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Some Sandwich Type Theorems for Analytic Functions Involving the Dziok-Srivastava Operator and Other Related Linear Operators *

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Abstract

In this paper, we obtain some applications of first order differential subordination and superordination results defined by Dzoik-Srivastava operator and other linear operators for certain normalized analytic functions.

Keywords and Phrases: *Analytic function, Hadamard product, Differential subordination, Superordination, Dzoik-Srivastava operator.*

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1. Introduction

Let $H(U)$ be the class of analytic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} ... \quad (a \in \mathbb{C}). \tag{1.1}$$

Also, let $\mathcal{A}$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + a_2 z^2 + ..., \tag{1.2}$$

and Let S^* denote the starlike subclass of $\mathcal{A}$.If f, $g \in H(U)$, we say that f is subordinate to g or f is superordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such that $f(z) = g(\omega(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[3], [13] and [14]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential superordination (1.4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant.

For complex parameters $a_1, ..., a_q; b_1, ..., b_s$ $(bj \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\};\ j = 1, ..., s\ )$, we define the generalized hypergeometric function ${}_qF_s\,(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z)$ by (see [18])the following infinite series:

$$ {}_qF_s\,(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_q)_n}{(b_1)_n \dots (b_s)_n} \frac{z^n}{n!} \tag{1.5} $$

$$ (q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\,; z \in U)\,, $$

where $(x)_n$ is the Pochhammer symbol (or the shift factorial) defined, in terms of the Gamma function Γ, by

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n=0), \\ \quad x\,(x+1)\dots(x+n-1) & (n \in \mathbb{N}). \end{cases} $$

Dziok and Srivastava [7] (see also [8]) considered a linear operator

$$ H(a_1, ..., a_q; b_1, ..., b_s) : \mathcal{A} \to \mathcal{A} $$

defined by the following Hadamard product:

$$ H(a_1, ..., a_q; b_1, ..., b_s)f\,(z) = h\,(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z) * f\,(z)\,, \tag{1.6} $$

where

$$ h\,(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z) = z\ {}_qF_s\,(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z) \tag{1.7} $$

$$ (q \leq s+1; q, s \in \mathbb{N}_0; z \in U)\,. $$

if $f\,(z) \in \mathcal{A}$ is given by (1.2) ,then we have

$$ H(a_1, ..., a_q; b_1, ..., b_s)f(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \dots (a_q)_{n-1}}{(b_1)_{n-1} \dots (b_s)_{n-1}\,(1)_{n-1}} a_{\ n} z^{\ n}, \tag{1.8} $$

If, for convenience, we write

$$ H_{q,s}\,[a_1; b_1] = H(a_1, ..., a_q; b_1, ..., b_s), $$

then one can easily verify from the definition (1.6) *or* (1.8) that

$$ z\,(H_{q,s}\,[a_1; b_1]\,f(z))' = a_1 H_{q,s}\,[a_1+1; b_1]\,f(z) - (a_1-1) H_{q,s}\,[a_1; b_1]\,f(z), \tag{1.9} $$

and

$$z\left(H_{q,s}\left[a_1;b_1+1\right]f(z)\right)' = b_1 H_{q,s}\left[a_1;b_1\right]f(z) - (b_1-1)H_{q,s}\left[a_1;b_1+1\right]f(z). \tag{1.10}$$

It should be remarked that the linear operator $H_{q,s}\left[a_1;b_1\right]$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$, we have

(*i*) $H_{2,1}(a,b;c)f(z) = \left(I_c^{a,b}f\right)(z)\left(a,b\in\mathbb{C};c\notin\mathbb{Z}_0^-\right)$, where the linear operator $I_c^{a,b}$ was investigated by Hohlov [9];

(*ii*) $H_{2,1}(\delta+1,1;1)f(z) = D^\delta f(z)(\delta > -1)$, where D^δ is the Ruscheweyh derivative of $f(z)$ (see [16]);

(*iii*) $H_{2,1}(\mu+1,1;\mu+2)f(z) = \mathcal{F}_\mu(f)(z) = \frac{\mu+1}{z^{\ \mu}}\int_0^z t^{mu-1}f(t)dt$ $(\mu > -1)$, where $\mathcal{F}_\mu$ is the Libera integral operator (see [11] and [1]);

