Tamsui Oxford Journal of Information and Mathematical Sciences **29(2)** (2013) 201-217
Aletheia University

Stability of Additive Mappings In Generalized Normed Spaces *

Alireza Mohammadpour[†]and Nabi Shobe[‡]
Department of Mathematics, Islamic azad university-Babol Branch, Babol, Iran

Received January 18, 2012, Accepted April 8, 2013.

Abstract

In this paper, we introduce the concept of a Generalized normed space and prove a theorem for existence of an additive mapping in this space. We show that our results extend some of the known results in literature.

Keywords and Phrases: *Generalized normed space, Additive mappings, Cauchy function.*

1. Introduction and Preliminaries

It is well known that the Ulam's [12] question in 1940: "Under what conditions does there exist an additive mapping near an approximately additive mapping ?", is the origin of the stability problem of functional equations. Many authors have extended, generalized and improved the answer to Ulam's question such as, Hyers [4] in the context of Banach space, K. Ravi, R. Murali and M. Arunkumar [10] for quadratic functional equation, T. Aoki [1] for additive mappings and Th.M. Rassias [8] for linear mappings in 1978 by considering the unbounded Cauchy difference. It states as follows:

---

*2000 *Mathematics Subject Classification.* Primary 39B82, 39B70, 39B52.

[†]Corresponding author. E-mail: mohammadpour_ar@yahoo.com
[‡]E-mail: nabi_shobe@yahoo.com

Theorem 1.1. *([8]) Let E, E' be two Banach spaces and let $\theta \in [0, \infty)$ and $p \in [0, 1)$. If a function $f : E \longrightarrow E'$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta[\|x\|^p + \|y\|^p]$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\,\|x\|^p$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$ then T is linear.

In the following theorem J. M. Rassias replaced the sum by the product of powers of norms.

Theorem 1.2. *([7]) Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\,\|x\|^p\,\|y\|^p \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p \leq \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \longrightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2-2^{2p}}\,\|x\|^{2p} \tag{1.2}$$

for all $x \in E$. If $p < 0$, then the inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. If $p > \frac{1}{2}$ then the inequality (1.1) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n \longrightarrow \infty} 2^n f(\frac{x}{2^n})$$

exists for all $x \in E$ and $A : E \longrightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p}-2}\,\|x\|^{2p}$$

for all $x \in E$. If in addition $f : E \longrightarrow E'$ is a mapping such that the transformation $t \rightarrow f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}-$linear mapping.

Also, the topic of stability of functional equations has been studied by a number of mathematicians (see [6, 2, 3, 9, 5] for more detailed information). Before giving the main results, we recall the definition of normed-binary operation and some examples and lemmas were used in [11].

Definition 1.3. ([11])A normed-binary operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ which satisfies the following conditions:
(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,\infty)$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,\infty)$.

Example 1.4. ([11])Let $a, b \in [0,\infty)$. Five typical examples of $\diamond$ are:
(a) $a \diamond_1 b = \max\{a, b\}$
(b) $a \diamond_2 b = \sqrt{a^2 + b^2}$
(c) $a \diamond_3 b = a + b$
(d) $a \diamond_4 b = ab + a + b$
(e) $a \diamond_5 b = (\sqrt{a} + \sqrt{b})^2$.
For $a, b \in [0,\infty)$, straight forward calculations lead to the following relations among normed binary operations giving above
$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b \leq a \diamond_4 b,$
and
$a \diamond_3 b \leq a \diamond_5 b.$

The follwing lemma defines a normed binary operation exploting some properties of a self map on $[0,\infty)$.

Lemma 1.5. *([11])Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a continuous, onto, and increasing map. Let $\diamond : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be defined by*
$a \diamond b = f^{-1}(f(a) + f(b))$ for $a, b \in [0,\infty)$,
then $\diamond$ is a normed binary operation.

Example 1.6. ([11])Let $f : [0,\infty) \longrightarrow [0,\infty)$ defined by $f(x) = e^x - 1$. Then $a \diamond b = Ln(e^a + e^b - 1)$ for $a, b \in [0,\infty)$ defines a normed binary operation.

We have the following simple observations about normed binary operation.

Lemma 1.7. *(**[11]**) (i) If $r, r' \geq 0$, then $r \leq r \diamond r'$.*
(ii) If $\delta \in (0, r)$, there exist $\delta' \in (0, r)$ such that $\delta' \diamond \delta < r$.
(iii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$.

