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Relations for Marginal and Joint Moment Generating Functions of Extended Type I Generalized Logistic Distribution based on Lower Generalized Order Statistics and Characterization *

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Abstract

In this study we give exact expressions and some recurrence relations for marginal and joint moment generating functions of lower generalized order statistics from extended type I generalized logistic distribution. The results for order statistics and lower record values are deduced from the relations derived. Further two characterization Theorems of this distribution has been considered on using conditional expectation and recurrence relations for marginal moment generating functions of the lower generalized order statistics is presented.

Keywords and Phrases: *Lower generalized order statistics, Order statistics, Lower record values, Extended type I generalized logistic distribution, Marginal and joint moment generating function, Recurrence relations and characterization.*

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1. Introduction

Kamps [18] introduced the concept of generalized order statistics (*gos*). It is know that order statistics, upper record values and sequential order statistics are special cases of *gos*. In this paper we will consider the lower generalized order statistics (*lgos*). Which is given as

Let $n \in N$, $k \geq 1$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) > 0, \quad \text{for all } \ 1 \leq r \leq n.$$

Then $X^*(1,n,m,k), \ldots, X^*(n,n,m,k)$ are n *lgos* from an absolutely continuous distribution function (*df*) $F(x)$ with the corresponding probability density function (*pdf*) $f(x)$ if their joint *pdf* has the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[F(x_i)]^{m_i}f(x_i)\Big)[F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

The marginal *pdf* of the $r-$th *lgos*, $X^*(r,n,m,k)$ is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1}f(x)g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint *pdf* of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$, $1 \leq r < s \leq n$ is expressed from (1.1) as

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1}f(y), \quad x > y, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i \ , \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -lnx, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

We shall also take $X^*(0,n,m,k) = 0$. If $m = 0$, $k = 1$, then $X^*(r,n,m,k)$ reduced to the $(n-r+1)-$th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X^*(r,n,m,k)$ reduced to the $r-$ th lower k record value (Pawlas and Szynal, [14]). The work of Burkschat *et al.* [11] may also refer for lower generalized order statistics.

Saran and Pandey [7] and Khan *et al.* [16] have established recurrence relations for marginal and joint moment generating functions of *lgos* from power function and generalized exponential distributions. Ahsanullah and Raqab [10], Raqab and Ahsanullah [12, 13] and Kumar [3] have established recurrence relations for moment generating functions of record values from Pareto, Gumble, power function, extreme value and generalized logistic distributions. Recurrence relations for marginal and joint moment generating functions of from power function and Erlange-truncated exponential distribution are derived by Saran and Singh [8] and Kumar *et al.* [4]. Al-Hussaini *et al.* [5, 6] have established recurrence relations for conditional and joint moment generating functions of *gos* based on mixed population, respectively. Khan *et al.* [15] have established explicit expressions and some recurrence relations for moment generating function of generalized order statistics from Gompertz distribution. Kamps [19] investigated the importance of recurrence relations of order statistics in characterization.
In the present study, we have established exact expressions and some recurrence relations for marginal and joint moment generating functions of *lgos* from extended type I generalized logistic distribution. Results for order statistics and lower record values are deduced as special cases and a characterization of extended type I generalized logistic distribution has been obtained on using conditional expectation and recurrence relation for marginal moment generating functions.
A random variable X is said to have extended type I generalized logistic distribution if its *pdf* is of the form

$$f(x) = \frac{\alpha\lambda^{\alpha}e^{-x}}{(\lambda + e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0 \tag{1.4}$$

and the corresponding *df* is

$$F(x) = \Big(\frac{\lambda}{\lambda + e^{-x}}\Big)^{\alpha}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0 \tag{1.5}$$

When $\alpha = \lambda = 1$, we have the ordinary logistic distribution and when $\lambda = 1$, we have the type I generalized logistic distribution.
For more details on this distribution and its applications one may refer to Olapade [1, 2].

