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# Qualitative Theory for Fractional Order Riemann-Liouville Integral Equations in Two Independent Variables *

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Dedicated to the memory of Professor B.G. Pachpatte

**Abstract**

In this paper, we present some results concerning the existence and uniqueness and global asymptotic stability of solutions for a functional integral equation of Riemann-Liouville fractional order, by using some fixed point theorems for the existence and uniqueness of the solution and by using some techniques of Pachpatte concerning the estimate on the solution.

**Keywords and Phrases:** *Functional integral equation, Left-sided mixed Riemann-Liouville integral of fractional order, Solution, Estimation, asymptotic stability, fixed point.*

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# 1. Introduction

Fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport and the kinetic theory of gases and others [5, 9, 10, 11, 15, 16]. However, many researchers remain unaware of this field. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [4], Baleanu *et al.* [6], Diethelm [13], Kilbas *et al.* [17], Miller and Ross [18], Podlubny [24], Samko *et al.* [25]. Recently some results on the existence and the attractivity of the solutions of various classes of integral equations have been obtained by Abbas *et al.* [1, 2, 3], Banaś and Zając [7], Darwish *et al.* [12], Pachpatte [19, 20, 21, 22, 23] and the references therein. In most of the above cited papers the main tool was the measure of noncompactness. In [23], Pachpatte proved some results concerning some basic qualitative properties of solutions of the following general partial integral equation of Barbashin type of the form

$$x(t,x)=h(t,x)+\int_0^t f(t,x,s,u(s,x))ds+\int_0^t\int_B g(t,x,s,y,u(s,y))dyds; \quad (1)$$

for $(t,x)\in E$, where $h:\mathbb{R}_+\times B\to\mathbb{R},\ f:E_1\times\mathbb{R}\to\mathbb{R},\ g:E^2\times\mathbb{R}\to\mathbb{R}$ are given functions continuous functions, $\mathbb{R}_+=[0,+\infty),\ B=\prod_{i=1}^m[a_i,b_i]\subset\mathbb{R}^m(a_i<b_i), E=\mathbb{R}_+\times B,\ E_1=\{(t,x,s):0\le s\le t<\infty,\ x\in B\}$. To establish the results, he obtains and uses a variant of a certain integral inequality with explicit estimate.

In this paper, by means of integral inequalities and the fixed point approach, we improve the above results for the following partial integral equation of Riemann-Liouville fractional order of the form

$$u(t,x)=\mu(t,x)+\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}f(t,x,s,u(s,x))ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}g(t,x,s,y,u(s,y))dyds;\ (t,x)\in J,\quad (2)$$

where $J=\mathbb{R}_+\times[0,b],\ b>0,\ r=(r_1,r_2),\ r_1,r_2\in(0,\infty),\ \mu:J\to\mathbb{R},\ f:J_1\times\mathbb{R}\to\mathbb{R},\ g:J_2\times\mathbb{R}\to\mathbb{R}$ are given continuous functions,

$$J_1=\{(t,x,s):0\le s\le t<\infty,\ x\in[0,b]\},$$

$$J_2 = \{(t,x,s,y) : 0 \le s \le t < \infty,\ x \in [0,b],\ y \in [0,b)\},$$

and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt,\ \xi > 0$.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $L^1([0,a]\times[0,b]);\ a,b > 0$ we denote the space of Lebesgue-integrable functions $u : [0,a]\times[0,b] \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(t,x)|dxdt.$$

As usual, by $C := C(J)$ we denote the space of all continuous functions from $J$ into $\mathbb{R}$. By $BC := BC(J)$ we denote the Banach space of all bounded and continuous functions from $J$ into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x)\in J} |u(t,x)|.$$

**Definition 1.** ([25]) Let $r \in (0,\infty)$. For $u \in L^1([0,b]);\ b > 0$ the expression

$$(I_0^r u)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1}u(s)ds,$$

is called the left-sided mixed Riemann-Liouville integral of order $r$.

In particular,

$$(I_0^0 u)(t) = u(t),\ (I_0^1 u)(t) = \int_0^t u(s)ds;\ \text{for almost all } t \in [0,b].$$

For instance, $I_0^r u$ exists for all $r > 0$, when $u \in L^1([0,b])$. Note also that when $u \in C([0,b])$, then $(I_0^r u) \in C([0,b])$,

**Example 2.1.** *Let $\omega \in (-1,0)\cup(0,\infty)$ and $r \in (0,\infty)$, then*

$$I_0^r t^\omega = \frac{\Gamma(1+\omega)}{\Gamma(1+\omega+r)} t^{\omega+r},\ \textit{for almost all } t \in [0,b].$$

**Definition 2.** [25] Let $r \in (0,\infty)$ and $u \in L^1([0,a]\times[0,b])$; $a,b>0$. The partial Riemann-Liouville integral of order $r$ of $u(t,x)$ with respect to $x$ is defined by the expression

$$I_{0,t}^r u(t,x) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s,x)ds,$$

for almost all $(t,x)\in[0,a]\times[0,b]$.