In this paper all vector spaces are real.

Now we are set to generalize the concept of a normed space.

Definition 1.8. Let X be vector space and $\diamond$ be a binary operation. A generalized norm on X is a function: $N : X \longrightarrow \mathbb{R}$ that satisfies the following properties:

(1) $N(x) \geq 0$ for each x in X,

(2) $N(x) = 0$ if and only if $x = 0$,

(3) $N(\alpha x) = |\alpha|^t N(x)$ for some $t \in (0, \infty)$, for each x in X and every $\alpha \in \mathbb{R}$.

(4) $N(x + y) \leq N(x) \diamond N(y)$, for each $x, y \in X$.

The $3-$*tuple* $(X, N, \diamond)$ is called a generalized normed space or a $G-$*normed* space.

Example 1.9. Let $(X, \|.\|)$ be a normed space, $a, b \in [0, \infty)$, and $x \in X$. If we define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$,

(i) $a \diamond b = a + b$, and N is defind by $N(x) = \|x\|$, then $(X, N, \diamond)$ is a $G - normed$ space for $t = 1$.

(ii) $a \diamond b = \sqrt{a^2 + b^2}$, and N is defined by $N(x) = \sqrt{\|x\|}$, then $(X, N, \diamond)$ is a $G - normed$ space for $t = \frac{1}{2}$.

(iii) $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, and N is defined by $N(x) = \|x\|^2$, then $(X, N, \diamond)$ is a $G - normed$ space for $t = 2$.

Remark 1.10. From Example 1.9 (i), we see that:
every normed space is a $G - normed$ space.

Remark 1.11. In (3) of Definition 1.8 t is unique.

Example 1.12. Let $X = \mathbb{R}^2$, if we define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ by
$a \diamond b = (\sqrt[4]{a} + \sqrt[4]{b})^4$ for $a, b \in [0, \infty)$, and define $N : X \longrightarrow R$ by
$N(x, y) = x^4 + y^4$ for $x, y \in \mathbb{R}$,
then $(X, N, \diamond)$ is a $G - normed$ space for $t = 4$.

Definition 1.13. Let $(X, N, \diamond)$ be a $G - normed$ space. For $r > 0$, the ball $B_N(x, r)$ with center $x \in X$ and radius r is defined by

$$B_N(x, r) = \{y \in X : N(x - y) < r\}.$$

Definition 1.14. Let $(X, N, \diamond)$ be a $G - normed$ space. A subset $A \subseteq X$ is open if for every $x \in A$, there exists $r > 0$ such that $B_N(x, r) \subseteq A$.

Let τ be the set of all open subsets $A \subseteq X$. It can be verified that τ is a topology on $X,$ called a topology induced by generalized norme N.

Lemma 1.15. *Let $(X, N, \diamond)$ be a $G-normed$ space. Then*
(i) $N(ax) \leq N(x)$ *for all real scalars a with $|a| \leq 1$.*
(ii) *if X is convex, then we get*

$$N(ax + (1-a)y) \leq N(x) \diamond N(y)$$

for all $x, y \in X$ and every $a \in (0, 1)$.

Proof. Proof immediately follows from Definition 1.8. □

Definition 1.16. Let $(X, N, \diamond)$ be a $G - normed$ space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$n \geq n_0 \Longrightarrow N(x_n - x) < \epsilon.$
We denote this by $N(x_n - x) \longrightarrow 0$ as $n \longrightarrow \infty$ or $\lim\limits_{n \longrightarrow \infty} x_n = x.$

Definition 1.17. Let $(X, N, \diamond)$ be a $G - normed$ space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0,$ there exists $n_0 \in \mathbb{N}$ such that $N(x_n - x_m) < \epsilon$ for each $n, m \geq n_0$.

The generalized normed space $(X, N, \diamond)$ is said to be generalized Banach space or $G - Banach$ space if every Cauchy sequence is convergent in X.

Now we prove the following basic lemmas needed in the sequel.