2. Relations for Marginal Moment Generating Function

Note that for extended type I generalized logistic distribution defined in (1.4)

$$\alpha F(x) = (1 + \lambda e^{x}) f(x). \tag{2.1}$$

The relation in (2.1) will be used to derive some recurrence relations for the moment generating function of *lgos* from extended type I generalized logistic distribution.
Let us denote the marginal moment generating functions of $X^*(r,n,m,k)$ by $M_{X^*(r,n,m,k)}(t)$ and its $j-$th derivative by $M^{(j)}_{X^*(r,n,m,k)}(t)$.
We shall first establish the explicit expressions for $M_{X^*(r,n,m,k)}(t)$. Using (1.2), we obtain when $m \neq -1$

$$\begin{aligned} M_{X^*(r,n,m,k)}(t) &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx \\ &= \frac{C_{r-1}}{(r-1)!} I_j(\gamma_r - 1, r-1) \end{aligned} \tag{2.2}$$

where

$$I_j(a,b) = \int_{-\infty}^{\infty} e^{tx} [F(x)]^{a} f(x) g_m^{b}(F(x)) dx \tag{2.3}$$

Further, on using the binomial expansion, we can rewrite (2.3) as

$$I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{a+(m+1)u} f(x) dx \tag{2.4}$$

Making the substitution $z = [F(x)]^{1/\alpha}$, (2.4) reduces to

$$I_j(a,b) = \frac{\alpha}{\lambda^t (m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_{0}^{1} (1-z)^{-t} z^{\alpha(a+(m+1)u+1)+t-1} dz.$$

On using the Maclaurine series expansion $(1-z)^{-t}=\sum_{p=0}^{\infty}\frac{(t)_{(p)}z^p}{p!}$, where

$$(t)_p=\begin{cases} t(t+1)...(t+p-1), & p=1,2,... \\ 1, & p=0 \end{cases}$$

and integrating the resulting expression, we obtain

$$I_j(a,b)=\frac{\alpha}{\lambda^t(m+1)^b}\sum_{p=0}^{\infty}\sum_{u=0}^{b}(-1)^u\binom{b}{u}\frac{(t)_{(p)}}{p![\alpha(a+(m+1)u+1)+t+p]},\ t\neq 0 \tag{2.5}$$

and when $m=-1$ that

$$I_j(a,b)=\frac{0}{0}, \qquad as \qquad \sum_{u=0}^{b}(-1)^u\binom{b}{u}=0.$$

Since $I_j(a,b)$ is of the form $\frac{0}{0}$ at $m=-1$, therefore we have

$$I_j(a,b)=\frac{\alpha}{\lambda^t(m+1)^b}\sum_{p=0}^{\infty}\sum_{u=0}^{b}(-1)^u\binom{b}{u}\frac{(t)_{(p)}}{p![\alpha(a+(m+1)u+1)+t+p]}. \tag{2.6}$$

Differentiating numerator and denominator of (2.6) b times with respect to m, we get

$$I_j(a,b)=\frac{\alpha}{\lambda^t}\sum_{p=0}^{\infty}\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{(t)_{(p)}\alpha^b u^b}{p![\alpha(a+(m+1)u+1)+t+p]^{b+1}}.$$

On applying L' Hospital rule, we have

$$\lim_{m\to -1} I_j(a,b)=\frac{\alpha^{b+1}}{\lambda^t}\sum_{p=0}^{\infty}\frac{(t)_{(p)}}{p![\alpha(a+1)+t+p]^{b+1}}\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}u^b, \quad b>0.$$

But for all integers $n\geq 0$ and for all real numbers x, we have Ruiz [17]

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^n=n!. \tag{2.7}$$

Therefore,

$$\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}u^b=b!.$$

Hence

$$\lim_{m\to-1} I_j(a,b) = \frac{b!\alpha^{b+1}}{\lambda^t}\sum_{p=0}^{\infty}\frac{(t)_{(p)}}{p![\alpha(a+1)+t+p]^{b+1}}. \tag{2.8}$$

Now substituting the above expressions for $I_j(\gamma_r-1, r-1)$ in (2.2) and simplifying, we obtain when $m \neq -1$ that

$$M_{X^*(r,n,m,k)}(t) = \frac{\alpha C_{r-1}}{\lambda^t (r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}\frac{(t)_{(p)}}{p!(\alpha\gamma_{r-u}+t+p)}. \tag{2.9}$$

and when $m = -1$ that

$$M_{X^*(r,n,-1,k)}(t) = \frac{(\alpha k)^r}{\lambda^t}\sum_{p=0}^{\infty}\frac{(t)_{(p)}}{p!(\alpha k+t+p)^r}, \quad t \neq 0. \tag{2.10}$$

Applying D' Alembert's Ratio test for convergence, it can easily be seen that $M_{X^*(r,n,-1,k)}(t)$ exist $\forall\ t, \quad -\infty < t < \infty$ and is analytic in $\forall\ t$.
Differentiating (2.9) and (2.10) and evaluating at $t = 0$, we get the mean of the $r-$th *lgos*.

