

# Sufficient Conditions for Hypergeometric Functions to be in A Certain Class of Holomorphic Functions \*

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## Abstract

In the present investigation our main objective is to find coefficient estimates, sufficient condition for the function  $f(z) \in \mathcal{A}$  to belong to the class  $R_\gamma^r(A, B)$  and finding connections between the classes  $R_\gamma^r(A, B)$  and  $k - \mathcal{UCV}$  by making use of the Hohlov operator [5].

**Keywords and Phrases:** *Analytic functions, Subordination, Schwarz functions, Gaussian Hypergeometric functions,  $k - \mathcal{UCV}$  functions, Starlike functions, Convex functions, Univalent functions.*

## 1. Introduction

Let  $\mathcal{A}$  denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

which are analytic in the open unit disk  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and  $\mathcal{S}$  denote the subclass of  $\mathcal{A}$  that are univalent in  $\mathbb{U}$ . A function  $f(z)$  in  $\mathcal{A}$  is said

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to be in class  $\mathcal{S}^*$  of starlike functions of order zero in  $\mathbb{U}$ , if  $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0$  for  $z \in \mathbb{U}$ . Let  $\mathcal{K}$  denote the class of all functions  $f \in \mathcal{A}$  that are convex. Also,  $f$  is convex if and only if  $zf'(z)$  is starlike. A function  $f \in \mathcal{A}$  is said to be close-to-convex of order  $\alpha$  ( $0 \leq \alpha < 1$ ) with respect to a fixed starlike function  $g \in \mathcal{S}^*$  if and only if  $\Re\left(\frac{zf'(z)}{g'(z)}\right) > \alpha$  for  $z \in \mathbb{U}$ . For more details about these classes see [3]. Furthermore,  $f \in \mathcal{A}$ , then  $f \in k - \mathcal{UCV}$  iff

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}, 0 \leq k < \infty). \quad (1.2)$$

The class  $k - \mathcal{UCV}$  was introduced by Kanas and Wisniowska [6], where its geometric definition and connection with the conic domains were considered. In particular  $0 - \mathcal{UCV} = \mathcal{K}$ .

If  $f, g \in \mathcal{H}$ , where  $\mathcal{H}$  denote the class of holomorphic functions on unit disk  $\mathbb{U}$ , then the function  $f$  is said to be subordinate to  $g$ , written as  $f(z) \prec g(z)$  ( $z \in \mathbb{U}$ ), if there exists a Schwarz function  $w \in \mathcal{H}$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in \mathbb{U}$ ) such that  $f(z) = g(w(z))$ .

In particular, if  $g$  is univalent in  $\mathbb{U}$ , then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Gaussian hypergeometric function defined by the series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad (1.3)$$

is analytic in the unit disk  $\mathbb{U}$ . It arises naturally in the study of second order linear differential equations with regular singular points. In (1.3),  $(a)_0 = 1$  for  $a \neq 0$  and for each positive integer  $n$ ,  $(a)_n = a(a+1)\dots(a+n-1)$  is the Pochhammer symbol. To avoid division by 0, the parameter  $c$  in (1.3) should be neither zero nor a negative integer. If  $a$  or  $b$  is 0 or a negative integer, then the power series reduces to a polynomial. Results regarding  ${}_2F_1(a, b; c; z)$  when  $\Re(c - a - b)$  is positive, zero or negative are abundant in the literature. In particular when  $\Re(c - a - b) > 0$ , the function  ${}_2F_1(a, b; c; z)$  is bounded. This and the zero balanced case  $\Re(c - a - b) = 0$  are discussed in detail by many authors (see [9, 13]). For interesting results regarding  $\Re(c - a - b) < 0$ , see [14].

The hypergeometric function  ${}_2F_1(a, b; c; z)$  has been extensively studied by

various authors and play an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters  $a, b$ , and  $c$ . We refer to [2, 4, 10, 13] and reference therein for some important results.

The normalized hypergeometric function  ${}_2F_1(a, b; c; z)$  has a series expansion of the form

$${}_2F_1(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n. \quad (1.4)$$

Consider the convolution operator by taking the convolution between functions  $f(z)$  of the form (1.1) and a normalized hypergeometric functions of the form  ${}_2F_1(a, b; c; z)$ :

$$H_{a,b,c}(f)(z) = {}_2F_1(a, b; c; z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \quad (1.5)$$

which was investigated by Hohlov [5]. This three-parameter family of operators given by (1.5) contains most of the known linear integral or differential operators as special cases. In particular, if  $a = 1$  in (1.5), then  $H_{1,b,c}$  is the operator  $\mathcal{L}(b, c)$  due to Carlson and Shaffer [2] which was defined by

$$\mathcal{L}(b, c)f(z) = {}_2F_1(1, b; c; z) * f(z). \quad (1.6)$$

Note that  ${}_2F_1(1, b; c; z) = \phi(b; c; z)$  is known as incomplete beta function.

