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A Companion for the Generalized Ostrowski and the Generalized Trapezoid Type Inequalities *

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Abstract

In this paper we will establish a companion for the generalized Ostrowski and the generalized trapezoid inequalities, including functions of bounded variation, Lipschitzian, convex and absolutely continuous functions, which generalizes Barnett et al.'s some results (N.S. Barnett et al., Math. Comput. Modeling 50 (2009) 179-187). Applications for weighted means are also given.

Keywords and Phrases: *Generalized Ostrowski type inequalities, Generalized trapezoid type inequalities, Riemann-Stieltjes integral.*

1. Introduction

In 2000, Dragomir [2] answered to the problem of approximating the Stieltjes integral $\int_a^b f(x)du(x)$ by the quantity $[u(b) - u(a)]f(x)$, which is a natural generalization of the Ostrowski problem [3] analysed in 1937. He obtained the

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following result:

$$\left|\int_a^b f(t)du(x) - [u(b)-u(a)]f(x)\right|$$
$$\leq H\left[(x-a)^r \bigvee_a^x(f) + (b-x)^r \bigvee_x^b(f)\right] \tag{1.1}$$
$$\leq H\begin{cases} [(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]; \\ [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}}, \\ \text{if } \ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(f), \end{cases}$$

for each $x \in [a,b]$, provided f is of bounded variation on $[a,b]$, while $u:[a,b]\to\mathbb{R}$ is r-H-Hölder continuous, i.e., we recall that:

$$|u(x)-u(y)| \leq H|x-y|^r \quad \text{for each } \ x,y\in[a,b].$$

From a different view point, the problem of approximating the Stieltjes integral $\int_a^b f(x)du(x)$ by the generalized trapezoid rule $[(u(b)-u(x))f(b)+(u(x)-u(a))f(a)]$ was considered by Dragomir et al. [4]. The following inequality was obtained:

$$\left|\int_a^b f(x)du(x) - [(u(b)-u(x))f(b)+(u(x)-u(a))f(a)]\right|$$
$$\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(f) \leq H(b-a)^r\bigvee_a^b(f)$$

for each $x \in [a,b]$, provided f is of bounded variation on $[a,b]$ while $u:[a,b]\to\mathbb{R}$ is r-H-Hölder continuous.

For a Riemann-Stieltjes integrable function $f:[a,b]\to\mathbb{R}$ and for a given $x\in[a,b]$, it is natural to investigate the distances between the quantities

$$f(x),\ \frac{1}{u(b)-u(a)}\int_a^b f(x)du(x) \text{ and } \frac{(u(b)-u(x))f(b)+(u(x)-u(a))f(a)}{u(b)-u(a)} \tag{1.2}$$

respectively, and to seek sharp upper bounds for these distances in terms of different measure that can be associated with f, where f is restricted to particular classes of functions including functions of bounded variation, Lipschitzian, convex and absolutely continuous functions.

The authors of [2, 4] have been given sharp upper bounds for absolute value between the first quantity and the second, the second and the third in (1.2).

The main aim of this paper is to provide sharp upper bounds for absolute value of the remaining difference between the first quantity and the third in (1.2), that is,

$$\Psi_f(x) := f(x) - \frac{(u(b)-u(x))f(b)+(u(x)-u(a))f(a)}{u(b)-u(a)}, \quad x \in [a,b]. \tag{1.3}$$

As applications, some bounds for the absolute value of the difference

$$\Phi_f(x) := \sum_{i=1}^{n} p_i f(x_i) - \frac{(u(b)-\sum_{i=1}^{n} p_i u(x_i))f(b)+(\sum_{i=1}^{n} p_i u(x_i)-u(a))f(a)}{u(b)-u(a)}, \tag{1.4}$$

where $x_i \in [a,b]$, $p_i \geq 0, i \in \{1,2,\cdots,n\}$ and $\Sigma_{i=1}^{n} p_i = 1$, are also given.

Remark Using the Stieltjes integral by Dragomir [2], generalization of the Ostrowski problem [3] was considered, so our results are natural to generalize some results obtained by Barnett et al.'s some results [1].

2. The case when f is of bounded variation and u Hölder continuous

The following representation holds.