Special cases

i) Putting $m = 0$, $k = 1$ in (2.9), the explicit formula for marginal moment generating functions of order statistics of the extended type I generalized logistic distribution can be obtained as

$$M_{X_{n-r+1:n}}(t) = \frac{\alpha C_{r:n}}{\lambda^t}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}\frac{(t)_{(p)}}{p![\alpha(n-r+1+u)+t+p]}.$$

That is

$$M_{X_{r:n}}(t) = \frac{\alpha C_{r:n}}{\lambda^t}\sum_{p=0}^{\infty}\sum_{u=0}^{n-r}(-1)^u \binom{n-r}{u}\frac{(t)_{(p)}}{p![\alpha(r+u)+t+p]},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

ii) Setting $k = 1$ in (2.10), we get the explicit expression for marginal moment generating function of lower record values from extended type I generalized logistic distribution can be obtained as

$$M_{X_{L(r)}}(t) = \frac{\alpha^r}{\lambda^t} \sum_{p=0}^{\infty} \frac{(t)_{(p)}}{p![\alpha + t + p]^r}, \quad t \neq 0.$$

A recurrence relation for marginal moment generating function for *lgos* from *df* (1.5) can be obtained in the following theorem.

Theorem 2.1. *For the distribution given in (1.5) and for* $2 \le r \le n,\ n \ge 2$ *and* $k = 1, 2...$

$$\Big(1 + \frac{t}{\alpha\gamma_r}\Big) M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) - \frac{j}{\alpha\gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t)$$
$$- \frac{\lambda}{\alpha\gamma_r} \Big\{ t M^{(j)}_{X^*(r,n,m,k)}(t+1) + j M^{(j-1)}_{X^*(r,n,m,k)}(t+1) \Big\}. \tag{2.11}$$

Proof. From (1.2), We have

$$M_{X^*(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{2.12}$$

Integrating by parts treating $[F(x)]^{\gamma_r - 1} f(x)$ for integration and rest of the integrand for differentiation, we get

$$M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t) - \frac{t C_{r-1}}{\gamma_r (r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx,$$

the constant of integration vanishes since the integral considered in (2.12) is a definite integral. On using (2.1), we obtain

$$M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t)$$
$$- \frac{t C_{r-1}}{\gamma_r (r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} \Big\{ \frac{(1 + \lambda e^x)}{\alpha} f(x) \Big\} g_m^{r-1}(F(x)) dx$$
$$= M_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha\gamma_r} \Big\{ M_{X^*(r,n,m,k)}(t) - \lambda M_{X^*(r,n,m,k)}(t+1) \Big\}. \tag{2.13}$$

Differentiating both the sides of (2.13) j times with respect to t, we get

$$M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha\gamma_r} M^{(j)}_{X^*(r,n,m,k)}(t)$$

$$-\frac{j}{\alpha\gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t) - \frac{\lambda t}{\alpha\gamma_r} M^{(j)}_{X^*(r,n,m,k)}(t+1) - \frac{\lambda j}{\alpha\gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t+1).$$

The recurrence relation in equation (2.11) is derived simply by rewriting the above equation.

By differentiating both sides of equation (2.11) with respect to t and then setting $t = 0$, we obtain the recurrence relations for moments of *lgos* from extended type I generalized logistic distribution in the form

$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)]$$

$$-\frac{j}{\alpha\gamma_r}\Big\{E[X^{*j-1}(r,n,m,k)] + \lambda E[\phi(X^*(r,n,m,k))]\Big\}, \tag{2.14}$$

where

$$\phi(x) = x^{j-1}e^x.$$

Remark 2.1. Putting $m = 0$, $k = 1$ in (2.11), we obtain the recurrence relation for marginal moment generating function of order statistics for extended type I generalized logistic distribution in the form

$$\Big(1 + \frac{t}{\alpha(n-r+1)}\Big) M^{(j)}_{X_{n-r+1:n}}(t) = M^{(j)}_{X_{n-r+2:n}}(t) - \frac{j}{\alpha(n-r+1)} M^{(j-1)}_{X_{n-r+1:n}}(t)$$

$$-\frac{\lambda}{\alpha(n-r+1)}\Big\{t M^{(j)}_{X_{n-r+1:n}}(t+1) + j M^{(j-1)}_{X_{n-r+1:n}}(t+1)\Big\}.$$