Lemma 1.18. *Let $(X, N, \diamond)$ be a $G - normed$ space. If $r > 0,$ then the ball $B_N(x, r)$ is open.*

Proof. Let $y \in B_N(x, r)$, so that we have $N(x - y) < r$. Put, $N(x - y) = \delta$ then by Lemma 1.7 there exists $\delta' > 0$ such that $\delta' \diamond \delta < r$. Now, we prove that $B_N(y, \delta') \subseteq B_N(x, r)$. For this, let $z \in B_N(y, \delta')$. By triangle inequality we have

$$N(x - z) \leq N(x - y) \diamond N(y - z) < \delta \diamond \delta' < r.$$

This implies that

$$B_N(y, \delta') \subseteq B_N(x, r).$$

Hence $B_N(x, r)$ is an open set. □

Lemma 1.19. *Every $G - normed$ space $(X, N, \diamond)$ is a Hausdorff space.*

Proof. Let $x, y \in X$ and $x \neq y$. If we set $N(x-y) = r$ then for $0 < \delta < r$ by Lemma 1.7 there exists $0 < \delta' < r$ such that $\delta' \diamond \delta < r$. We prove that $B_N(x, \delta) \cap B_N(y, \delta') = \varnothing$. Let $z \in B_N(x, \delta) \cap B_N(y, \delta')$. Now, by triangle inequality, we get that

$$r = N(x-y) \leq N(x-z) \diamond N(z-y) < \delta \diamond \delta' < r,$$

which is a contradiction. Hence $(X, N, \diamond)$ is a Hausdorff space. □

Lemma 1.20. *Let $(X, N, \diamond)$ be a $G-$normed space, then every convergent sequence in X is Cauchy in X.*

Proof. Let $\{x_n\}$ be a sequence in X which converges to $x \in X$. For $\epsilon > 0$, by Lemma 1.7 we can choose a $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Since $x_n \longrightarrow x$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we obtain that $N(x_n - x) < \delta$.

Thus for every $n, m \geq n_0$, we have

$$N(x_n - x_m) \leq N(x_n - x) \diamond N(x - x_m) < \delta \diamond \delta < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Lemma 1.21. *Let $(X, N, \diamond)$ be a $G-$normed space, then addition $+: X \times X \longrightarrow X$ defined by $+(x, y) = x + y$ and scalar multiplication $\cdot : \mathbb{R} \times X \longrightarrow X$ defined by $\cdot(\alpha, x) = \alpha \cdot x$ are continuous.*

Proof. First we prove continuity of addition. Let $x_n \longrightarrow x$, $y_n \longrightarrow y$. By Lemma 1.7 for each $\epsilon > 0$ there exists $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Also, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow N(x_n - x) < \delta,$$

and

$$n \geq n_0 \Longrightarrow N(y_n - y) < \delta.$$

By triangle inequality we have

$$N((x_n + y_n) - (x + y)) \leq N(x_n - x) \diamond N(y_n - y) < \delta \diamond \delta < \epsilon.$$

Now we prove that scalar multiplication is continuous. Let $\alpha_n \longrightarrow \alpha$, and $x_n \longrightarrow x$(which means that $\lim_{n \longrightarrow \infty} N(x_n - x) = 0$).

Triangle inequality gives that

$$N(\alpha_n{\cdot}x_n-\alpha{\cdot}x) = N(\alpha_n{\cdot}(x_n-x)+(\alpha_n-\alpha){\cdot}x) \leq |\alpha_n|^t\, N(x_n-x)\diamond|\alpha_n-\alpha|^t\, N(x).$$

and so

$$\limsup_{n\longrightarrow\infty} N(\alpha_n{\cdot}x_n-\alpha{\cdot}x) \leq \lim_{n\longrightarrow\infty}|\alpha_n|^t\, N(x_n-x)\diamond \lim_{n\longrightarrow\infty}|\alpha_n-\alpha|^t\, N(x) = 0.$$

Hence

$$\lim_{n\longrightarrow\infty} \alpha_n{\cdot}x_n = \alpha{\cdot}x.$$ □

Example 1.22. Let $a\diamond b = \max\{a,b\}$, then there is not any $t\in(0,\infty)$ such that

$$N(\alpha{\cdot}x) = |\alpha|^t\cdot N(x).$$

Because If we assume that (on contrary), there exists $t\in(0,\infty)$ and $N(\alpha{\cdot}x) = |\alpha|^t{\cdot}N(x)$. Then by taking $\alpha=2$ we obtain:

$$|2|^t\cdot N(x) = N(2x) = N(x+x) \leq N(x)\diamond N(x) = N(x),$$

which is a contradiction.

