,\alpha,n}(x)\, \mathcal{F}_{\alpha,n}(f)(\lambda).
\end{aligned}$$

(ii) From (16) and (35) we have

$$\begin{aligned}
\int_0^\infty T^x_{\alpha,n} f(y)\, g(y) y^{2\alpha+1} dy &= x^{2n} \int_0^\infty T^x_{\alpha+2n}\big(\mathcal{M}_n^{-1} f\big)(y)\, \mathcal{M}_n^{-1} g(y)\, y^{2\alpha+4n+1} dy \\
&= x^{2n} \int_0^\infty \mathcal{M}_n^{-1} f(y)\, T^x_{\alpha+2n}\big(\mathcal{M}_n^{-1} g\big)(y)\, y^{2\alpha+4n+1} dy \\
&= \int_0^\infty f(y) T^x_{\alpha,n} g(y)\, y^{2\alpha+1} dy. \quad \square
\end{aligned}$$

Definition 4.3. For $f \in \mathcal{D}_n(\mathbb{R})$ and $g \in \mathcal{E}_n(\mathbb{R})$, the generalized convolution product $f *_{\alpha,n} g$ is defined by

$$f *_{\alpha,n} g(x) = \int_0^\infty T^x_{\alpha,n} f(y) g(y) y^{2\alpha+1} dy, \quad x \in \mathbb{R}. \tag{36}$$

Proposition 4.6. *For $f \in \mathcal{D}_n(\mathbb{R})$ and $g \in \mathcal{E}_n(\mathbb{R})$, we have*

$$f *_{\alpha,n} g = \mathcal{M}_n \left[\left(\mathcal{M}_n^{-1} f\right) *_{\alpha+2n} \left(\mathcal{M}_n^{-1} g\right)\right]. \tag{37}$$

Proof. By (17), (35) and (36) it follows that

$$\begin{aligned} f *_{\alpha,n} g(x) &= \int_0^\infty T^x_{\alpha,n} f(y) g(y) y^{2\alpha+1} dy \\ &= x^{2n} \int_0^\infty T^x_{\alpha+2n}\left(\mathcal{M}_n^{-1} f\right)(y)\, g(y)\, y^{2\alpha+2n+1} dy \\ &= x^{2n} \int_0^\infty T^x_{\alpha+2n}\left(\mathcal{M}_n^{-1} f\right)(y)\, \mathcal{M}_n^{-1} g(y)\, y^{2\alpha+4n+1} dy \\ &= x^{2n} \left(\mathcal{M}_n^{-1} f\right) *_{\alpha+2n} \left(\mathcal{M}_n^{-1} g\right)(x), \end{aligned}$$

and the result follows. □

Theorem 4.6. *(i) Let $f \in \mathcal{D}_{a,n}(\mathbb{R})$ and $g \in \mathcal{D}_{b,n}(\mathbb{R})$. Then $f *_{\alpha,n} g \in \mathcal{D}_{a+b,n}(\mathbb{R})$ and*

$$\mathcal{F}_{\alpha,n}(f *_{\alpha,n} g)(\lambda) = \mathcal{F}_{\alpha,n}(f)(\lambda)\, \mathcal{F}_{\alpha,n}(g)(\lambda), \quad \lambda \in \mathbb{C}. \tag{38}$$

(ii) For all $f, g \in \mathcal{D}_n(\mathbb{R})$,

$$W_{\alpha,n}(f *_{\alpha,n} g) = W_{\alpha,n}(f) * W_{\alpha,n}(g), \tag{39}$$

where $*$ is the symmetric convolution product on $\mathbb{R}$ given by (18).

Proof. Identity (38) follows by combining (32), (37) and Proposition 2.2(i). Identity (39) follows by applying the cosine transform to both its sides and by using formulas (33) and (38). □

5. The generalized Sonine integral transform

Throughout this section assume $\beta > \alpha > -1/2$ and m, n two non-negative integers such that $m \geq n$.