Replacing $(n-r+1)$ by $r-1$, we have

$$M^{(j)}_{X_{r:n}}(t) = \Big(1 + \frac{t}{\alpha(r-1)}\Big) M^{(j)}_{X_{r-1:n}}(t) + \frac{j}{\alpha(r-1)} M^{(j)}_{X_{r-1:n}}(t)$$

$$+\frac{\lambda}{\alpha(r-1)}\Big\{t M^{(j)}_{X_{r-1:n}}(t+1) + j M^{(j-1)}_{X_{r-1:n}}(t+1)\Big\}.$$

Remark 2.2. Setting $m = -1$ and $k \geq 1$, in Theorem 2.1, we get a recurrence relation for marginal moment generating function of lower k record values for extended type I generalized logistic distribution in the form

$$\Big(1+\frac{t}{\alpha k}\Big)M^{(j)}_{X^*(r,n,-1,k)}(t) = M^{(j)}_{X^*(r-1,n,-1,k)}(t) - \frac{j}{\alpha k}M^{(j-1)}_{X^*(r,n,-1,k)}(t)$$
$$-\frac{\lambda}{\alpha k}\Big\{tM^{(j)}_{X^*(r,n,-1,k)}(t+1) + jM^{(j-1)}_{X^*(r-1,n,-1,k)}(t+1)\Big\}.$$

3. Relations for Joint Moment Generating Function

On using (1.3) and binomial expansion, the explicit expression for the joint moment generating of *lgos* $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ $1 \leq r < s \leq n$, can be obtained when $m \neq -1$ as

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times \sum_{a=o}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\begin{pmatrix} r-1 \\ a \end{pmatrix}\begin{pmatrix} s-r-1 \\ b \end{pmatrix}$$
$$\times \int_{-\infty}^{\infty} e^{t_1 x}[F(x)]^{(s-r+a)(m+1)-1} f(x) I(x) dx, \quad x > y \tag{3.1}$$

where

$$I(x) = \int_{-\infty}^{x} e^{t_2 y}[F(y)]^{\gamma_{s-b}-1} f(y) dy. \tag{3.2}$$

By setting $z = [F(y)]^{1/\alpha}$ in (3.2), we obtain

$$I(x) = \frac{\alpha}{\lambda^{t_2}}\sum_{p=0}^{\infty}\frac{(t_2)_{(p)}[F(x)]^{\gamma_{s-b}+(p+t_2)/\alpha}}{p!(\alpha\gamma_{s-b}+p+t_2)}.$$

On substituting the above expression of $I(x)$ in (3.1), we find that

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{\alpha C_{s-1}}{\lambda^{t_2}(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{p=0}^{\infty} \sum_{a=o}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a} \binom{s-r-1}{b}$$

$$\times \frac{(t_2)_{(p)}}{p!(\alpha\gamma_{s-b}+p+t_2)} \int_{-\infty}^{\infty} e^{t_1 x} [F(x)]^{\gamma_{r-a}-1+(p+t_2)/\alpha} f(x) dx$$

$$= \frac{\alpha^2 C_{s-1}}{\lambda^{t_1+t_2}(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{a=o}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a}$$

$$\times \binom{s-r-1}{b} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha\gamma_{s-b}+p+t_2)(\alpha\gamma_{r-a}+p+q+t_1+t_2)} \tag{3.3}$$

and for $s = r+1$

$$M_{X^*(r,n,m,k),X^*(r+1,n,m,k)}(t_1,t_2)$$

$$= \frac{\alpha^2 C_r}{\lambda^{t_1+t_2}(r-1)!(m+1)^{r-1}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{a=o}^{r-1} (-1)^a \binom{r-1}{a}$$

$$\times \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha\gamma_{r+1}+p+t_2)(\alpha\gamma_{r-a}+p+q+t_1+t_2)}. \tag{3.4}$$