Lemma 2.1 *Let f is of bounded function on $[a,b]$ and let $T : [a,b]^2 \to \mathbb{R}$ be given by*

$$T(x,s) := \begin{cases} u(x)-u(a), & if \;\; s \in [a,x], \\ u(x)-u(b), & if \;\; s \in [x,b]. \end{cases} \tag{2.1}$$

Then we have the following representation,

$$\Psi_f(x) = \frac{1}{u(b)-u(a)} \int_a^b T(x,s)df(s), \quad x \in [a,b], \tag{2.2}$$

where the integral is considered in the Riemann-Stieltjes sense.

Proof. If f is bounded on $[a,b]$, the for any $t \in [a,b]$, the Riemann-Stieltjes integral $\int_a^x df(s) = f(x) - f(a)$, $\int_x^b df(s) = f(b) - f(x)$. It follows that

$$\begin{aligned}\int_a^b T(x,s)df(s) &= (u(x)-u(a))\int_a^x df(s) + (u(x)-u(b))\int_x^b df(s) \\ &= (u(b)-u(a))\Psi_f(x),\end{aligned}$$

for any $t \in [a,b]$. $\square$

The following provides a sharp bound for the absolute value of Ψ_f where f is of bounded variation and u is r-H-Hölder continuous.

Theorem 2.2 *If $f : [a,b] \to \mathbb{R}$ is of bounded variation and $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous on the interval $[a,b]$, i.e.,*

$$|u(x)-u(y)| \le H|x-y|^r \quad \text{for each } x,y \in [a,b].$$

Then

$$|\Psi_f(x)| \le \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\bigvee_a^x(f) + |u(x)-u(b)|\bigvee_x^b(f)\right] \tag{2.3}$$

$$\le \frac{H}{|u(b)-u(a)|}\left[(x-a)^r\bigvee_a^x(f) + (b-x)^r\bigvee_x^b(f)\right] \tag{2.4}$$

$$\le \frac{H}{|u(b)-u(a)|}\begin{cases} [(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right]; \\ [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p + \left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}}, \\ \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \end{cases} \tag{2.5}$$

for any $x \in [a,b]$. The constant $1/2$ is also the best possible in both branches of (2.5).

Proof. Utilizing the representation (2.2), we have

$$\begin{aligned}
|\Psi_f(x)| &= \frac{1}{|u(b)-u(a)|}\left|(u(x)-u(a))\int_a^x df(s) + (u(x)-u(b))\int_x^b df(s)\right| \\
&\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\left|\int_a^x df(s)\right| + |u(x)-u(b)|\left|\int_x^b df(s)\right|\right] \\
&\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\bigvee_a^x(f) + |u(x)-u(b)|\bigvee_x^b(f)\right] \\
&\leq \frac{H}{|u(b)-u(a)|}\left[(x-a)^r\bigvee_a^x(f) + (b-x)^r\bigvee_x^b(f)\right]
\end{aligned}$$

which implies the inequalities (2.3) and (2.4).

Combination inequality (1.1) and the above inequality, we have inequality (2.5).

Now, we prove that The constant $1/2$ is also the best possible in both branches of (2.5). Consider the function $f_0(t) = |t-(a+b)/2|$ which is of bounded variation on $[a,b]$, with $f_0(a) = f_0(b) = (b-a)/2$ and $\bigvee_a^b(f_0) = b-a$. And $u_0(x) = x$ which is 1-1-Hölder continuous. According to the proof of the best possibility of the constant in Theorem 1 in [1], the sharpness of the constant $1/2$ in the inequality (2.5) is the best possible. □

As application, we give the case when f and u have some slight variations as follows.

Corollary 2.3 *If $f : [a,b] \to \mathbb{R}$ is L_1-Lipschitzian on $[a,x]$ and L_2-Lipschitzian on $[x,b]$, $L_1, L_2 > 0$, $x \in [a,b]$, while the function $u : [a,b] \to \mathbb{R}$ satisfies some local Hölder continuous, namely,*