Henceforth, we assume that the normed binary operation $\diamond$ on $[0,\infty)\times[0,\infty)$ satisfy the following properties:

(PI) : $\alpha{\cdot}(a\diamond b) = \alpha{\cdot}a\diamond\alpha{\cdot}b$ for every $\alpha\in\mathbb{R}^+$ and

(PII) : there exists $h\geq 0$ such that $1\diamond 1\diamond\cdots\diamond 1\leq n^h$, for every $n\in N$.

In the following example, we give some normed binary operations $\diamond$ on $[0,\infty)\times[0,\infty)$ with properties (PI) and $P(II)$.

Example 1.23. Let $a\diamond b=\max\{a,b\}$ or $a\diamond b=\sqrt{a^2+b^2}$ or $a\diamond b = a+b$ or $a\diamond b = (\sqrt{a}+\sqrt{b})^2$

then in each case, $\diamond$ satisfies properties (PI) and (PII).

The next example includes a normed binary operation $\diamond$ on $[0,\infty)\times[0,\infty)$ which does not satisfy (PI) and $P(II)$ properties.

Example 1.24. Define $\diamond : [0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ by $a\diamond b = a+b+ab$, for $a,b\in[0,\infty)$. Obviously $\diamond$ is not have (PI) and (PII) properties.

2. Main Results

In the rest of this paper, we will assume that $(X, N', \diamond)$ is $G-normed$ space and $(Y, N, \diamond)$ is $G-Banach$ space.

Let ϕ be a function from $X \times X$ to X. A mapping $f : X \longrightarrow Y$ is called a $\phi-approximately\ Cauchy\ function$, if

$$N(f(x+y) - f(x) - f(y)) \leq N'(\phi(x,y)) \tag{2.1}$$

for all $x, y \in X$.

Example 2.1. Let $X = Y = \mathbb{R}$ and N, N' be usual norm. Let ϕ be a function from $X \times X \longrightarrow X$ defined by $\phi(x,y) = xy(\frac{x+y}{2})$.

Let mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = \sin x$. Then one can easily see that f is a $\phi-approximately\ Cauchy\ function$, because:

$$\begin{aligned} |f(x+y) - f(x) - f(y)| &= \left| -4\sin(\frac{x+y}{2}) \cdot \sin\frac{x}{2} \cdot \sin\frac{y}{2} \right| \\ &\leq 4\left| \frac{x+y}{2} \cdot \frac{x}{2} \cdot \frac{y}{2} \right| \\ &= \frac{|x+y|\,|xy|}{2} = |\phi(x,y)| \end{aligned}$$

for all $x, y \in \mathbb{R}$.

Hence $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\phi-approximately\ Cauchy\ function.$

In the sequel, all binary operation $\diamond$ satisfy (PI) and (PII) properties.

Theorem 2.2. *Let $\phi : X \times X \longrightarrow X$ be a function and $f : X \longrightarrow Y$ be a $\phi-approximately\ Cauchy\ function$ and for some $0 < \alpha < \frac{1}{2}$ assume that,*

$$N'(\phi(\frac{x}{2}, \frac{y}{2})) \leq N'(\alpha\phi(x,y))$$

then there exists an additive mapping $T : X \longrightarrow Y$.
Morover, if $a \diamond b \leq a + b$ for every $a, b \in [0, \infty)$, then

$$N(f(x) - T(x)) \leq \frac{\alpha^t}{1 - (2\alpha)^t} N'(\phi(x,x)) \tag{2.2}$$

for all $x \in X$ and T is unique.

Proof. Since f is $\phi-approximately\ Cauchy\ function$, put $x=y$ in (2.1) to obtain
$$N(f(2x)-2f(x)) \leq N'(\phi(x,x)) \quad (x\in X) \tag{2.3}$$

Replacing x by $2^{-n-1}x$ in inequality (2.3) we get

$$\begin{aligned} N(f(\frac{x}{2^n})-2f(\frac{x}{2^{n+1}}) &\leq N'(\phi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}})) \\ &\leq N'(\alpha\phi(\frac{x}{2^n},\frac{x}{2^n})) \\ &\leq |\alpha|^t\, N'(\phi(\frac{x}{2^n},\frac{x}{2^n})) \\ &\ \ \vdots \\ &\leq |\alpha|^{nt}\, N'(\phi(x,x)) \end{aligned}$$