At $m = -1$, (3.3) is of the form $\frac{0}{0}$ as $\sum_{a=0}^{r-1}(-1)^a \binom{r-1}{a} = 0.$
Therefore applying L' Hospital rule and using (2.7), we have

$$M_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2)$$

$$= \frac{(\alpha k)^s}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha k+p+t_2)^{s-r}(\alpha k+p+q+t_1+t_2)^r} \tag{3.5}$$

and for $s = r+1$

$$M_{X^*(r,n,-1,k),X^*(r+1,n,-1,k)}(t_1,t_2)$$

$$= \frac{(\alpha k)^{r+1}}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha k+p+t_2)(\alpha k+p+q+t_1+t_2)^r}. \tag{3.6}$$

Differentiating (3.3), (3.4), (3.5) and (3.6) and evaluating at $t_1 = t_2 = 0$, we get the mean of the $r-$th $lgos$.

Special cases

i) Putting $m = 0, \quad k = 1$, in (3.3), the explicit formula for joint moment generating functions of order statistics for extended type I generalized logistic distribution can be obtained as

$$M_{X_{n-r+1,n-s+1:n}}(t_1, t_2)$$

$$= \frac{\alpha^2 C_{r,s:n}}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{a=o}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a} \binom{s-r-1}{b}$$

$$\times \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q![\alpha(n-s+1+b)+p+t_2][\alpha(n-r+1+a)+p+q+t_1+t_2]}.$$

That is

$$M_{X_{r,s:n}}(t_1, t_2) = \frac{\alpha^2 C_{r,s:n}}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{a=o}^{r-1} \sum_{b=0}^{s-r-1} (-1)^{a+b} \binom{r-1}{a} \binom{s-r-1}{b}$$

$$\times \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q![\alpha(r+b)+p+t_2][\alpha(s+a)+p+q+t_1+t_2]},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii) Setting $k = 1$ in (3.5), we get the explicit expression for joint moment generating function of lower record value for extended type I generalized logistic distribution can be obtained as

$$M_{X_{L(r)},X_{L(s)}}(t_1, t_2) = \frac{\alpha^s}{\lambda^{t_1+t_2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)_{(p)}(t_1)_{(q)}}{p!q!(\alpha+p+t_2)^{s-r}(\alpha+p+q+t_1+t_2)^r}.$$

Making use of (2.1), we can derive the recurrence relations for joint moment generating function of *lgos* from (1.5).

Theorem 3.1. *For the distribution given in (1.5) and for* $1 \le r < s \le n$, $n \ge 2$ *and* $k = 1, 2...$

$$\left(1 + \frac{t_2}{\alpha\gamma_s}\right) M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1, t_2)$$

$$-\frac{j}{\alpha\gamma_s}M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{\lambda}{\alpha\gamma_s}\Big\{t_2M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)$$

$$+jM^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)\Big\}. \tag{3.7}$$

Proof. Using (1.3), the joint moment generating function of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is given by

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}[F(x)]^m f(x)g_m^{r-1}(F(x))G(x)dx \tag{3.8}$$

where

$$G(x)=\int_{-\infty}^{x}e^{t_1x+t_2y}[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy.$$

Solving the integral in $G(x)$ by parts and substituting the resulting expression in (3.8), we get

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)=M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{t_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x}e^{t_1x+t_2y}[F(x)]^m f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}dydx,$$

the constant of integration vanishes since the integral in $G(x)$ is a definite integral. On using the relation (2.1), we obtain

$$M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)=M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$

$$-\frac{t_2}{\alpha\gamma_s}M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$

$$-\frac{\lambda t_2}{\alpha\gamma_s}M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1). \tag{3.9}$$

Differentiating both the sides of (3.9) i times with respect to t_1 and then j times with respect to t_2, we get

$$M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)$$
$$-\frac{t_2}{\alpha\gamma_s}M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$
$$-\frac{j}{\alpha\gamma_s}M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2)$$
$$-\frac{\lambda t_2}{\alpha\gamma_s}M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1)$$
$$-\frac{j\lambda}{\alpha\gamma_s}M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2+1),$$

which, when rewritten gives the recurrence relation in (3.7).
One can also note that Theorem 2.1 can be deduced from Theorem 3.1 by letting t_1 tends to zero.
By differentiating both sides of equation (3.7) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the recurrence relations for product moments of *lgos* from extended type I generalized logistic distribution in the form