(*iv*) $H_{2,1}(a,1;c)f(z) = L(a,c)f(z)(a\in\mathbb{R};c\in\mathbb{R}\backslash\mathbb{Z}_0^-)$, where $L(a,c)$ is the Carlson-Shaffer operator (see [4]);

(*vi*) $H_{2,1}(\lambda+1,c;a)f(z) = I^\lambda(a,c)f(z)(a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-;\lambda>-1)$, where $I^\lambda(a,c)f(z)$ is the Cho–Kwon–Srivastava operator (see [5]);

(*vii*) $H_{2,1}(\mu,1;\lambda+1)f(z) = I_{\lambda,\mu}f(z)(\lambda > -1;\mu > 0)$, where $I_{\lambda,\mu}f(z)$ is the Choi–Saigo–Srivastava operator [6] which is closely related to the Carlson–Shaffer [4] operator $L(\mu,\lambda+1)f(z)$.

(*vii*) $H_{2,1}(1,1;n+1)f(z) = I_nf(z)(n\in\mathbb{N}_0)$, where $I_nf(z)$ is Noor operator of $n-th$ order (see [15]).

Definition 1. The function $f\in\mathcal{A}$ belongs to the class $S_{q.s}^*\left(a_1;b_1\right)$ if and only if $H_{q,s}\left[a_1;b_1\right]f(z)\in S^*$ for $z\in U$.

Definition 2. The function $f\in\mathcal{A}$ belongs to the class $C_{q.s}\left(a_1;b_1\right)$ if and only if there exists $g\in S_{q.s}^*\left(a_1;b_1\right)$ such that

$$\Re\left\{\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\right\} > 0 \quad (z\in U).$$

In this paper, we obtain sufficient conditions for normalized analytic functions f, g satisfy

$$q_1(z) \prec \frac{z\left(H_{q,s}[a_1; b_1] f(z)\right)'}{H_{q,s}[a_1; b_1] g(z)} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U.

2. Definitions and Preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 3. [14]. Denote by Q, the set of all functions f that are analytic and injective on $U \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1 [14]. *Let $q(z)$ be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \tag{2.1}$$

Suppose that

(i) $\psi(z)$ is starlike univalent in U,

(ii) $\Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ for $z \in U$.

If $p(z)$ is analytic with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1, Shanmugam et al. [17] obtained the following lemma.

Lemma 2 [17]. *Let $q(z)$ be univalent in U with $q(0)=1$. Let $\alpha \in \mathbb{C}$; $\gamma \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, further assume that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{\alpha}{\gamma}\right)\right\}. \tag{2.3}$$

If $p(z)$ is analytic in U, and

$$\alpha p(z) + \gamma z p'(z) \prec \alpha q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 [2]. *Let $q(z)$ be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\Re\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,

(ii) $\Psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in H[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.4}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Taking $\vartheta(w) = \alpha w$ and $\phi(w) = \gamma$ in Lemma 3, Shanmugam et al. [17] obtained the following lemma.

Lemma 4 [17]. *Let $q(z)$ be convex univalent in U, $q(0)=1$. Let $\alpha \in \mathbb{C}$; $\gamma \in \mathbb{C}^*$ and $\Re\left(\frac{\alpha}{\gamma}\right) > 0$. If $p(z) \in H[q(0),1] \cap Q$, $\alpha p(z) + \gamma z p'(z)$ is univalent in U and*

$$\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z),$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Sandwich Results

Theorem 1. *Let $q(z)$ be univalent in U with $q(0)=1$, and $\gamma \in \mathbb{C}^*$. Further assume that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{3.1}$$

If $f, g \in \mathcal{A}$, $H_{q,s}[a_1; b_1]\, g(z) \neq 0$, *satisfy the following subordination condition:*

$$\frac{z(H_{q,s}[a_1;b_1]f(z))'}{H_{q,s}[a_1;b_1]g(z)} \left\{1+\gamma\left[1+\frac{z(H_{q,s}[a_1;b_1]f(z))''}{(H_{q,s}[a_1;b_1]f(z))'} - \frac{z(H_{q,s}[a_1;b_1]g(z))'}{H_{q,s}[a_1;b_1]g(z)}\right]\right\} \prec q(z) + \gamma z q'(z), \tag{3.2}$$

then

$$\frac{z\left(H_{q,s}[a_1; b_1]\, f(z)\right)'}{H_{q,s}[a_1; b_1]\, g(z)} \prec q(z) \tag{3.3}$$

and $q(z)$ *is the best dominant.*

Proof. Define a function $p(z)$ by

$$p(z) = \frac{z\left(H_{q,s}[a_1; b_1]\, f(z)\right)'}{H_{q,s}[a_1; b_1]\, g(z)} \qquad (z \in U). \tag{3.4}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, differentiating (3.4) logarithmically with respect to z and using the the subordination condition (3.2), we get

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).$$

Therefore, the assertion (3.3) of Theorem 1 now follows by an application of Lemma 2. □

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 1, we have the following corollary.