5.1 The classical Sonine integral transform

The classical Sonine integral formula may be formulated as follows :

$$j_\beta(z) = c(\alpha,\beta) \int_0^1 j_\alpha(zt)(1-t^2)^{\beta-\alpha-1}\, t^{2\alpha+1}\, dt \quad (z \in \mathbb{C}), \tag{40}$$

where

$$c(\alpha,\beta) = \frac{2\,\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}. \tag{41}$$

Using this integral formula, Trimeche [3] has introduced the so-called Sonine integral transform

$$S_{\alpha,\beta} f(x) = c(\alpha,\beta) \int_0^1 f(tx)(1-t^2)^{\beta-\alpha-1}\, t^{2\alpha+1}\, dt, \quad x \in \mathbb{R}, \tag{42}$$

and obtained the following result.

Theorem 5.1. *The Sonine integral transform $S_{\alpha,\beta}$ is the unique automorphism of $\mathcal{E}(\mathbb{R})$ satisfying*

$$S_{\alpha,\beta} \circ L_\alpha(f) = L_\beta \circ S_{\alpha,\beta}(f) \qquad \textit{and} \qquad S_{\alpha,\beta} f(0) = f(0), \tag{43}$$

for all $f \in \mathcal{E}(\mathbb{R})$. Moreover,

$$S_{\alpha,\beta}(f) = R_\beta \circ R_\alpha^{-1}(f) \qquad \textit{for all}\, f \in \mathcal{E}(\mathbb{R}). \tag{44}$$

The dual Sonine integral transform is defined by

$${}^tS_{\alpha,\beta}(f)(y) = c(\alpha,\beta) \int_{|y|}^\infty f(x)\,(x^2-y^2)^{\beta-\alpha-1}\, x\, dx, \quad y \in \mathbb{R}. \tag{45}$$

The transforms $S_{\alpha,\beta}$ and ${}^tS_{\alpha,\beta}$ are transposed by virtue of the relation

$$\int_0^\infty S_{\alpha,\beta}(f)(x) g(x) x^{2\beta+1} dx = \int_0^\infty f(y)\, {}^tS_{\alpha,\beta}(g)(y) y^{2\alpha+1} dy$$

valid for any $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$.

Theorem 5.2. *The dual Sonine integral transform ${}^tS_{\alpha,\beta}$ is an automorphism of $\mathcal{D}(\mathbb{R})$ satisfying the intertwining relation*

$${}^tS_{\alpha,\beta} \circ L_\beta(f) = L_\alpha \circ {}^tS_{\alpha,\beta}(f) \qquad \textit{for all}\, f \in \mathcal{D}(\mathbb{R}), \tag{46}$$

and admits the factorization

$${}^tS_{\alpha,\beta}(f) = W_\alpha^{-1} \circ W_\beta(f) \qquad \textit{for all}\, f \in \mathcal{D}(\mathbb{R}). \tag{47}$$

Remark 5.1. *$S_{\alpha,\beta}$ (resp. ${}^tS_{\alpha,\beta}$) is a transmutation operator between L_α and L_β on $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{D}(\mathbb{R})$).*

5.2 The generalized Sonine integral transform

An easy combination of (27) and (40) leads to

$$\varphi_{\lambda,\beta,m}(x) = c(\alpha+2n,\beta+2m)\,x^{2(m-n)} \times \\ \times \int_0^1 \varphi_{\lambda,\alpha,n}(xt)\,(1-t^2)^{\beta-\alpha+2(m-n)-1}\,t^{2\alpha+2n+1}\,dt, \tag{48}$$

where $c(\alpha+2n,\beta+2m)$ is given by (41).