$$E[X^{*i}(r,n,m,k),X^{*j}(s,n,m,k)] = E[X^{*i}(r,n,m,k),X^{*j}(s-1,n,m,k)]$$
$$-\frac{j}{\alpha\gamma_s}\Big\{E[X^{*i}(r,n,m,k),X^{*j-1}(s,n,m,k)]$$
$$+\lambda E[\phi(X^*(r,n,m,k),X^*(s,n,m,k))]\Big\}, \quad (3.10)$$

where

$$\phi(x,y) = x^i y^{j-1} e^y.$$

Remark 3.1. Putting $m = 0$, $k = 1$ in (3.7), we obtain the recurrence relations for joint moment generating function of order statistics for extended type I generalized logistic distribution in the form

$$\Big(1+\frac{t_2}{\alpha(n-s+1)}\Big)M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2) = M^{(i,j)}_{X_{n-r+1,n-s+2:n}}(t_1,t_2)$$
$$-\frac{j}{\alpha(n-s+1)}M^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2)$$

$$-\frac{\lambda}{\alpha(n-s+1)}\Big\{t_2 M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2+1)+jM^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1,t_2+1)\Big\}.$$

That is

$$M^{(i,j)}_{X_{r,s:n}}(t_1,t_2)=\Big(1+\frac{t_1}{\alpha(r-1)}\Big)M^{(i,j)}_{X_{r-1,s:n}}(t_1,t_2)+\frac{i}{\alpha(r-1}M^{(i-1,j)}_{X_{r-1,s:n}}(t_1,t_2)$$

$$+\frac{\lambda}{\alpha(r-1)}\Big\{t_1 M^{(i,j)}_{X_{r-1,s:n}}(t_1+1,t_2)+iM^{(i-1,j)}_{X_{r-1,s:n}}(t_1,t_2)\Big\}.$$

Remark 3.2. Substituting $m=-1$ and $k\geq 1$, in Theorem 3.1, we get recurrence relation for joint moment generating function of lower k record values for extended type I generalized logistic distribution in the form

$$\Big(1+\frac{t_2}{\alpha k}\Big)M^{(i,j)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2)=M^{(i,j)}_{X^*(r,n,-1,k),X^*(s-1,n,-1,k)}(t_1,t_2)$$

$$-\frac{j}{\alpha k}M^{(i,j-1)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2)$$

$$-\frac{\lambda}{\alpha k}\Big\{t_2 M^{(i,j)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2+1)$$

$$+jM^{(i,j-1)}_{X^*(r,n,-1,k),X^*(s,n,-1,k)}(t_1,t_2+1)\Big\}.$$

4. Characterization

Let $X^*(r,n,m,k)$, $r=1,2,\ldots,n$ be *lgos* from a continuous population with *df* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X^*(s,n,m,k)$ given $X^*(r,n,m,k)=x$, $1\leq r<s\leq n$, in view of (1.2) and (1.3), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x)=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_r+1}$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y). \tag{4.1}$$

Theorem 4.1. *Let X be an absolutely continuous random variable df $F(x)$ and pdf $f(x)$ on the support $(-\infty,\infty)$ then*

$$E[e^{tX^*(s,n,m,k)}|X^*(l,n,m,k)=x]$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \prod_{j=1}^{s-l} \Big(\frac{\gamma_{l+j}}{\gamma_{l+j} - p/\alpha}\Big), \quad l = r, r+1 \tag{4.2}$$

if and only if

$$F(x) = \Big(\frac{\lambda}{\lambda + e^{-x}}\Big)^{\alpha}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0.$$

Proof. From (4.1), we have

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}}$$

$$\times \int_{-\infty}^{x} e^{ty} \Big[1 - \Big(\frac{F(y)}{F(x)}\Big)^{m+1}\Big]^{s-r-1} \Big(\frac{F(y)}{F(x)}\Big)^{\gamma_s - 1} \frac{f(y)}{F(x)} dy. \tag{4.3}$$

By setting $u = \frac{F(y)}{F(x)} = \Big(\frac{\lambda + e^{-x}}{\lambda + e^{-y}}\Big)^{\alpha}$ from (1.5) in (4.3), we obtain

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}}$$

$$\times \int_0^1 [(\lambda + e^{-x}) u^{-1/\alpha} - \lambda]^{-t} u^{\gamma_s - 1} (1 - u^{m+1})^{s-r-1} du$$

$$= \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p$$

$$\times \int_0^1 u^{\gamma_s - (p/\alpha) - 1} (1 - u^{m+1})^{s-r-1} du. \tag{4.4}$$