$$|u(t)-u(a)| \leq H_1|t-a|^{r_1} \quad \text{for any } t \in [a,x] \tag{2.6}$$

and

$$|u(b)-u(t)| \leq H_2|b-t|^{r_2} \quad \text{for any } t \in [x,b] \tag{2.7}$$

where $H_1, H_2 > 0$, $r_1, r_2 \in (-1, +\infty)$, *then*

$$
\begin{aligned}
|\Psi_f(x)| &\le \frac{1}{|u(b)-u(a)|}\left[L_1|u(x)-u(a)|(x-a)+L_2|u(x)-u(b)|(b-x)\right] && (2.8)\\
&\le \frac{1}{|u(b)-u(a)|}\left[H_1L_1(x-a)^{r_1+1}+H_2L_2(b-x)^{r_2+1}\right] && (2.9)\\
&\le \frac{1}{|u(b)-u(a)|}\begin{cases}\max\{H_1L_1,H_2L_2\}\left[(x-a)^{r_1+1}+(b-x)^{r_2+1}\right];\\ \left[(H_1L_1)^p+(H_2L_2)^p\right]^{\frac{1}{p}}\left[(x-a)^{qr_1}+(b-x)^{qr_2}\right]^{\frac{1}{q}},\\ \text{if } \ p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \max\{(x-a)^{r_1+1},(b-x)^{r_2+1}\}(H_1L_1+H_2L_2),\end{cases} && (2.10)
\end{aligned}
$$

for any $x \in [a, b]$.

Proof. It is well known that if $g : [\alpha, \beta] \to \mathbb{R}$ is L-Lipschitzian, then g is of bounded variation and $\bigvee_{\alpha}^{\beta}(g) \le L(\beta - \alpha)$. Therefore, by the first inequality (2.4), we get the corresponding inequality (2.8). Using the local Hölder continuity of the function u, we have inequality (2.9) from (2.8). The other inequalities follow by the Hölder inequality and the details are omitted. $\square$

Corollary 2.4 *If* $f : [a, b] \to \mathbb{R}$ *is monotonic nondecreasing, while* $u : [a, b] \to \mathbb{R}$ *is* L*-Lipschitzian on* $[a, b]$*, where* $L > 0$*, then*

$$
\begin{aligned}
|\Psi_f(x)| &\le \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|(f(x)-f(a))+|u(x)-u(b)|(f(b)-f(x))\right]\\
&\le \frac{L}{|u(b)-u(a)|}\left[(x-a)(f(x)-f(a))+(b-x)(f(b)-f(x))\right]\\
&\le \frac{L}{|u(b)-u(a)|}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)];\\ \left[(x-a)^p+(b-x)^p\right]^{\frac{1}{p}}\left[(f(x)-f(a))^q+(f(b)-f(x))^q\right]^{\frac{1}{q}},\\ \text{if } \ p>1, \frac{1}{p}+\frac{1}{q}=1;\\ (b-a)\left[\frac{1}{2}(f(b)-f(a))+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right],\end{cases}
\end{aligned}
$$

for any $x \in [a, b]$.

Proof. It is easy to observe that we obtain Corollary 2.4 by using Theorem 2.2 and Hölder inequality, so the details are omitted. $\square$

3. The case when f is absolutely continuous and u Hölder continuous

When f is absolutely continuous, the following representation holds.

Lemma 3.1 *If f is of bounded function on $[a,b]$. Then we have the following representation,*

$$\Psi_f(x) = \frac{1}{u(b)-u(a)} \int_a^b T(x,s) f'(s) ds, \quad x \in [a,b], \tag{3.1}$$

where the integral is considered in the Lebesgue sense and where the kernel $T : [a,b]^2 \to \mathbb{R}$ has been defined in (2.1).

We cite the following Lebesgue norms defined in Section 3 in [1] as follows.

$$\| f' \|_{[a,b],\infty} := ess \sup_{x\in[a,b]} |f'(x)|, \quad \| f' \|_{[a,b],p} := \left(\int_a^b |f'(x)|^p dx \right)^{\frac{1}{p}}, \ p \geq 1.$$