Analogously, we define the integral

$$I_{0,t}^r u(x,t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(x,s)ds,$$

for almost all $(x,t)\in[0,a]\times[0,b]$.

**Definition 3.** ([26]) Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u \in L^1([0,a]\times[0,b])$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}u(s,y)dyds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x)$$

$$= \int_0^t\int_0^x u(s,y)dyds;\ \text{for almost all } (t,x)\in[0,a]\times[0,b],$$

where $\sigma=(1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1,r_2>0$, when $u\in L^1([0,a]\times[0,b])$. Moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t\in[0,a],\ x\in[0,b].$$

**Example 2.2.** *Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then*

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2},$$

*for almost all* $(t,x)\in[0,a]\times[0,b]$.

Let $G$ be an operator from $\Omega \subset BC;\ \Omega \neq \emptyset$ into itself and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{3}$$

Now we review the concept of attractivity of solutions for equation (1) (see [3]).

**Definition 4.** Solutions of equation (3) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space $BC$ such that for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equations (3) belonging to $B(u_0,\eta) \cap \Omega$ we have that for each $x \in [0,b]$

$$\lim_{t\to\infty}(v(t,x) - w(t,x)) = 0. \tag{4}$$

When the limit (4) is uniform with respect to $B(u_0,\eta)\cap\Omega$, solutions of equation (3) are said to be uniformly locally attractive (or equivalently that solutions of (3) are locally asymptotically stable).

**Definition 5.** The solution $v = v(t,x)$ of equation (3) is said to be globally attractive if (4) hold for each solution $w = w(t,x)$ of (3). If condition (4) is satisfied uniformly with respect to the set $\Omega$, solutions of equation (3) are said to be globally asymptotically stable (or uniformly globally attractive).

Denote by $D_1 := \frac{\partial}{\partial t}$, the partial derivative of a function defined on $J_1$ (or $J_2$) with respect to the first variable. In the sequel we will make use of the following Lemma due to Pachpatte.

**Lemma 2.3.** *([23]) Let $u \in C(J),\ q, D_1 q \in C(J_1), k, D_1 k \in C(J_2)$ be positive functions, and $c \geq 0$ is a constant. If*

$$u(t,x) \leq c + \int_0^t q(t,x,s)u(s,x)ds + \int_0^t \int_0^b k(t,x,s,y)u(s,y)dyds;\ (t,x) \in J, \tag{5}$$

*then,*

$$u(t,x) \leq cP(t,x)\exp\Big(\int_0^t A(\sigma,x)d\sigma\Big);\ (t,x) \in J, \tag{6}$$

*where*

$$P(t,x) = \exp(Q(t,x)), \tag{7}$$

*in which*

$$Q(t,x) = \int_0^t \Big[q(\eta,x,\eta) + \int_0^\eta D_1 q(\eta,x,\xi)d\xi\Big]d\eta, \tag{8}$$

*and*

$$A(t,x)=\int_0^b k(t,x,t,y)P(t,y)dy+\int_0^t\int_0^b P(s,y)D_1k(t,x,s,y)dyds;\ (t,x)\in J. \tag{9}$$

## 3. Main Results

Let us start by defining what we mean by a solution of equation (2).

**Definition 6.** A function $u \in BC$ is said to be a solution of (2) if $u$ satisfies the equation (2) on $J$.

Our first result is about the existence and uniqueness of the solution of equation (2).

**Theorem 3.1.** *Assume that following hypotheses hold*

$(H_1)$ *The function $\mu$ is continuous and bounded with*

$$\mu^* = \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} |\mu(t,x)|.$$

$(H_2)$ *There exists a positive function $q \in BC(J_1)$ such that*

$$|f(t,x,s,u)-f(t,x,s,v)| \le q(t,x,s)|u-v|,$$

$$\text{for each}(t,x,s)\in J_1 \text{and } u,v\in\mathbb{R}.$$

*Moreover, assume that the function $t \to \int_0^t (t-s)^{r_1-1} f(t,x,s,0)ds$ is bounded on $J$ with*

$$f^* = \sup_{(t,x)\in J} \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds.$$