Corollary 1. *Let* $\gamma \in \mathbb{C}^*$ *and*

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f, g \in \mathcal{A}$, $H_{q,s}[a_1; b_1]\, g(z) \neq 0$, *satisfy the following subordination condition:*

$$\frac{z(H_{q,s}[a_1;b_1]f(z))'}{H_{q,s}[a_1;b_1]g(z)} \left\{1+\gamma\left[1+\frac{z\left(H_{q,s}[a_1;b_1]\, f(z)\right)''}{\left(H_{q,s}[a_1;b_1]\, f(z)\right)'} - \frac{z\left(H_{q,s}[a_1;b_1]\, g(z)\right)'}{H_{q,s}[a_1;b_1]\, g(z)}\right]\right\} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)\,z}{(1+Bz)^2},$$

then

$$\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Taking $A=1$, $B=-1$ and $g \in S^*_{q.s}\left(a_1;b_1\right)$ in Corollary 1, we obtain

Corollary 2. *Let* $\gamma \in \mathbb{C}^*$ *with* $\Re\left(\bar{\gamma}\right) > 0$. *If* $g \in \mathcal{A}$ *such that* $g \in S^*_{q.s}\left(a_1;b_1\right)$, *and* $f, g \in \mathcal{A}$ *satisfy the following subordination condition:*

$$\begin{aligned} &\tfrac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)} \quad \left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)''}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'} - \frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\} \\ &\prec \qquad \frac{1+z}{1-z}+\gamma\frac{z}{\left(1-z\right)^2}, \end{aligned}$$

then $f(z) \in C_{q,s}\left(a_1;b_1\right)$ *and this result best possible.*

For $q=2, s=1, a_1=a$ and $b_1=c$ $\left(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-\right)$ in Theorem 1, we have the following subordination for Carlson-Shaffer operator.

Corollary 3. *Let* $q\left(z\right)$ *be univalent in* U *with* $q(0)=1$, *and* $\gamma \in \mathbb{C}^*$. *Further assume that* (3.1) *holds. If* $f, g \in \mathcal{A}$, $L\left(a,c\right)g(z) \neq 0$, *satisfy the following subordination condition:*

$$\frac{z\left(L\left(a,c\right)f(z)\right)'}{L\left(a,c\right)g(z)}\left\{1+\gamma\left[1+\frac{z\left(L\left(a,c\right)f(z)\right)''}{\left(L\left(a,c\right)f(z)\right)'} - \frac{z\left(L\left(a,c\right)g(z)\right)'}{L\left(a,c\right)g(z)}\right]\right\}$$

$$\prec q\left(z\right)+\gamma z q'\left(z\right),$$

then

$$\frac{z\left(L\left(a,c\right)f(z)\right)'}{L\left(a,c\right)g(z)} \prec q\left(z\right)$$

and $q\left(z\right)$ *is the best dominant.*

For $q=2, s=1, a_1=\lambda+1, a_2=c$ and $b_1=a$ $\left(a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > -1\right)$ in Theorem 1, we obtain the following subordination for Cho-Kwon-Srivastava operator.

Corollary 4. *Let $q(z)$ be univalent in U with $q(0)=1$, and $\gamma \in \mathbb{C}^*$. Further assume that (3.1) holds. If $f, g \in \mathcal{A}$, $I^{\lambda}(a,c)\,g(z) \neq 0$, satisfy the following subordination condition:*

$$\frac{z\left(I^{\lambda}(a,c)\,f(z)\right)'}{I^{\lambda}(a,c)\,g(z)}\left\{1+\gamma\left[1+\frac{z\left(I^{\lambda}(a,c)\,f(z)\right)''}{\left(I^{\lambda}(a,c)\,f(z)\right)'}-\frac{z\left(I^{\lambda}(a,c)\,g(z)\right)'}{I^{\lambda}(a,c)\,g(z)}\right]\right\}$$

$$\prec q(z)+\gamma z q'(z),$$

then

$$\frac{z\left(I^{\lambda}(a,c)\,f(z)\right)'}{I^{\lambda}(a,c)\,g(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

For $q=2, s=1, a_1=\mu$,$a_2=1$ and $b_1=\lambda+1$ $(\lambda>-1; \mu>0)$ in Theorem 1, we have the following subordination for Choi-Saigo-Srivastava operator.