Theorem 3.2 *If f is absolutely continuous on $[a,b]$, $u : [a,b] \to \mathbb{R}$ is r-H-Hölder continuous on $[a,b]$, where $H > 0$ and $r \in (-1, \infty)$. Then we have the following inequalities:*

$$|\Psi_f(x)| \leq \frac{1}{|u(b)-u(a)|} \left[|u(x)-u(a)| \, \| f' \|_{[a,x],1} + |u(b)-u(x)| \, \| f' \|_{[x,b],1} \right] \tag{3.2}$$

$$\leq \frac{H}{|u(b)-u(a)|} \left[(x-a)^r \, \| f' \|_{[a,x],1} + (b-x)^r \, \| f' \|_{[x,b],1} \right] \tag{3.3}$$

$$\leq \frac{H}{|u(b)-u(a)|} W(x), \quad x \in [a,b], \tag{3.4}$$

where $W(x)$ is defined by

$$W(x) := \begin{cases} (x-a)^{r+1} \, \| f' \|_{[a,x],\infty}, & if \ \ f' \in L_\infty[a,b] \\ (x-a)^{r+\frac{1}{q}} \, \| f' \|_{[a,x],p}, & if \ \ f' \in L_p[a,b], \ \ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$
$$+ \begin{cases} (b-x)^{r+1} \, \| f' \|_{[x,b],\infty}, & if \ \ f' \in L_\infty[a,b], \\ (b-x)^{r+\frac{1}{\beta}} \, \| f' \|_{[x,b],\alpha}, & if \ \ f' \in L_\alpha[a,b], \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and $W(x)$ should be seen as all four possible combinations.

Proof. By Lemma 3.1, we have

$$\begin{aligned}
|\Psi_f(x)| &= \left|\frac{1}{u(b)-u(a)}\left[(u(x)-u(a))(f(x)-f(a))+(u(b)-u(x))(f(b)-f(x))\right]\right| \\
&= \left|\frac{1}{u(b)-u(a)}\left[(u(x)-u(a))\int_a^x f'(s)ds+(u(b)-u(x))\int_x^b f'(s)ds\right]\right| \\
&\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)|\int_a^x |f'(s)|ds+|u(b)-u(x)|\int_x^b |f'(s)|ds\right] \\
&\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)| \parallel f' \parallel_{[a,x],1} +|u(b)-u(x)| \parallel f' \parallel_{[x,b],1}\right] \\
&\leq \frac{H}{|u(b)-u(a)|}\left[(x-a)^r \parallel f' \parallel_{[a,x],1} +(b-x)^r \parallel f' \parallel_{[x,b],1}\right],
\end{aligned}$$

for $x \in [a,b]$, which implies inequalities (3.2) and (3.3).

Utilizing (3.4) and (3.5) in [1] and the above inequality, we obtain the desired inequality (3.4). □

Corollary 3.3 *If f is absolutely continuous on $[a,b]$, the function $u:[a,b]\to\mathbb{R}$ satisfies some local Hölder continuous defined by (2.6) and (2.7). Then we have*

$$\begin{aligned}
|\Psi_f(x)| &\leq \frac{1}{|u(b)-u(a)|}\left[|u(x)-u(a)| \parallel f' \parallel_{[a,x],1} +|u(b)-u(x)| \parallel f' \parallel_{[x,b],1}\right] \\
&\leq \frac{1}{|u(b)-u(a)|}\left[H_1(x-a)^{r_1} \parallel f' \parallel_{[a,x],1} +H_2(b-x)^{r_2} \parallel f' \parallel_{[x,b],1}\right] \\
&\leq \frac{1}{|u(b)-u(a)|}W(x), \quad x\in[a,b],
\end{aligned}$$

where $W(x)$ is defined by

$$\begin{aligned}
W(x) &:= \begin{cases} H_1(x-a)^{r_1+1} \parallel f' \parallel_{[a,x],\infty}, & if\ \ f'\in L_\infty[a,b] \\ H_1(x-a)^{r_1+\frac{1}{q}} \parallel f' \parallel_{[a,x],p}, & if\ \ f'\in L_p[a,b],\ \ p>1, \frac{1}{p}+\frac{1}{q}=1; \end{cases} \\
&+\begin{cases} H_2(b-x)^{r_2+1} \parallel f' \parallel_{[x,b],\infty}, & if\ \ f'\in L_\infty[a,b], \\ H_2(b-x)^{r_2+\frac{1}{\beta}} \parallel f' \parallel_{[x,b],\alpha}, & if\ \ f'\in L_\alpha[a,b],\ \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \end{cases}
\end{aligned}$$

and $W(x)$ should be seen as all four possible combinations.