$(H_3)$ *There exists a positive function $k \in BC(J_2)$ such that*

$$|g(t,x,s,y,u)-g(t,x,s,y,v)| \le k(t,x,s,y)|u-v|,$$

*for each $(t,x,s,y)\in J_2$ and $u,v\in\mathbb{R}$. Moreover, assume that the function $t\to\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}g(t,x,s,y,0)dyds$ is bounded on $J$ with*

$$g^* = \sup_{(t,x)\in J}\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds.$$

*If*

$$q^* + k^* < 1, \tag{10}$$

*where*

$$q^* = \sup_{(t,x)\in J}\left[\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)ds\right],$$

*and*

$$k^* = \sup_{(t,x)\in J}\left[\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)dyds\right],$$

*then equation (2) has a unique solution on $J$.*

**Proof.** Let us define the operator $N : BC \to BC$, such that for each $(t,x) \in J$,

$$(Nu)(t,x) = \mu(t,x) + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}f(t,x,s,u(s,x))ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}g(t,x,s,y,u(s,y))dyds;\ (t,x)\in J. \tag{11}$$

It is clear that the function $(t,x) \mapsto N(u)(t,x)$ is continuous on $J$. Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $(t,x) \in J$ we have

$$
\begin{aligned}
|(Nu)(t,x)| &= \Big|\mu(t,x) + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1} f(t,x,s,u(s,x))ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} g(t,x,s,y,u(s,y))dyds\Big| \\
&\le |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds \\
&\le |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1} q(t,x,s)|u(s,x)|ds \\
&+ \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} k(t,x,s,y)|u(s,y)|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds \\
&\le \mu^* + f^* + g^* + (q^* + k^*)\|u\|_{BC}.
\end{aligned}
$$

Hence $N(u) \in BC$. Let $u,\ v \in BC$. Using the hypotheses, for each $(t,x) \in J$, we have

$$
\begin{aligned}
&|(Nu)(t,x)-(Nv)(t,x)| \\
&\leq \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x))-f(t,x,s,v(s,x))|ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y))-g(t,x,s,y,v(s,y))|dyds \\
&\leq \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)|u(s,x)-v(s,x)|ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)|u(s,y)-v(s,y)|dyds \\
&\leq \sup_{(t,x)\in J}\Big[\frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)ds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)dyds\Big]\|u-v\|_{BC} \\
&\leq (q^*+k^*)\|u-v\|_{BC}.
\end{aligned}
$$

From (10), it follows that $N$ has a unique fixed point in $BC$ by Banach contraction principle. The fixed point of $N$ is however a solution of equation (2).

Now, we shall prove the following theorem concerning the estimate on the solution of equation (2).

**Theorem 3.2.** *Set*

$$d=\mu^*+f^*+g^*. \tag{12}$$

*Assume that* $(H_1)-(H_3)$ *and the following hypothesis holds*

$(H_4)$ $q_1, D_1q_1 \in BC(J_1)$ *and* $k_1, D_1k_1 \in BC(J_2)$, *where*

$$q_1(t,x,s)=\frac{1}{\Gamma(r_1)}(t-s)^{r_1-1}q(t,x,s)$$

*and*

$$k_1(t,x,s,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}(t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y).$$

*If $u$ is any solution of (2) on $J$, then*

$$|u(t,x)| \leq dP_1(t,x)\exp\Big(\int_0^t A_1(\sigma,x)d\sigma\Big);\ (t,x)\in J, \tag{13}$$

*where*

$$P_1(t,x) \leq \exp(Q_1(t,x)), \tag{14}$$

*in which*

$$Q_1(t,x) \leq \int_0^t \Big[q_1(\eta,x,\eta) + \int_0^\eta D_1 q_1(\eta,x,\xi)d\xi\Big]d\eta, \tag{15}$$

*and*

$$A_1(t,x) \leq \int_0^b k_1(t,x,t,y)P_1(t,y)dy + \int_0^t\int_0^b P_1(s,y)D_1k_1(t,x,s,y)dyds. \tag{16}$$

**Proof.** Using the fact that $u$ is a solution of (2) and hypotheses, then for each $(t,x)\in J$, we have

$$\begin{aligned}
|u(t,x)| &\leq |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,0)|ds\\
&+ \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,0)|ds\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}\\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}|g(t,x,s,y,0)|dyds\\
&\leq d + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,0)|ds\\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}\\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds\\
&\leq d + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)|u(x,s)|ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)|u(s,y)|dyds.
\end{aligned} \tag{17}$$

Now an application of Lemma 2.3, to (17) yields (13).