Again by setting $z = u^{m+1}$ in (4.4), we get

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r}}$$

$$\times \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \int_0^1 z^{\frac{\alpha k - p}{\alpha(m+1)} + n - s - 1} (1 - z)^{s-r-1} dz$$

$$= \frac{C_{s-1}}{C_{r-1} (m+1)^{s-r}} \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \frac{\Gamma\Big(\frac{\alpha k - p}{\alpha(m+1)} + n - s\Big)}{\Gamma\Big(\frac{\alpha k - p}{\alpha(m+1)} + n - r\Big)}$$

$$= \frac{C_{s-1}}{C_{r-1}} \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \frac{1}{\prod_{j=1}^{s-r} (\gamma_{r+j} - p/\alpha)}$$

and hence the result given in (4.2).
To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \int_{-\infty}^{\infty} e^{t_2 y} [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}$$
$$\times [F(y)]^{\gamma_s - 1} f(y) dy = [F(x)]^{\gamma_{r+1}} H_r(x), \tag{4.5}$$

where

$$H_r(x) = \sum_{p=0}^{\infty} \frac{(-1)^t (t)_{(p)}}{\lambda^t p!} \Big(\frac{\lambda + e^{-x}}{\lambda}\Big)^p \prod_{j=1}^{s-r} \Big(\frac{\gamma_{r+j}}{\gamma_{r+j} - p/\alpha}\Big).$$

Differentiating (4.5) both sides with respect to x, we get

$$\frac{C_{s-1} [F(x)]^m f(x)}{(s-r-2)! C_{r-1} (m+1)^{s-r-2}} \int_{-\infty}^{\infty} e^{t_2 y} [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2}$$
$$\times [F(y)]^{\gamma_s - 1} f(y) dy = H_r'(x) [F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x)$$
$$\gamma_{r+1} H_{r+1}(x) [F(x)]^{\gamma_{r+2} + m} f(x)$$
$$= H_r'(x) [F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{H_r'(x)}{\gamma_{r+1} [H_{r+1}(x) - H_r(x)]} = \frac{\alpha}{(1 + \lambda e^x)}$$

which proves that

$$F(x) = \Big(\frac{\lambda}{\lambda + e^{-x}}\Big)^{\alpha}, \quad -\infty < x < \infty, \quad \alpha, \lambda > 0.$$

Theorem 4.2. *Let X be an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all x, then*

$$M_{X^*(r,n,m,k)}(t) - M_{X^*(r-1,n,m,k)}(t)$$

$$-\frac{t}{\alpha\gamma_r}\Big\{M_{X^*(r,n,m,k)}(t)+\lambda M_{X^*(r-1,n,m,k)}(t+1)\Big\}. \tag{4.6}$$

if and only if

$$F(x)=\Big(\frac{\lambda}{\lambda+e^{-x}}\Big)^{\alpha}, \quad -\infty<x<\infty, \quad \alpha,\lambda>0.$$

Proof. The necessary part follows immediately from equation (2.11). On the other hand if the recurrence relation in equation (4.6) is satisfied, then on using equation (1.2), we have

$$\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$=\frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx$$

$$-\frac{tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$-\frac{\lambda tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{(t+1)x}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{4.7}$$

Integrating the first integral on the right hand side of equation (4.7), by parts, we get

$$\frac{tC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}e^{tx}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\Big\{F(x)-\frac{1}{\alpha}f(x)-\frac{\lambda e^{x}}{\alpha}f(x)\Big\}dx=0. \tag{4.8}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [9] to equation (4.8), we get

$$\frac{f(x)}{F(x)}=\frac{\alpha}{(1+\lambda e^{x})}$$

which proves that

$$F(x)=\Big(\frac{\lambda}{\lambda+e^{-x}}\Big)^{\alpha}, \quad -\infty<x<\infty, \quad \alpha,\lambda>0.$$

5. Conclusion

This paper deals with the lower generalized order statistics from the extended type I generalized logistic distribution. Some explicit expressions and recurrence relations for marginal and joint moment generating functions are derived. Characterizing results of the extended type I generalized logistic distribution has been obtained by using conditional expectation and recurrence relation of lower generalized order statistics. Special cases are also deduced.

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