If set $a_n(x)=2^n f(2^{-n}x)$, we have

$$\begin{aligned} N(a_n(x)-a_{n+1}(x)) &= N(2^n f(2^{-n}x)-2^{n+1}f(2^{-n-1}x)) \\ &= 2^{nt}N(f(2^{-n}x)-2f(2^{-n-1}x)) \\ &\leq |2\alpha|^{nt}\, N'(\phi(x,x)) \end{aligned}$$

Also for $n\leq m$ $(n,m\in\mathbb{N})$

$$\begin{aligned} N(a_m(x)-a_n(x)) &\leq N(a_{n+1}(x)-a_n(x))\diamond N(a_{n+2}(x)-a_{n+1}(x)) \\ &\quad \diamond\cdots\diamond N(a_m(x)-a_{m-1}(x)) \\ &\leq |2\alpha|^{nt}\, N'(\phi(x,x))\diamond|2\alpha|^{(n+1)t}\, N'(\phi(x,x)) \\ &\quad \diamond\cdots\diamond|2\alpha|^{(m-1)t}\, N'(\phi(x,x)) \\ &\leq |2\alpha|^{nt}\, N'(\phi(x,x))(\underbrace{1\diamond 1\diamond\cdots 1}_{(m-1)-n}) \\ &\leq |2\alpha|^{nt}\, N'(\phi(x,x))(\underbrace{1\diamond 1\diamond\cdots 1}_{m}) \\ &\leq |2\alpha|^{nt}\, N'(\phi(x,x))\cdot m^h \end{aligned}$$

It is easy to see that for every $m\geq n$, there exists $s>0$ such that $m\leq n^s$. Thus
$$N(a_m(x)-a_n(x))\leq|2\alpha|^{nt}\cdot n^s\cdot N'(\phi(x,x))\longrightarrow 0.$$

Which implies that $a_n(x) = 2^n f(2^{-n}x)$ is a Cauchy sequence. Since Y is $G - Banach$ space, hence, for every $x \in X$ there exists $y_v \in Y$ such that $\lim_{n\longrightarrow\infty} a_n(x) = y_v(x)$. Indeed we can define a mapping $T : X \longrightarrow Y$ by $T(x) = \lim_{n\longrightarrow\infty} a_n(x)$. That is

$$\lim_{n\longrightarrow\infty} N(2^n f(2^{-n}x) - T(x)) = 0(x \in X).$$

Now, we show that T is an additive mapping. We have

$$\begin{aligned}
N(T(x+y) - Tx - Ty) \;&\leq\; N(T(x+y) - 2^n f(2^{-n}(x+y))) \\
&\diamond N(2^n f(2^{-n}x) - Tx) \diamond N(2^n f(2^{-n}y - Ty) \\
&\diamond N(2^n f(2^{-n}(x+y) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y) \\
&\leq\; N(T(x+y) - 2^n f(2^{-n}(x+y))) \diamond N(2^n f(2^{-n}x) - Tx) \\
&\diamond N(2^n f(2^{-n}y - Ty) \diamond |2\alpha|^{nt}\, N'(\phi(x,y)).
\end{aligned}$$

As $n \longrightarrow \infty$ we get

$$N(T(x+y) - Tx - Ty) \longrightarrow 0.$$

Hence $T(x+y) = T(x) + T(y)$. Now we show that the mapping T satisfies in the inequality (2.2).

We have

$$N(f(x) - T(x)) \leq N(f(x) - 2^n f(2^{-n}x)) \diamond N(2^n f(2^{-n}x) - T(x)).$$