**Theorem 3.3.** *Set*

$$\overline{d} := f^* + g^* + \mu^*(q^* + k^*). \tag{18}$$

*Assume that* $(H_1) - (H_3)$ *hold. If* $u$ *is any solution of (2) on* $J$*, then*

$$|u(t,x) - \mu(t,x)| \leq \overline{d} P_1(t,x) \exp\Big(\int_0^t A_1(\sigma, x) d\sigma\Big); \ (t,x) \in J, \tag{19}$$

*where* $P_1$ *and* $A_1$ *are given by (14) and (16), respectively.*

**Proof.** Let $h(t,x) = |u(t,x) - \mu(t,x)|$. Using the fact that $u$ is a solution of (2) and from the hypotheses, for each $(t,x) \in J$, we have

$$\begin{aligned}
h(t,x) \leq\ & \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} |f(t,x,s,u(s,x)) - f(t,x,s,\mu(s,x))| ds \\
& + \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} |f(t,x,s,\mu(s,x))| ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^b (t-s)^{r_1-1} (b-y)^{r_2-1} \\
& \times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,\mu(s,x))| dyds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^b (t-s)^{r_1-1} (b-y)^{r_2-1} |g(t,x,s,y,\mu(s,x))| dyds \\
& \leq \overline{d} + \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} |f(t,x,s,u(s,x)) - f(t,x,s,\mu(s,x))| ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^b (t-s)^{r_1-1} (b-y)^{r_2-1} \\
& \times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,\mu(s,x))| dyds \\
& \leq \overline{d} + \frac{1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} q(t,x,s) h(x,s) ds
\end{aligned}$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^b (t-s)^{r_1-1} (b-y)^{r_2-1} k(t,x,s,y) h(s,y) dyds. \tag{20}$$

Now an application of Lemma 2.3, to (20) yields (19).

We next prove under more appropriate conditions on the functions involved in (2) that the solutions tends exponentially toward zero as $t \to \infty$.

**Theorem 3.4.** *Assume that the following hypotheses hold*

$(H_5)$ *There exist constants* $\alpha > 0$ *and* $M \geq 0$ *such that*

$$|\mu(t,x)| \leq Me^{-\alpha t}; \tag{21}$$

$$|f(t,x,s,u) - f(t,x,s,v)| \leq q(t,x,s)e^{-\alpha(t-s)}|u-v|; \tag{22}$$

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \leq k(t,x,s,y)e^{-\alpha(t-s)}|u-v|; \tag{23}$$

*and* $f(t,x,s,0) = g(t,s,s,y,0) = 0$; *for each* $(t,x) \in J$, $(t,x,s) \in J_1$, $(t,x,s,y) \in J_2$, $u,v \in \mathbb{R}$, *and the functions* $q$, $k$ *be as in Theorem 3.1,*

$(H_6)$ $\sup_{(t,x)\in J} Q_1(t,x) < \infty$, $\int_0^\infty A_1(\sigma,x)d\sigma < \infty$, *where* $Q_1$ *and* $A_1$ *are given by (15) and (16).*

*If* $u$ *is any solution of (2) on* $J$, *then all solutions of equation (2) are uniformly globally attractive on* $J$.

**Proof.** From the hypotheses, for each $(t,x) \in J$, we have that

$$\begin{aligned}
|u(t,x)| &\leq |\mu(t,x)| + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}|f(t,x,s,u(s,x)) - f(t,x,s,0)|ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1} \\
&\times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds \\
&\leq Me^{-\alpha t} + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)e^{-\alpha(t-s)}|u(x,s)|ds
\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)e^{-\alpha(t-s)}|u(s,y)|dyds. \tag{24}$$

From (24), we get

$$|u(t,x)|e^{\alpha t} \leq M + \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1}q(t,x,s)e^{\alpha s}|u(x,s)|ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)e^{\alpha s}|u(s,y)|dyds. \tag{25}$$

Now an application of Lemma 2.3 to (25) yields

$$|u(t,x)|e^{\alpha t} \leq MP_1(t,x)\exp\Big(\int_0^t A_1(\sigma,x)d\sigma\Big);\ (t,x)\in J. \tag{26}$$

Multiplying both sides of (26) by $e^{-\alpha t}$ and in view of $(H_6)$, we get

$$\lim_{t\to\infty}|u(t,x)| \leq \lim_{t\to\infty} MP_1(t,x)\exp\Big(-\alpha t+\int_0^t A_1(\sigma,x)d\sigma\Big)=0.$$

Hence, the solution $u$ tends to zero as $t\to\infty$. Consequently, all solutions of equation (2) are uniformly globally attractive on $J$.