In particular, the restriction  $b = 1 + \delta, c = 2 + \delta$  with  $\Re \delta > -1$  on the operator  $\mathcal{L}(b, c)f(z)$  gives the Bernardi operator

$$\mathcal{B}_\delta(f)(z) = \mathcal{L}(\delta + 1, \delta + 2)(f)(z) = (1 + \delta) \int_0^1 t^{\delta-1} f(tz) dt, \quad (1.7)$$

which reduces to the Alexander and Libera transforms, respectively, for  $\delta = 1$  and  $\delta = 2$ . It is interesting to note that these operators are all example of the zero-balanced case  $\Re(c - a - b) = 0$  in  $H_{1,b,c}(f)(z)$ .

Throughout this work, we frequently use the well-known formula

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)}, \quad (\Re(c - a - b) > 0, c \in \mathbb{C} \setminus \mathbb{Z}_0^-). \quad (1.8)$$

Motivated by the class introduced by Swaminathan [16], Bansal [1] introduced the class  $R_\gamma^\tau(A, B)$  as follows:

**Definition 1.1.** Let  $0 \leq \gamma \leq 1$ ,  $\tau \in \mathbb{C} \setminus \{0\}$ . A function  $f \in \mathcal{A}$  is in the class  $R_\gamma^\tau(A, B)$ , if

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}), \quad (1.9)$$

which is equivalent to saying that

$$\left| \frac{f'(z) + \gamma z f''(z) - 1}{\tau(A - B) - B(f'(z) + \gamma z f''(z) - 1)} \right| < 1. \quad (1.10)$$

We list few particular cases of this class discussed in the literature [1]  $R_\gamma^\tau(1 - 2\beta, -1) = R_\gamma^\tau(\beta)$  for  $0 \leq \beta < 1$ ,  $\tau = \mathbb{C} \setminus \{0\}$  was discussed recently by Swaminathan [16].

[2] The class  $R_\gamma^\tau(1 - 2\beta, -1)$  for  $\tau = e^{i\eta} \cos \eta$  where  $-\pi/2 < \eta < \pi/2$  is considered in [11] (see also [12]).

[3] The class  $R_1^\tau(0, -1)$  with  $\tau = e^{i\eta} \cos \eta$  was considered in [7] with reference to the univalence of partial sums.

[4]  $f \in R_\gamma^{e^{i\eta} \cos \eta}(1 - 2\beta, -1)$  whenever  $z f'(z) \in P_\gamma^\tau(\beta)$ , the class considered in [17].

For geometric aspects of these classes see the corresponding references.

Our main objective in the present paper is to find coefficient estimates, sufficient condition for the functions of the form (1.1) to belong to the class  $R_\gamma^\tau(A, B)$  and finding connections between the classes  $R_\gamma^\tau(A, B)$  and  $k - \mathcal{UCV}$  by making use of the Hohlov operator defined by (1.5). Each of the following lemmas will be required in our investigation.

**Lemma A.** (See [15]). *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \prec 1 + \sum_{n=1}^{\infty} C_n z^n = H(z) \quad (z \in \mathbb{U}). \quad (1.11)$$

*If the function  $H$  is univalent in  $\mathbb{U}$  and  $H(\mathbb{U})$  is a convex set, then*

$$|c_n| \leq |C_1|. \quad (1.12)$$

**Lemma B.** (See [6]). *Let  $f \in \mathcal{A}$  be of the form (1.1). If for some  $k(0 \leq k < \infty)$ , the following inequality:*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2} \quad (1.13)$$

holds true, then  $f \in k - \mathcal{UCV}$ . The number  $\frac{1}{k+2}$  cannot be increased.

**Lemma C.** (See [8]). Let  $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ , then

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z} \quad (z \in \mathbb{U}).$$

## 2. Main Results

We first give the following result related to the coefficient of  $f(z) \in R_\gamma^\tau(A, B)$ .

**Theorem 2.1.** Let  $f(z) \in \mathcal{A}$  is of the form (1.1). If  $f(z)$  is in  $R_\gamma^\tau(A, B)$ , then

$$|a_n| \leq \frac{|\tau|(A - B)}{n[1 + \gamma(n - 1)]} \quad (n \in \mathbb{N} \setminus \{1\}). \quad (2.1)$$

**Proof.** If  $f(z)$  of the form (1.1) belongs to in  $R_\gamma^\tau(A, B)$ , then by definition

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1 + Az}{1 + Bz} = h(z) \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}), \quad (2.2)$$

where  $h(z)$  is obviously convex univalent in  $\mathbb{U}$  under the stated conditions on  $A$  and  $B$ . Using (1.1) and doing Binomial expansion of  $(1 + Bz)^{-1}$  in (2.2), we have

$$\begin{aligned} & 1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(1+n\gamma)}{\tau} a_{n+1} z^n \prec 1 + (A - B)z - B(A - B)z^2 + \dots (z \in \mathbb{U}). \end{aligned}$$

Now, by applying Lemma A we get the desired result.  $\square$

It is easy to find the sufficient condition for  $f(z)$  to be in  $R_\gamma^\tau(A, B)$  under standard techniques. Hence we state the following result without proof.