Proof. It is similar to the proof of Theorem 3.2, so the details are omitted. □

4. The case when f is convex and u monotonic nondecreasing and bi-Hölder

Before giving the case when f is convex and u is monotonic nondecreasing and bi-Hölder, we establish sharp lower and upper bounds for the remaining differences as follows:

$$\Omega_1(x) := \int_a^b f(x)du(x) - (u(b) - u(a))f(x) \tag{4.1}$$

and

$$\Omega_2(x) := [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] - \int_a^b f(x)du(x). \tag{4.2}$$

Theorem 4.1 *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ with $f'_-(b)$ and $f'_+(a)$ finite, and $u : [a, b] \to \mathbb{R}$ is monotonic nondecreasing and bi-Hölder function on $[a, b]$, that is,*

$$\mathscr{L}_1(y - x)^r \leq u(y) - u(x) \leq \mathscr{L}_2(y - x)^r, \quad \text{for} \quad x \leq y, \quad x, y \in [a, b], \tag{4.3}$$

where $\mathscr{L}_1, \mathscr{L}_2 > 0$ and $r > -1$. Then we have the following inequalities:

$$\begin{aligned} &\frac{1}{r+1}[\mathscr{L}_1(b-x)^{r+1}f'_+(x) - \mathscr{L}_2(x-a)^{r+1}f'_-(x)] \\ &\qquad \leq \Omega_1(x) \leq \frac{1}{r+1}[\mathscr{L}_2(b-x)^{r+1}f'_-(b) - \mathscr{L}_1(x-a)^{r+1}f'_+(a)] \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\frac{1}{r+1}[\mathscr{L}_1(b-x)^{r+1}f'_+(x) - \mathscr{L}_2(x-a)^{r+1}f'_-(x)] \\ &\qquad \leq \Omega_2(x) \leq \frac{1}{r+1}[\mathscr{L}_2(b-x)^{r+1}f'_-(b) - \mathscr{L}_1(x-a)^{r+1}f'_+(a)], \end{aligned} \tag{4.5}$$

where $\Omega_1(x)$ and $\Omega_2(x)$ are defined by (4.1) and (4.2). The constant $1/(r+1)$ is sharp in both inequalities.

Proof. First of all, we give the proof of inequality (4.4). It is easy to see that for any locally absolutely continuous function $f : [a, b] \to \mathbb{R}$, we have the identity

$$\int_a^x (u(t) - u(a))f'(t)dt + \int_x^b (u(t) - u(b))f'(t)dt = (u(b) - u(a))f(x) - \int_a^b f(t)du(t) \tag{4.6}$$

for any $x \in (a, b)$, where f' is the derivation of f which exists a.e. on (a, b).

Since f is convex, then it is locally Lipschitzian and thus (4.6) holds. Moreover, for any $x \in (a, b)$, we have the inequalities

$$f'(t) \le f'_-(x) \quad \text{for a.e.} \quad t \in [a, x] \tag{4.7}$$

and

$$f'(t) \ge f'_+(x) \quad \text{for a.e.} \quad t \in [x, b]. \tag{4.8}$$

If we multiply (4.7) by $u(t) - u(a) \ge 0$, $t \in [a, x]$ and integrate on $[a, x]$, by (4.3), we get

$$\int_a^x (u(t) - u(a))f'(t)dt \le f'_-(x) \int_a^x (u(t) - u(a))dt \le \frac{1}{r+1} \mathscr{L}_2 (x-a)^{r+1} f'_-(x) \tag{4.9}$$

and if we multiply (4.8) by $u(b) - u(x) \ge 0$, $t \in [x, b]$ and integrate on $[x, b]$, by (4.3), we get

$$\int_x^b (u(b) - u(t))f'(t)dt \ge f'_+(x) \int_x^b (u(b) - u(t))dt \ge \frac{1}{r+1} \mathscr{L}_1 (b-x)^{r+1} f'_+(x). \tag{4.10}$$

If we subtract (4.10) from (4.9) and use the representation (4.6), we deduce the first inequality in (4.4).