Corollary 5. *Let $q(z)$ be univalent in U with $q(0)=1$, and $\gamma \in \mathbb{C}^*$. Further assume that (3.1) holds. If $f, g \in \mathcal{A}$, $I_{\lambda,\mu}g(z) \neq 0$, satisfy the following subordination condition:*

$$\frac{z\left(I_{\lambda,\mu}f(z)\right)'}{I_{\lambda,\mu}g(z)}\left\{1+\gamma\left[1+\frac{z\left(I_{\lambda,\mu}f(z)\right)''}{\left(I_{\lambda,\mu}f(z)\right)'}-\frac{z\left(I_{\lambda,\mu}g(z)\right)'}{I_{\lambda,\mu}g(z)}\right]\right\} \prec q(z)+\gamma z q'(z),$$

then

$$\frac{z\left(I_{\lambda,\mu}f(z)\right)'}{I_{\lambda,\mu}g(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Remark 1. Taking $q=2, s=1, a_1=a_2=1$ and $b_1=n+1$ $(n \in \mathbb{N}_0)$ in Theorem 1, we obtain the subordination result of Ibrahim and Darus [10,Theorem 2] for the Noor operator.

Now, by appealing to Lemma 4 it can be easily prove the following theorem.

Theorem 2. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma \in \mathbb{C}$*

with $\Re(\bar{\gamma}) > 0$. *If* $f, g \in \mathcal{A}$ *such that* $\frac{z(H_{q,s}[a_1;b_1]f(z))'}{H_{q,s}[a_1;b_1]g(z)} \in H[1,1] \cap Q$,

$$\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)''}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}-\frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\}$$

is univalent in U, *and the following superordination condition*

$$q(z)+\gamma z q'(z)$$

$$\prec \frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)''}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}-\frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\}$$

holds, then

$$q(z) \prec \frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}$$

and $q(z)$ *is the best subordinant.*

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we have the following corollary.

Corollary 6. *Let* $\gamma \in \mathbb{C}$ *with* $\Re(\bar{\gamma}) > 0$. *If* $f, g \in \mathcal{A}$ *such that*

$$\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)} \in H[1,1] \cap Q,$$

$$\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)''}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}-\frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\}$$

is univalent in U, *and the following superordination condition*

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}$$

$\prec \frac{z(H_{q,s}[a_1;b_1]f(z))'}{H_{q,s}[a_1;b_1]g(z)}\left\{1+\gamma\left[1+\frac{z(H_{q,s}[a_1;b_1]f(z))''}{(H_{q,s}[a_1;b_1]f(z))'}-\frac{z(H_{q,s}[a_1;b_1]g(z))'}{H_{q,s}[a_1;b_1]g(z)}\right]\right\}$ *holds, then*

$$\frac{1+Az}{1+Bz} \prec \frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)'}{H_{q,s}\left[a_1;b_1\right]g(z)}$$

and $q(z)$ *is the best subordinant.*

For $q = 2, s = 1, a_1 = a$ and $b_1 = c$ $\left(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-\right)$ in Theorem 1, we have the following superordination result for Carlson-Shaffer operator.

Corollary 7. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z(L(a,c)f(z))'}{L(a,c)g(z)} \in H[1,1] \cap Q$,*

$$\frac{z\left(L(a,c)\,f(z)\right)'}{L(a,c)\,g(z)}\left\{1+\gamma\left[1+\frac{z\left(L(a,c)\,f(z)\right)''}{\left(L(a,c)\,f(z)\right)'}-\frac{z\left(L(a,c)\,g(z)\right)'}{L(a,c)\,g(z)}\right]\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma z q'(z)$$
$$\prec \frac{z\left(L(a,c)\,f(z)\right)'}{L(a,c)\,g(z)}\left\{1+\gamma\left[1+\frac{z\left(L(a,c)\,f(z)\right)''}{\left(L(a,c)\,f(z)\right)'}-\frac{z\left(L(a,c)\,g(z)\right)'}{L(a,c)\,g(z)}\right]\right\}$$

holds,
then

$$q(z) \prec \frac{z\left(L(a,c)\,f(z)\right)'}{L(a,c)\,g(z)}$$

and $q(z)$ is the best subordinant.