This identity enables us to define a generalized Sonine integral transform by putting

$$S^{m,n}_{\alpha,\beta}(f)(x) = c(\alpha+2n,\beta+2m)\,x^{2(m-n)} \times \\ \times \int_0^1 f(tx)\,(1-t^2)^{\beta-\alpha+2(m-n)-1}\,t^{2\alpha+2n+1}\,dt. \tag{49}$$

Remark 5.2. *(i) By (48) and (49) it follows that*

$$\varphi_{\lambda,\beta,m} = S^{m,n}_{\alpha,\beta}(\varphi_{\lambda,\alpha,n}).$$

(ii) By a change of variables we have for $x > 0$,

$$S^{m,n}_{\alpha,\beta}(f)(x) = \frac{c(\alpha+2n,\beta+2m)}{x^{2\beta+2m}} \int_0^x f(y)(x^2-y^2)^{\beta-\alpha+2(m-n)-1}\,y^{2\alpha+2n+1}\,dy. \tag{50}$$

Theorem 5.3. *The generalized Sonine integral transform $S^{m,n}_{\alpha,\beta}$ is an isomorphism from $\mathcal{E}_n(\mathbb{R})$ onto $\mathcal{E}_m(\mathbb{R})$ satisfying the intertwining relation*

$$S^{m,n}_{\alpha,\beta} \circ L_{\alpha,n}(f) = L_{\beta,m} \circ S^{m,n}_{\alpha,\beta}(f) \qquad \text{for all } f \in \mathcal{E}_n(\mathbb{R}).$$

Moreover, we have the factorization

$$S^{m,n}_{\alpha,\beta}(f) = R_{\beta,m} \circ R^{-1}_{\alpha,n}(f) \qquad \text{for all } f \in \mathcal{E}_n(\mathbb{R}). \tag{51}$$

Proof. By (42) and (49) observe that

$$S_{\alpha,\beta}^{m,n} = \mathcal{M}_m \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}_n^{-1}. \tag{52}$$

So by Lemma 3.1(i) and Theorem 5.1, $S_{\alpha,\beta}^{m,n}$ is an isomorphism from $\mathcal{E}_n(\mathbb{R})$ onto $\mathcal{E}_m(\mathbb{R})$. Moreover, by (20), (43) and (52),

$$\begin{aligned} S_{\alpha,\beta}^{m,n} \circ L_{\alpha,n} &= \mathcal{M}_m \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}_n^{-1} \circ L_{\alpha,n} \\ &= \mathcal{M}_m \circ S_{\alpha+2n,\beta+2m} \circ L_{\alpha+2n} \circ \mathcal{M}_n^{-1} \\ &= \mathcal{M}_m \circ L_{\beta+2m} \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}_n^{-1} \\ &= L_{\beta,m} \circ \mathcal{M}_m \circ S_{\alpha+2n,\beta+2m} \circ \mathcal{M}_n^{-1} \\ &= L_{\beta,m} \circ S_{\alpha,\beta}^{m,n}. \end{aligned}$$

Finally, identity (51) follows readily by combining (23), (44) and (52). □

Remark 5.3. *By Theorem 5.3, $S_{\alpha,\beta}^{m,n}$ is a transmutation operator between $L_{\alpha,n}$ and $L_{\beta,m}$.*

Define the dual generalized Sonine transform ${}^tS_{\alpha,\beta}^{m,n}$ on $\mathcal{D}_m(\mathbb{R})$ by

$${}^tS_{\alpha,\beta}^{m,n}(f)(y) = c(\alpha+2n,\beta+2m)\, y^{2n} \int_{|y|}^{\infty} f(x)\,(x^2-y^2)^{\beta-\alpha+2(m-n)-1}\,\frac{dx}{x^{2m-1}}, \tag{53}$$

where $c(\alpha+2n,\beta+2m)$ is given by (41).