Since

$$\begin{aligned}
N(f(x)-2^n f(2^{-n}x)) &= N(\sum_{i=0}^{n-1} 2^i f(2^{-i}x) - 2^{i+1} f(2^{-i-1}x)) \\
&\leq N(f(x)-2f(2^{-1}x)) \diamond N(2f(2^{-1}x)-2^2 f(2^{-2}x)) \\
&\quad \diamond N(2^{n-1}f(2^{n-1}x)-2^n f(2^{-n}x)) \\
&\leq N'(\phi(\frac{x}{2},\frac{x}{2})) \diamond 2^t N'(\phi(\frac{x}{4},\frac{x}{4})) \diamond 2^{2t} N'(\phi(\frac{x}{8},\frac{x}{8})) \\
&\quad \diamond \cdots \diamond 2^{(n-1)t} N'(\phi(\frac{x}{2^n},\frac{x}{2^n})) \\
&\leq \alpha^t N'(\phi(x,x)) \diamond 2^t \alpha^{2t} N'(\phi(x,x)) \diamond 2^{2t}\alpha^{3t} N'(\phi(x,x)) \\
&\quad \diamond \cdots \diamond 2^{(n-1)t}\alpha^{nt} N'(\phi(x,x)) \\
&\leq \alpha^t N'(\phi(x,x))(1 \diamond (2\alpha)^t \diamond (2\alpha)^{2t} \diamond \cdots \diamond (2\alpha)^{(n-1)t} \\
&\leq \alpha^t N'(\phi(x,x))(1+(2\alpha)^t+(2\alpha)^{2t}+\cdots+(2\alpha)^{(n-1)t}) \\
&\leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)).
\end{aligned}$$

Therefore

$$N(f(x)-T(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)) \diamond N(2^n f(2^{-n}x)-T(x)).$$

As n tends to infinity we have

$$N(f(x)-T(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x)).$$

For uniqueness, suppose $T' : X \longrightarrow X$ is another additive mapping such that $T' \neq T$ and

$$N(f(x)-T'(x)) \leq \frac{\alpha^t}{1-(2\alpha)^t} N'(\phi(x,x))$$

for every $x \in X$.

Also, since T and T' are additive we have

$$T(x) = 2^n T(2^{-n}x) \text{ and } T'(x) = 2^n T'(2^{-n}x)$$

Hence we get

$$\begin{aligned} N(Tx-T'x) &\leq N(2^nT(2^{-n}x)-2^nf(2^{-n}x))\diamond N(2^nf(2^{-n}x)-2^nT'(2^{-n}x)) \\ &\leq 2^{nt}N(T(2^{-n}x)-f(2^{-n}x))\diamond 2^{nt}N(f(2^{-n}x)-T'(2^{-n}x)) \\ &\leq 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}N'(\phi(\frac{x}{2^n},\frac{x}{2^n}))\diamond 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}N'(\phi(\frac{x}{2^n},\frac{x}{2^n})) \\ &\leq 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}\cdot\alpha^{nt}N'(\phi(x,x))\diamond 2^{nt}\frac{\alpha^t}{1-(2\alpha)^t}\cdot\alpha^{nt}N'(\phi(x,x)) \\ &\leq (2\alpha)^{nt}\frac{\alpha^t}{1-(2\alpha)^t}N'(\phi(x,x))(1\diamond 1)\longrightarrow 0. \end{aligned}$$

It follwes that $T=T'$. $\square$

Corollary 2.3. *Let $(X,\|\cdot\|_2)$ be normed space and $(Y,\|\cdot\|_1)$ be Banach Space. Let ϕ be a function from $X\times X$ to X, and mapping $f:X\longrightarrow Y$ be a $\phi-approximately\ Cauchy\ function$. If for some $0<\alpha<\frac{1}{2}$ assume that,*

$$\left\|\phi(\frac{x}{2},\frac{y}{2})\right\|_2\leq\|\alpha\phi(x,y)\|_2$$

then there exists a unique additive mapping $T:X\longrightarrow Y$ such that

$$\|f(x)-T(x)\|_1\leq\frac{\alpha}{1-2\alpha}\|\phi(x,x)\|_2$$

for all $x\in X$.

Proof. If we take $(X,\|\cdot\|_2)=(X,N',\diamond)$ and $(Y,\|\cdot\|_1)=(Y,N,\diamond)$ we get the proof by Theorem 2.2 and Remark 1.10. $\square$

Example 2.4. Let $X=Y=\mathbb{R}$ and N,N' be usual norm. Let ϕ be a function from $\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by $\phi(x,y)=xy(\dfrac{x+y}{2})$. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defind by $f(x)=\sin x$. By Example 2.1 f is a $\phi-approximately\ Cauchy\ function$ and

$$\left|\phi(\frac{x}{2},\frac{y}{2})\right|\leq|\alpha\phi(x,y)|\quad\text{for }\frac{1}{8}\leq\alpha<\frac{1}{2}.$$