For $q=2, s=1, a_1=\lambda+1, a_2=c$ and $b_1=a$ $\left(a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda>-1\right)$ in Theorem 2, we obtain the following superordination result for Cho-Kwon-Srivastava operator.

Corollary 8. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z\left(I^\lambda(a,c)f(z)\right)'}{I^\lambda(a,c)g(z)} \in H[1,1] \cap Q$,*

$$\frac{z\left(I^\lambda(a,c)\,f(z)\right)'}{I^\lambda(a,c)\,g(z)}\left\{1+\gamma\left[1+\frac{z\left(I^\lambda(a,c)\,f(z)\right)''}{\left(I^\lambda(a,c)\,f(z)\right)'}-\frac{z\left(I^\lambda(a,c)\,g(z)\right)'}{I^\lambda(a,c)\,g(z)}\right]\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma z q'(z)$$
$$\prec \frac{z\left(I^\lambda(a,c)\,f(z)\right)'}{I^\lambda(a,c)\,g(z)}\left\{1+\gamma\left[1+\frac{z\left(I^\lambda(a,c)\,f(z)\right)''}{\left(I^\lambda(a,c)\,f(z)\right)'}-\frac{z\left(I^\lambda(a,c)\,g(z)\right)'}{I^\lambda(a,c)\,g(z)}\right]\right\}$$

holds, then

$$q(z) \prec \frac{z\left(I^{\lambda}(a,c)\, f(z)\right)'}{I^{\lambda}(a,c)\, g(z)}$$

and $q(z)$ is the best subordinant.

For $q=2, s=1, a_1=\mu$,$a_2=1$ and $b_1=\lambda+1$ $(\lambda>-1; \mu>0)$ in Theorem 2, we have the following superordination result for Choi-Saigo-Srivastava operator.

Corollary 9. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z\left(I_{\lambda,\mu} f(z)\right)'}{I_{\lambda,\mu} g(z)} \in H[1,1] \cap Q$,*

$$\frac{z\left(I_{\lambda,\mu} f(z)\right)'}{I_{\lambda,\mu} g(z)}\left\{1+\gamma\left[1+\frac{z\left(I_{\lambda,\mu} f(z)\right)''}{\left(I_{\lambda,\mu} f(z)\right)'}-\frac{z\left(I_{\lambda,\mu} g(z)\right)'}{I_{\lambda,\mu} g(z)}\right]\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma z q'(z) \prec \frac{z\left(I_{\lambda,\mu} f(z)\right)'}{I_{\lambda,\mu} g(z)}\left\{1+\gamma\left[1+\frac{z\left(I_{\lambda,\mu} f(z)\right)''}{\left(I_{\lambda,\mu} f(z)\right)'}-\frac{z\left(I_{\lambda,\mu} g(z)\right)'}{I_{\lambda,\mu} g(z)}\right]\right\}$$

holds, then

$$q(z) \prec \frac{z\left(I_{\lambda,\mu} f(z)\right)'}{I_{\lambda,\mu} g(z)}$$

and $q(z)$ is the best subordinant.

Remark 2. Taking $q=2, s=1, a_1=a_2=1$ and $b_1=n+1$ $(n \in \mathbb{N}_0)$ in Theorem 2, we obtain the superordination result of Ibrahim and Darus [10,Theorem 4] for the Noor operator.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