**Theorem 2.2.** Let  $f(z) \in \mathcal{A}$ . Then a sufficient condition for  $f(z)$  to be in  $R_\gamma^\tau(A, B)$  is

$$\sum_{n=2}^{\infty} n[1 + \gamma(n - 1)] |a_n| \leq \frac{|\tau|(A - B)}{|B| + 1}. \quad (2.3)$$

The result is sharp for the function

$$f(z) = z + \frac{|\tau|(A - B)}{n[1 + \gamma(n - 1)](1 + |B|)} z^n \quad (n \in \mathbb{N} \setminus \{1\}) \quad (2.4)$$

**Remark 2.1.** For  $B = -1$  and  $A = 1 - 2\beta$  ( $0 \leq \beta < 1$ ) Theorem 2.1 and 2.2 gives corresponding result of [16].

**Theorem 2.3.** Let  $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ . Then

$$R_\gamma^r(A_1, B_1) \subset R_\gamma^r(A_2, B_2). \quad (2.5)$$

**Proof.** Let  $f \in R_\gamma^r(A_1, B_1)$  then by Definition 1.1 of the class  $f \in R_\gamma^r(A_1, B_1)$  we have

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

Since  $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ , by Lemma C, we have

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Which implies that  $R_\gamma^r(A_1, B_1) \subset R_\gamma^r(A_2, B_2)$ .  $\square$

**Theorem 2.4.** Suppose that  $a, b \in \mathbb{C} \setminus \{0\}$  and  $\Re(c) > |a| + |b|$ . If  $f \in R_\gamma^r(A, B)$  and the inequality

$${}_2F_1(|a|, |b|; \Re(c); 1) \leq \frac{|B| + 2}{|B| + 1} \quad (2.6)$$

holds true, then  $z {}_2F_1(a, b; c; z) * f(z) \in R_\gamma^r(A, B)$ .

**Proof.** Using Theorem 2.2 and (1.5) it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[1 + \gamma(n-1)] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| |a_n| \leq \frac{|\tau|(A-B)}{1+|B|}.$$

Applying Theorem 2.1, for  $f \in R_\gamma^r(A, B)$ , we have

$$\begin{aligned} \sum_{n=2}^{\infty} n[1 + \gamma(n-1)] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| |a_n| &\leq |\tau|(A-B) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}} \\ &= |\tau|(A-B) [{}_2F_1(|a|, |b|; \Re(c); 1) - 1] \leq \frac{|\tau|(A-B)}{|B|+1} \quad (\text{In view of (2.6)}). \square \end{aligned}$$

If we set  $\gamma = 0, B = -1, A = 1 - 2\alpha$  ( $0 \leq \alpha < 1$ ) and  $\tau = 1$ , we get the functions in the class  $R_\gamma^r(A, B)$  satisfying the analytic criterion  $\Re(f') > \alpha$  which implies that  $f(z)$  is close-to-convex of order  $\alpha$  with respect to the starlike function  $g(z) = z$ . Hence the following result is immediate:

**Corollary 2.1.** *Suppose that  $a, b \in \mathbb{C} \setminus \{0\}$  and  $\Re(c) > |a| + |b|$ . If  $f \in \mathcal{A}$  of form (1.1) satisfying  $\Re(f') > \alpha$ , and the inequality*

$$\frac{\Gamma(\Re(c) - |a| - |b|)\Gamma(\Re(c))}{\Gamma(\Re(c) - a)\Gamma(\Re(c) - b)} \leq \frac{3}{2} \quad (2.7)$$

*holds true, then  $z {}_2F_1(a, b; c; z) * f(z)$  is close-to-convex of order  $\alpha$  with respect to the starlike function  $g(z) = z$ .*

**Theorem 2.5.** *Let  $a, b, c$  and  $\gamma$  satisfy the hypergeometric inequality*

$${}_2F_1(|a|, |b|; \Re(c); 1) \left[ 1 + \frac{(1+2\gamma)|ab|}{\Re(c) - |a| - |b| - 1} + \frac{\gamma(|a|)_2(|b|)_2}{(\Re(c) - |a| - |b| - 1)(\Re(c) - |a| - |b| - 2)} \right]^{-1} \\ \leq \frac{|\tau|(A-B)}{|B|+1}, \quad (2.8)$$