Remark 5.4. *(i) By (45) and (53) notice that*

$${}^tS_{\alpha,\beta}^{m,n} = \mathcal{M}_n \circ {}^tS_{\alpha+2n,\beta+2m} \circ \mathcal{M}_m^{-1}. \tag{54}$$

(ii) Let $f \in \mathcal{E}_n(\mathbb{R})$ and $g \in \mathcal{D}_m(\mathbb{R})$. A combination of (50), (53) and Fubini's theorem yields the relation

$$\int_0^{\infty} S_{\alpha,\beta}^{m,n}(f)(x)g(x)x^{2\beta+1}dx = \int_0^{\infty} f(y)\,{}^tS_{\alpha,\beta}^{m,n}(g)(y)y^{2\alpha+1}dy$$

which means that the transforms $S_{\alpha,\beta}^{m,n}$ and ${}^tS_{\alpha,\beta}^{m,n}$ are transposed.

We can now state

Theorem 5.4. *The dual generalized Sonine transform* ${}^t\mathcal{S}^{m,n}_{\alpha,\beta}$ *is an isomorphism from* $\mathcal{D}_m(\mathbb{R})$ *onto* $\mathcal{D}_n(\mathbb{R})$ *satisfying the intertwining relation*

$${}^tS^{m,n}_{\alpha,\beta} \circ L_{\beta,m}(f) = L_{\alpha,n} \circ {}^tS^{m,n}_{\alpha,\beta}(f) \qquad \text{for all } f \in \mathcal{D}_m(\mathbb{R}).$$

Moreover,

$${}^tS^{m,n}_{\alpha,\beta}(f) = W^{-1}_{\alpha,n} \circ W_{\beta,m}(f) \qquad \text{for all } f \in \mathcal{D}_m(\mathbb{R}). \tag{55}$$

Proof. It is clear from (54), Lemma 3.1(i) and Theorem 5.2, that ${}^tS^{m,n}_{\alpha,\beta}$ is an isomorphism from $\mathcal{D}_m(\mathbb{R})$ onto $\mathcal{D}_n(\mathbb{R})$. Furthermore, by (20), (46) and (54) it follows that

$$\begin{aligned}
{}^tS^{m,n}_{\alpha,\beta} \circ L_{\beta,m} &= \mathcal{M}_n \circ {}^tS_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_m \circ L_{\beta,m} \\
&= \mathcal{M}_n \circ {}^tS_{\alpha+2n,\beta+2m} \circ L_{\beta+2m} \circ \mathcal{M}^{-1}_m \\
&= \mathcal{M}_n \circ L_{\alpha+2n} \circ {}^tS_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_m \\
&= L_{\alpha,n} \circ \mathcal{M}_n \circ {}^tS_{\alpha+2n,\beta+2m} \circ \mathcal{M}^{-1}_m \\
&= L_{\alpha,n} \circ {}^tS^{m,n}_{\alpha,\beta}.
\end{aligned}$$

Finally, (55) is an easy consequence of (25), (47) and (54). □

The next statement provides formulas relating harmonic analysis tools tied to $L_{\alpha,n}$ with those tied to $L_{\beta,m}$, and involving the transform ${}^tS^{m,n}_{\alpha,\beta}$.

Proposition 5.1. *(i) For every* $f \in \mathcal{D}_m(\mathbb{R})$ *we have the identity*

$$\mathcal{F}_{\beta,m}(f) = \mathcal{F}_{\alpha,n} \circ {}^tS^{m,n}_{\alpha,\beta}(f). \tag{56}$$

(ii) Let $f, g \in \mathcal{D}_m(\mathbb{R})$*. Then*

$${}^tS^{m,n}_{\alpha,\beta}(f *_{\beta,m} g) = {}^tS^{m,n}_{\alpha,\beta}(f) *_{\alpha,n} {}^tS^{m,n}_{\alpha,\beta}(g). \tag{57}$$

Proof. Identity (56) follows readily by combining (33) and (55). Identity (57) follows by applying the generalized Fourier transform $\mathcal{F}_{\alpha,n}$ to both its sides and by using (38) and (56). □

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