Theorem 3. *Let $q_i(z)$ be convex univalent in U with $q_i(0)=1\,(i=1,2)$, $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in A$ such that $\frac{z\left(H_{q,s}[a_1;b_1] f(z)\right)'}{H_{q,s}[a_1;b_1] g(z)} \in H[1,1] \cap Q$,*

$$\frac{z\left(H_{q,s}[a_1;b_1]\, f(z)\right)'}{H_{q,s}[a_1;b_1]\, g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}[a_1;b_1]\, f(z)\right)''}{\left(H_{q,s}[a_1;b_1]\, f(z)\right)'}-\frac{z\left(H_{q,s}[a_1;b_1]\, g(z)\right)'}{H_{q,s}[a_1;b_1]\, g(z)}\right]\right\}$$

is univalent in U*, and*

$$\begin{aligned} & q_1(z)+\gamma z q_1^{'}(z) \\ \prec \quad & \frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{''}}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}-\frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\} \\ \prec \quad & q_2(z)+\gamma z q_2^{'}(z) \end{aligned}$$

holds, then

$$q_1(z)\prec\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Taking $q_i(z)=\frac{1+A_iz}{1+B_iz}$ $(i=1,2;-1\le B_2\le B_1<A_1\le A_2\le 1)$ in Theorem 3, we have the following corollary.

Corollary 10. *Let* $\gamma\in\mathbb{C}$ *with* $\Re(\bar{\gamma})>0$*. If* $f,g\in\mathcal{A}$ *such that* $\frac{z(H_{q,s}[a_1;b_1]f(z))^{'}}{H_{q,s}[a_1;b_1]g(z)}\in H[1,1]\cap Q$,

$$\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{''}}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}-\frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\}$$

is univalent in U*, and*

$$\begin{aligned} & \frac{1+A_1z}{1+B_1z}+\gamma\frac{(A_1-B_1)z}{(1+B_1z)^2} \\ \prec \quad & \frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{''}}{\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}-\frac{z\left(H_{q,s}\left[a_1;b_1\right]g(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\right]\right\} \\ \prec \quad & \frac{1+A_2z}{1+B_2z}+\gamma\frac{(A_2-B_2)z}{(1+B_2z)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{z\left(H_{q,s}\left[a_1;b_1\right]f(z)\right)^{'}}{H_{q,s}\left[a_1;b_1\right]g(z)}\prec\frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

Remark 3. Combining (i) Corollary 3 and Corollary 7; (ii) Corollary 4 and Corollary 8; (iii) Corollary 5 and Corollary 9, we obtain similar sandwich theorems for the corresponding linear operators.

Remark 4. Taking $q = 2, s = 1, a_1 = a_2 = 1$ and $b_1 = n + 1$ $(n \in \mathbb{N}_0)$ in Theorem 3, we obtain the sandwich result of Ibrahim and Darus [10,Theorem 6] for the Noor operator.

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References

[1] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135** (1969), 429-446.

[2] T. Bulboacă, Classes of first order differential superordinations, *Demonstratio Math.*, **35** no. 2 (2002), 287-292.

[3] T. Bulboacă, *Differential Subordinations and Superordinations*, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[4] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (1984), 737-745.

[5] N. E. Cho,O. H Kwon, and H. M. Srivastava, Inclusion and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, *J. Math. Anal. Appl.*, **292** (2004), 470-483.

[6] J. H. Choi, M. Saigo, and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, *J. Math. Anal. Appl.*, **276** (2002), 432-445.

[7] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, **103** (1999), 1-13.

[8] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform. Spec. Funct.*, **14** (2003), 7-18.

[9] Yu. E. Hohlov, Operators and operations in the class of univalent functions, *Izv. Vysšh. Učebn. Zaved. Mat.*, **10** (1978), 83-89 (in Russian).

[10] R. W. Ibrahim and M. Darus, On sandwich theorems of analytic functions involving Noor operator, *J. Math. & Stat.*, **4** no. 1 (2008), 32-36.

[11] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, **16** (1965), 755-658.

[12] S. S. Miller and P. T Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** no. 2 (1981), 157-171.

[13] S. S. Miller and P. T Mocanu, *Differential Subordination. Theory and Applications*, Marcel Dekker Inc., New York, 2000.

[14] S. S. Miller and P. T Mocanu, Subordinates of differential superordinations, *Complex Variables* **48** no. 10 (2003), 815-826.

[15] K. I. Noor, On new classes of integral operator, *J. Nate. Geom.*, **16** no 1-2 (1999), 71-80.

[16] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109-115.

[17] T. N. Shanmugam, V. Ravichandran, and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions, *Aust. J. Math. Anal. Appl.*, **3** no. 1 (2006), Art. 8, 11 pp. (electronic).

[18] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Ltd., Chichester, Halsted Press (John Wiley & Sons, Inc.), New York, 1985.