# 4. An Example

To illustrate our results, we consider the following partial integral equation of Riemann-Liouville fractional order of the form

$$u(t,x)=\frac{e^{x-t}}{1+t+x^2}+\frac{1}{\Gamma(r_1)}\int_0^t(t-s)^{r_1-1}f(t,x,s,u(s,x))ds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1(t-s)^{r_1-1}(1-y)^{r_2-1}g(t,x,s,y,u(s,y))dyds;\ (t,x)\in\mathbb{R}_+\times[0,1], \tag{27}$$

where $r_1,r_2\in(0,\infty)$,

$$\begin{cases} f(t,x,s,u)=\dfrac{x^2t^{-r_1}s^{-\frac{1}{2}}\sin s\sin t}{2c(1+t^{-\frac{1}{2}})(1+|u|)}; for (t,x,s)\in J_1,\ st\neq 0 and u\in\mathbb{R},\\ f(t,x,0,u)=f(0,x,0,u)=0,\end{cases}$$

$$J_1=\{(t,x,s):0\leq s\leq t<\infty,\ x\in[0,1]\},$$

$$c:=\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+r_1)}+\frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)},$$

$$\begin{cases} g(t,x,s,y,u)=\dfrac{t^{-r_1}s^{-\frac{1}{2}}e^{x-y-\frac{1}{s}-\frac{1}{t}}}{2c(1+t^{-\frac{1}{2}})(1+|u|)}; for (t,x,s,y)\in J_2,\ st\neq 0 and\ u\in\mathbb{R},\\ g(t,x,0,y,u)=g(0,x,0,y,u)=0,\end{cases}$$

and

$$J_2 = \{(t,x,s,y) : 0 \le s \le t < \infty,\ x \in [0,1],\ y \in [0,1)\}.$$

Set

$$\mu(t,x) = \frac{e^{x-t}}{1+t+x^2};\ (t,x) \in J.$$

We can see that the function $\mu$ is continuous and bounded with $\mu^* = e$.
For each $u, v \in \mathbb{R}$ and $(t,x,s) \in J_1$, we have

$$|f(t,x,s,u) - f(t,x,s,v)| \le \frac{1}{2c(1+t^{-\frac{1}{2}})}\left(x^2 t^{-r_1} s^{-\frac{1}{2}} |\sin s \sin t|\right)|u-v|,$$

and for each $u, v \in \mathbb{R}$ and $(t,x,s,y) \in J_2$, we have

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \le \frac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right)|u-v|.$$

Hence condition $(H_2)$ is satisfied with

$$\begin{cases} q(t,x,s) = \dfrac{1}{2c(1+t^{-\frac{1}{2}})}\left(x^2 t^{-r_1} s^{-\frac{1}{2}} |\sin s \sin t|\right);\ st \ne 0, \\ \quad q(t,x,0) = q(0,x,0) = 0, \end{cases}$$

and condition $(H_3)$ is satisfied with

$$\begin{cases} k(t,x,s,y) = \dfrac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right);\ st \ne 0, \\ \quad k(t,x,0,y) = k(0,x,0,y) = 0. \end{cases}$$

We shall show that condition (10) holds with $b = 1$. Indeed

$$\begin{aligned} & \frac{1}{\Gamma(r_1)}\int_0^t (t-s)^{r_1-1} q(t,x,s)\,ds \\ \le\ & \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)}\int_0^t (t-s)^{r_1-1} x^2 t^{-r_1} s^{-\frac{1}{2}}\,ds \\ =\ & x^2 t^{-r_1} t^{-\frac{1}{2}+r_1}\frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)} \\ \le\ & \frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)} t^{-\frac{1}{2}}, \end{aligned}$$

then

$$q^* \leq \frac{\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)}.$$

Also,

$$\begin{aligned}
&\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^b (t-s)^{r_1-1}(b-y)^{r_2-1}k(t,x,s,y)dyds \\
\leq\ & \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1}t^{-r_1}s^{-\frac{1}{2}}e^x dyds \\
\leq\ & e^x t^{-r_1}t^{-\frac{1}{2}+r_1}\frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)} \\
\leq\ & \frac{\Gamma(\frac{1}{2})et^{-\frac{1}{2}}}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)},
\end{aligned}$$

then

$$k^* \leq \frac{e\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}.$$

Thus,

$$q^*+k^* \leq \frac{1}{2c}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+r_1)}+\frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}\right)=\frac{1}{2}<1,$$

which is satisfied for each $r_1, r_2 \in (0,\infty)$. Consequently Theorem 3.1 implies that equation (27) has a unique solution defined on $\mathbb{R}_+ \times [0,1]$.

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