Since f is convex, then we have the inequalities

$$f'(t) \ge f'_+(a) \quad \text{for a.e.} \quad t \in [a, x] \tag{4.11}$$

and

$$f'(t) \le f'_-(b) \quad \text{for a.e.} \quad t \in [x, b]. \tag{4.12}$$

If we multiply (4.11) by $u(t)-u(a)\ge 0$, $t\in[a,x]$ and integrate on $[a,x]$, by (4.3), we get

$$\int_a^x (u(t)-u(a))f'(t)dt \ge f'_+(a)\int_a^x (u(t)-u(a))dt \ge \frac{1}{r+1}\mathscr{L}_1(x-a)^{r+1}f'_+(a) \quad (4.13)$$

and if we multiply (4.12) by $u(b)-u(x)\ge 0$, $t\in[x,b]$, integrate on $[x,b]$ and integrate on $[a,x]$, by (4.3), we get

$$\int_x^b (u(b)-u(t))f'(t)dt \le f'_-(b)\int_x^b (u(b)-u(t))dt \le \frac{1}{r+1}\mathscr{L}_2(b-x)^{r+1}f'_-(b). \quad (4.14)$$

If we subtract (4.14) from (4.13) and use the representation (4.6), we deduce the second inequality in (4.4).

Now we prove that the constant $1/(r+1)$ is also the best possible in inequalities (4.4). Consider the function $f_0(t)=k\,|t-(a+b)/2|$ which is a convex function on the interval $[a,b]$, where $k>0$, $t\in[a,b]$. Then

$$f'_{0-}\left(\frac{a+b}{2}\right)=-k,\quad f'_{0+}\left(\frac{a+b}{2}\right)=k \quad\text{and}\quad f_0\left(\frac{a+b}{2}\right)=0.$$

And $u_0(x)=x$, then $\mathscr{L}_1=\mathscr{L}_2=r=1$. Thus we have $\int_a^b f_0(t)dt=k(b-a)^2/2$. If in (4.4) we choose $f=f_0$, $u=u_0$ and $x=(a+b)/2$. According to the proof of the best possibility of the constant in Lemma 2.1 in [5], the sharpness of the constant $1/(r+1)$ in the inequality (4.4) is the best possible.

Secondly, we give the proof of inequality (4.5). It is easy to see that for any locally absolutely continuous function $f:[a,b]\to\mathbb{R}$, we have the identity

$$\begin{aligned}\int_a^b (u(t)-u(x))f'(t)dt &= (u(b)-u(x))f(b)\\ &\quad +(u(x)-u(a))f(a)-\int_a^b f(t)du(t) \quad (4.15)\end{aligned}$$

for any $x\in(a,b)$, where f' is the derivation of f which exists a.e. on (a,b).

Since f is convex, then it is locally Lipschitzian and thus (4.15) holds. The following proof is similar to the proof of inequalities (4.4) and Lemma 2.1 in [6], so the details are omitted. □

In the following we give sharp lower and upper bounds for the remaining difference (1.4) when f is convex and u is monotonic nondecreasing and bi-Hölder.

Theorem 4.2 *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ with $f'_-(b)$ and $f'_+(a)$ finite, and $u : [a, b] \to \mathbb{R}$ is monotonic nondecreasing and bi-Hölder function on $[a, b]$ defined by (4.3). Then we have the following inequalities:*

$$\frac{1}{u(b)-u(a)}[\mathscr{L}_1(x-a)^{r+1}f'_+(a) - \mathscr{L}_2(b-x)^{r+1}f'_-(b)] \leq \Psi_f(x) \leq \frac{1}{u(b)-u(a)}[\mathscr{L}_2(x-a)^{r+1}f'_-(x) - \mathscr{L}_1(b-x)^{r+1}f'_+(x)] \quad (4.16)$$

where $\Psi_f(x)$ is defined by (1.3). The constant 1 is the best possible on both sides of (4.16).

Proof. From Lemma 2.1,

$$(u(b)-u(a))\Phi_f(x) = (u(x)-u(a))(f(x)-f(a)) - (u(b)-u(x))(f(b)-f(x)), \quad x \in [a, b]. \quad (4.17)$$

Let $x \in (a, b)$, then, by the convexity of f, we have

$$(x-a)f'_-(x) \geq f(x) - f(a) \geq (x-a)f'_+(a) \quad (4.18)$$

and

$$(b-x)f'_-(b) \geq f(b) - f(x) \geq (b-x)f'_+(x). \quad (4.19)$$

If we multiply (4.18) by $u(x) - u(a) > 0$ and (4.19) by $u(b) - u(x) > 0$, we obtain

$$(u(x)-u(a))(x-a)f'_-(x) \geq (u(x)-u(a))(f(x)-f(a)) \geq (u(x)-u(a))(x-a)f'_+(a) \quad (4.20)$$

and

$$(u(b)-u(x))(b-x)f'_-(b) \geq (u(b)-u(x))(f(b)-f(x)) \geq (u(b)-u(x))(b-x)f'_+(x). \quad (4.21)$$