*with  $a, b \in \mathbb{C} \setminus \{0\}$  and  $\Re(c) - |a| - |b| - 2 > 0$ . Then  $z {}_2F_1(a, b; c; z)$  is in  $R_\gamma^r(A, B)$ .*

**Proof.** Using Theorem 2.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n [1 + \gamma(n-1)] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \leq \frac{|\tau|(A-B)}{|B|+1}.$$

It is easy to see that, the left hand side of the above inequality is

$$S = \sum_{n=2}^{\infty} n [1 + \gamma(n-1)] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \\ = \sum_{n=2}^{\infty} [1 + (1+2\gamma)(n-1) + \gamma(n-1)(n-2)] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \\ \leq \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}} + (1+2\gamma) \frac{|ab|}{\Re(c)} \sum_{n=2}^{\infty} \frac{(|a|+1)_{n-2}(|b|+1)_{n-2}}{(\Re(c)+1)_{n-2}(1)_{n-2}} \\ + \gamma \frac{(|a|)_2(|b|)_2}{(\Re(c))_2} \sum_{n=3}^{\infty} \frac{(|a|+2)_{n-3}(|b|+2)_{n-3}}{(\Re(c)+2)_{n-3}(1)_{n-3}} \\ = {}_2F_1(|a|, |b|; \Re(c); 1) \left[ 1 + \frac{(1+2\gamma)|ab|}{\Re(c) - |a| - |b| - 1} + \frac{\gamma(|a|)_2(|b|)_2}{(\Re(c) - |a| - |b| - 1)(\Re(c) - |a| - |b| - 2)} \right]^{-1} \\ \leq \frac{|\tau|(A-B)}{|B|+1} \quad (\text{In view of (2.8)}). \square$$

If we set  $\gamma = 0$ ,  $A = 1 - 2\alpha$  ( $0 \leq \alpha < 1$ ),  $B = -1$  and  $\tau = 1$  in Theorem 2.5, we get the following result:

**Corollary 2.2.** *Let  $a$ ,  $b$  and  $c$  satisfy the hypergeometric inequality*

$${}_2F_1(|a|, |b|; \Re(c); 1) \left[ 1 + \frac{|ab|}{\Re(c) - |a| - |b| - 1} \right] \leq 2 - \alpha, \quad (2.9)$$

with  $a, b \in \mathbb{C} \setminus \{0\}$  and  $\Re(c - |a| - |b| - 1) > 0$ , then,  $z {}_2F_1(a, b; c; z)$  is close-to-convex of order  $\alpha$  with respect to the starlike function  $g(z) = z$ .  $\square$

**Theorem 2.6.** *Suppose that  $a, b \in \mathbb{C} \setminus \{0\}$ ,  $|a| \neq 1, |b| \neq 1, \Re(c) \neq 1$  and  $\Re(c) > |a| + |b|$ .*

*If  $f \in R_1^\tau(A, B)$  and, for some  $k$  ( $0 \leq k < \infty$ ), the inequality*

$$\begin{aligned} {}_2F_1(|a|, |b|; \Re(c); 1) - \frac{\Re(c) - 1}{(|a| - 1)(|b| - 1)} ({}_2F_1(|a| - 1, |b| - 1; \Re(c) - 1; 1) - 1) \\ \leq \frac{1}{|\tau|(A - B)(k + 2)} \end{aligned} \quad (2.10)$$

holds true, then  $z {}_2F_1(a, b; c; z) * f(z) \in k - \mathcal{UCV}$ .

**Proof.** For  $f \in R_1^\tau(A, B)$  of form (1.1), by applying Theorem 2.1, we have

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| |a_n| &\leq \sum_{n=2}^{\infty} \frac{n(n-1)|\tau|(A-B)}{n^2} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}} \\ &= |\tau|(A-B) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_{n-1}} - |\tau|(A-B) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(\Re(c))_{n-1}(1)_n} \\ &= |\tau|(A-B) [{}_2F_1(|a|, |b|; \Re(c); 1) - 1] - |\tau|(A-B) \frac{\Re(c) - 1}{(|a| - 1)(|b| - 1)} \sum_{n=2}^{\infty} \frac{(|a| - 1)_n (|b| - 1)_n}{(\Re(c) - 1)_n (1)_n} \\ &= |\tau|(A-B) \left[ {}_2F_1(|a|, |b|; \Re(c); 1) - \frac{\Re(c) - 1}{(|a| - 1)(|b| - 1)} ({}_2F_1(|a| - 1, |b| - 1; \Re(c) - 1; 1) - 1) \right]. \end{aligned}$$

Finally, if we make use of (2.10) in above, we find that

$$\sum_{n=2}^{\infty} n(n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| |a_n| \leq \frac{1}{k+2} \quad (0 \leq k < \infty),$$

which, in view of (1.5) and Lemma B, immediately proves the inclusion property asserted by Theorem 2.6.  $\square$



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