Hence all conditions of Corollary 2.3 are hold. So there exists a unique additive mapping $T:\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$|f(x)-T(x)|\leq\frac{\alpha}{1-2\alpha}\left|x^3\right|.$$

where

$$T(x) = \lim_{n\longrightarrow\infty} 2^n f(2^{-n}x) = \lim_{n\longrightarrow\infty} 2^n \sin(\frac{x}{2^n}) = x.$$

And

$$|f(x) - T(x)| = |\sin x - x| \leq \frac{\alpha}{1-2\alpha} \left|x^3\right|.$$

We show that corollary (2.3) extends the theorems 1.1 ([8]) and 1.2 ([7]).

Corollary 2.5. *Let X, X' be two Banach spaces and let $\theta \in [0,\infty)$. If a function $f : X \longrightarrow X'$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta[\|x\|^p + \|y\|^p] \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \longrightarrow X'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2^p - 2} \|x\|^p, \ \textit{for} \ \ p > 1. \tag{2.5}$$

And

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p, \ \ \textit{for} \ \ 0 < p < 1. \tag{2.6}$$

Proof. We show that, if set

$$\phi(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{\theta}{\|x\|+\|y\|} \cdot (\|x\|^p + \|y\|^p)(x+y) & (x,y) \neq (0,0) \end{cases}$$

Then, all conditions of corollary (2.3) are established. Case of $(x, y) = (0, 0)$ is obviously. In case of $(x, y) \neq (0, 0)$ we have

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \|\phi(x,y)\| \\ &= \frac{\theta}{\|x\|+\|y\|} \cdot (\|x\|^p + \|y\|^p) \|x+y\| \\ &\leq \theta \cdot (\|x\|^p + \|y\|^p) \end{aligned}$$

Also we have

$$\begin{aligned} \left\|\phi(\frac{x}{2}, \frac{y}{2})\right\| &= \frac{\theta}{\|x\|+\|y\|} \cdot (\|x\|^p + \|y\|^p) \cdot \|x+y\| \cdot \frac{1}{2^p} \\ &= \theta \cdot (\|x\|^p + \|y\|^p) \cdot \|x+y\| \cdot \frac{1}{2^p} \\ &= \frac{1}{2^p} \|\phi(x,y)\| \end{aligned}$$

i) Let $p > 1$. If set $\alpha = \dfrac{1}{2^p}$, then $\alpha < \dfrac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha}\|\phi(x,x)\| \\ &= \frac{\frac{1}{2^p}}{1-\frac{2}{2^p}} \cdot \frac{\theta}{2\|x\|} \cdot 2\|x\|^p . \|2x\| \\ &= \frac{2\theta}{2^p - 2}\|x\|^p .\end{aligned}$$

ii) Let $0 < p < 1$. If set $\alpha = \dfrac{2^p}{4}$, then $\alpha < \dfrac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha}\|\phi(x,x)\| \\ &= \frac{\frac{2^p}{4}}{1-2\cdot\frac{2^p}{4}} \cdot \frac{\theta}{2\|x\|} \cdot 2\|x\|^p \cdot \|2x\| \\ &= \frac{2^p}{4 - 2.2^p} \cdot 2\theta\|x\|^p \\ &= \frac{2^{p-1}}{2 - 2^p} \cdot 2\theta\|x\|^p \leq \frac{2\theta}{2 - 2^p}\|x\|^p . \quad \text{(By considering } p < 1.)\end{aligned}$$

$\square$

Corollary 2.6. *Let $f : X \longrightarrow X'$ be a mapping from a normed vector space X into a Banach space X' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^p\|y\|^p$$

for all $x, y \in X$, where $\epsilon > 0$.

If $p > \dfrac{1}{2}$, then there exists $T : X \longrightarrow X'$ such that

$$T(x) = \lim_{n \longrightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$ *and* $T : X \longrightarrow X'$ *is the unique additive mapping which satisfies*

$$\|f(x) - T(x)\| \leq \frac{\epsilon}{2^{2p} - 2} \|x\|^{2p}.$$

If $0 < p < \frac{1}{2}$, *then there exists* $T : X \longrightarrow X'$ *such that*

$$T(x) = \lim_{n \longrightarrow \infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$ *and* $T : X \longrightarrow X'$ *is the unique additive mapping which satisfies*

$$\|f(x) - T(x)\| \leq \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p}.$$

Proof. We show that if set

$$\phi(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}}(x\|y\|^p + y\|x\|^p) & (x,y) \neq (0,0) \end{cases}$$

Then all conditions of corollary (2.3) are established.