By (4.3), the above inequalities can rewrite

$$\mathscr{L}_2(x-a)^{r+1}f'_-(x) \geq (u(x)-u(a))(f(x)-f(a)) \geq \mathscr{L}_1(x-a)^{r+1}f'_+(a) \quad (4.22)$$

and

$$-\mathscr{L}_1(b-x)^{r+1}f'_+(x) \geq -(u(b)-u(x))(f(b)-f(x)) \geq -\mathscr{L}_2(b-x)^{r+1}f'_-(b). \quad (4.23)$$

Finally, on adding (4.22) to (4.23), we deduce the desired result (4.16).

Now we prove that The constant 1 is also the best possible in inequalities (4.16). Consider the function $f_0(t) = k\,|t-(a+b)/2|$ which is a convex function on $[a,b]$, where $k>0$, $t \in [a,b]$. Then

$$f'_{0-}(b) = -k, \quad f'_{0+}(a) = k, \quad f'_{0-}\left(\frac{a+b}{2}\right) = -k, \quad f'_{0+}\left(\frac{a+b}{2}\right) = k,$$

$$f_0\left(\frac{a+b}{2}\right) = 0 \ \text{ and } \ f_0(a) = f_0(b) = \frac{k(b-a)}{2}.$$

And $u_0(x) = x$, then $\mathscr{L}_1 = \mathscr{L}_2 = r = 1$. If in (4.4) we choose $f = f_0$, $u = u_0$ and $x = (a+b)/2$. According to the proof of the best possibility of the constant in Theorem 3 in [1], the sharpness of the constant 1 in the inequality (4.16) is the best possible. □

5. Some applications

As applications, some bounds for the absolute value of the difference (1.4).

Proposition 5.1 *If $f:[a,b]\to\mathbb{R}$ is of bounded variation and $u:[a,b]\to\mathbb{R}$ is r-H-Hölder continuous on $[a,b]$. Then*

$$|\Phi_f(x)| \leq \frac{H}{|u(b)-u(a)|}\left[\frac{1}{2}(b-a) + \sum_{i=1}^{n} p_i \left|x_i - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(f), \quad (5.1)$$

where $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$, the constant $1/2$ is also the best possible in both branches of (5.1).

Proof. We use the third inequality in (2.5) to state:

$$\left|f(x_i) - \frac{(u(b)-u(x_i))f(b) + (u(x_i)-u(a))f(a)}{u(b)-u(a)}\right| \leq \frac{H}{|u(b)-u(a)|}\left[\frac{1}{2}(b-a) + \left|x_i - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(f) \quad (5.2)$$

for $i = 1, 2, \cdots, n$.

If we multiply (5.2) by $p_i \geq 0$, sum over $i = 1$ to n, we deduce the desired result (5.1).

The fact that 1/2 is the best possible follows from the fact that it is the best possible for $n = 1$. $\square$

In a similar manner, on utilizing the third inequality in (2.10), we can state the following result:

Proposition 5.2 *If $f : [a, b] \to \mathbb{R}$ is L-Lipschitzian on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ is r-H-Hölder continuous on $[a, b]$. Then*

$$|\Phi_f(x)| \leq \frac{HL(b-a)}{|u(b)-u(a)|}\left[\frac{1}{2}(b-a) + \sum_{i=1}^{n} p_i \left|x_i - \frac{a+b}{2}\right|\right]^r,$$

where $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$.

Finally, on utilizing the inequality in (3.3), we can also state that:

Proposition 5.3 *If f is absolutely continuous on $[a, b]$, $u : [a, b] \to \mathbb{R}$ is r-H-Hölder continuous on $[a, b]$. Then*

$$|\Phi_f(x)| \leq \frac{H}{|u(b)-u(a)|}\left[\sum_{i=1}^{n}(x_i - a)^r \parallel f' \parallel_{[a,x_i],1} + \sum_{i=1}^{n}(b - x_i)^r \parallel f' \parallel_{[x_i,b],1}\right],$$

where $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$.

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