Case of $(x,y) = (0,0)$ is obviously. In case of $(x,y) \neq (0,0)$ we have

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \|\phi(x,y)\| \\ &\leq \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|(x\|y\|^p + y\|x\|^p)\| \\ &\leq \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|x\|^p \|y\|^p \cdot (\|x\|^{1-p} + \|y\|^{1-p}) \\ &\leq \epsilon \|x\|^p \|y\|^p. \end{aligned}$$

Moreover

$$\begin{aligned} \left\|\phi(\frac{x}{2}, \frac{y}{2})\right\| &= \frac{\epsilon 2^{1-p}}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|(x\|y\|^p + y\|x\|^p)\| \cdot \frac{1}{2^{p+1}} \\ &= \frac{1}{2^{2p}} \cdot \frac{\epsilon}{\|x\|^{1-p} + \|y\|^{1-p}} \cdot \|(x\|y\|^p + y\|x\|^p)\| \\ &= \frac{1}{2^{2p}} \cdot \|\phi(x,y)\|. \end{aligned}$$

i) Let $p > \frac{1}{2}$. If set $\alpha = \frac{1}{2^{2p}}$, then $\alpha < \frac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha}\|\phi(x,x)\| \\
&= \frac{\frac{1}{2^{2p}}}{1-\frac{1}{2^{2p}}}\cdot\frac{\epsilon}{2\|x\|^{1-p}}\cdot 2\|x\|^{1+p} \\
&= \frac{\epsilon}{2^{2p}-2}\|x\|^{2p}.
\end{aligned}$$

ii) Let $0 < p < \frac{1}{2}$. If set $\alpha = \frac{2^{2p}}{4}$, then $\alpha < \frac{1}{2}$ and by corollary (2.3) there exists an additive mapping $T : X \longrightarrow Y$ such that

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \frac{\alpha}{1-2\alpha}\|\phi(x,x)\| \\
&= \frac{\frac{2^{2p}}{4}}{1-\frac{2^{2p}}{4}}\cdot\epsilon\|x\|^{2p} \\
&= \frac{2^{2p}}{4-2.2^{2p}}\cdot\epsilon\|x\|^{2p} \\
&= \frac{2^{2p-1}}{2-2^{2p}}\cdot\epsilon\|x\|^{2p} \\
&\leq \frac{\epsilon}{2-2^{2p}}\|x\|^{2p} \quad \text{(By considering } p < \tfrac{1}{2})
\end{aligned}$$

□

References

[1] T. Aoki, On the stability of the linear transformationin Banach spaces, *J. Math. Soc. Japan,* **2** (1950), 64-66.

[2] C. Baak, Cauchy-Rassias stability of Cauchy-Jensen additive mappings in Banach spaces, *Acta Mathematica Sience, English Series*, **Vol. 22** no. 6 (2006), 1789-1796.

[3] G. L. Forti, An existence and stability theorem for a class of functional equations, *Stochastica*, **4** (1980), 23-30.

[4] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.*, **27** (1941), 222-224.

[5] W. Jian, Some further generalizations of the Hyers-Ulam-Rassias stability of functional equations, *J. Math. Anal. Appl.*, **263** (2001), 406-423.

[6] S. M. Jung, M. S. Moslehian, and P. K. Sahoo, Stability of a generalized Jensen equation on restricted domains, *J. M. I*, **Vol. 4** no. 2 (2010), 191-206.

[7] J. M. Rassias, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal. USA*, **46** (1982), 126-130.

[8] TH. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

[9] TH. M. Rassias(ED.), *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[10] K. Ravi, R. Murali, and M. Arunkumar, The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation, *Journal of inequalities in pure and applied mathematics*, **Vol. 9**, iss. 1, art. 20, 2008.

[11] S. Sedghi, N.Shobe, K. P. R. Rao, and P. R. J. Rajendra, Extensions of Fixed Point Theorems with Respect to ω – T – Distance, *International Journal of Advances in Science and Technology* **Vol. 2** no. 6 (2011), 100-107.

[12] S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science Editions, Wiley, New York, 1964.