

A Study of New Mock Theta Functions *

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Abstract

The mock theta functions of Andrews and the mock theta functions of Bringmann et al are related by half-shift transformation. The generating functions for the partial mock theta functions are given. We extend these mock theta functions to give continued fraction representations. Some interesting expansions are also given in the end.

Keywords and Phrases: *Mock theta functions, Generating functions, Continued fractions.*

1. Introduction

The work of H.M. Srivastava [10],[11],[12] on generating functions motivated me to work on the generating functions. I had the new mock theta functions generated by G.E. Andrews in his paper [2] on orthogonal polynomials and two more mock theta functions generated by Bringmann, Hikami and Lovejoy [3]. The mock theta functions were there and my simple summation identity in [9] was a tool, to give generating functions for the partial mock theta functions. In partial mock theta functions we sum the defining series from 0 to N instead of from 0 to infinity, that is, for the mock theta function $\bar{\psi}_0(q)$,

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$$\overline{\psi}_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q; q)_{2n}},$$

the partial mock theta function will be defined and denoted as

$$\overline{\psi}_{0,N}(q) = \sum_{n=0}^N \frac{q^{2n^2}}{(-q; q)_{2n}}.$$

The second motivation was to apply the half-shift transformation on these functions. The half-shift transformation was introduced by Gordon and McIntosh [6] to develop eighth order mock theta functions. The application of this method shows that these functions are related to each by half-shift transformation. This is done in section 3.

In my earlier papers I have considered these functions in detail showing they belong to the class of F_q -functions, their integral representation etc.

In section 5, we give the generating functions for these partial mock theta functions.

In section 6, we represent the generalized functions as continued fraction. Some expansions for these mock theta functions are given in section 7.

The mock theta functions of Andrews [2]:

$$\overline{\psi}_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q; q)_{2n}} = {}_1\varphi_2 \left[\begin{matrix} q^2 \\ -q, -q^2 \end{matrix} ; q^2, q^2 \right], \quad (1)$$

$$\overline{\psi}_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q; q)_{2n+1}} = \frac{1}{(1+q)} {}_1\varphi_2 \left[\begin{matrix} q^2 \\ -q^2, -q^3 \end{matrix} ; q^2, q^4 \right], \quad (2)$$

$$\overline{\psi}_2(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n} (q; q^2)_n}{(q^2; q^2)_n (-q; q)_{2n}} = {}_1\varphi_2 \left[\begin{matrix} q \\ -q, -q^2 \end{matrix} ; q^2, q^4 \right], \quad (3)$$

$$\overline{\psi}_3(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q)_n^2}{(q; q)_{2n}} = {}_1\varphi_2 \left[\begin{matrix} -q \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}} \end{matrix} ; q, q \right], \quad (4)$$

and the mock theta functions of Bringmann, Hikami and Lovejoy [3]:

$$\overline{\phi}_0(q) = \sum_{n=0}^{\infty} q^n (-q; q)_{2n+1} = (1+q) {}_3\varphi_2 \left[\begin{matrix} q^2, -q^2, -q^3 \\ 0, 0 \end{matrix} ; q^2, q \right], \quad (5)$$

$$\bar{\phi}_1(q) = \sum_{n=0}^{\infty} q^n (-q; q)_{2n} = {}_3\varphi_2 \left[\begin{matrix} q^2, -q^2, -q \\ 0, 0 \end{matrix}; q^2, q \right]. \quad (6)$$

2. Basic Results

The following q -notations have been used

For $|q^k| < 1$,

$$(a; q^k)_n = \prod_{j=0}^{n-1} (1 - aq^{kj}), \quad n \geq 1$$

$$(a; q^k)_0 = 1,$$

$$(a; q^k)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^{kj}),$$

$$(a)_n = (a; q)_n,$$

$$(a_1, a_2, \dots, a_m; q^k)_n = (a_1; q^k)_n (a_2; q^k)_n \dots (a_m; q^k)_n.$$

A generalized basic hypergeometric series with base q is defined as

$$\begin{aligned} {}_r\varphi_s & [\ a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z \] \\ & = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \dots (a_r; q)_n z^n}{(b_1; q)_n \dots (b_s; q)_n (q; q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}} \right]^{1+s-r}, \end{aligned}$$

where $q \neq 0$ when $r > s + 1$.

3. Half-Shift Transformation

(i) We first obtain $\bar{\psi}_0(q)$ by applying left half-shift transformation on $\bar{\psi}_1(q)$.

Now

$$\begin{aligned}\bar{\psi}_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(-q;q)_{2n+1}} = \frac{1}{(-q;q)_{\infty}} \sum_{n=0}^{\infty} q^{2n^2+2n} (-q^{2n+2};q)_{\infty} \\ &= \sum_{n=0}^{\infty} a_n \quad (\text{say}),\end{aligned}\tag{7}$$

where a_n is defined for all real n . Making a left half-shift transformation and summing a_n over the positive half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, instead of the non-negative integers. Define $b_n = a_{n-\frac{1}{2}}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} b_n &= \sum_{n=0}^{\infty} a_{n-\frac{1}{2}} = \frac{q^{-\frac{1}{2}}}{(-q;q)_{\infty}} \sum_{n=0}^{\infty} q^{2n^2} (-q^{2n+1};q)_{\infty} \\ &= q^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} = q^{-\frac{1}{2}} \bar{\psi}_0(q).\end{aligned}\tag{8}$$

Thus by (7) and (8) we obtain $q^{-\frac{1}{2}} \bar{\psi}_0(q)$ by applying a left half-shift on $\bar{\psi}_1(q)$. This implies that $\bar{\psi}_0(q)$ and $\bar{\psi}_1(q)$ are related by a half-shift transformation.

(ii) Now we obtain $\bar{\varphi}_0(q)$ by applying left half-shift transformation on $\bar{\varphi}_1(q)$.

By definition

$$\begin{aligned}\bar{\varphi}_1(q) &= \sum_{n=0}^{\infty} q^n (-q;q)_{2n} = (-q;q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q^{2n+1};q)_{\infty}} \\ &= \sum_{n=0}^{\infty} a_n \quad (\text{say}),\end{aligned}\tag{9}$$

where a_n is defined for all real n . Making a left half-shift transformation and summing a_n over the positive half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, instead of the non-negative integers. Define $b_n = a_{n-\frac{1}{2}}$. Then

$$\sum_{n=0}^{\infty} b_n = (-q;q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n-\frac{1}{2}}}{(-q^{2n};q)_{\infty}} = q^{-\frac{1}{2}} \sum_{n=0}^{\infty} q^n (-q;q)_{2n-1}$$

$$\begin{aligned}
&= q^{-\frac{1}{2}} \sum_{n=1}^{\infty} q^{n+1} (-q; q)_{2n+1} \\
&= q^{-\frac{1}{2}} \left[\frac{1}{2} + \sum_{n=0}^{\infty} q^{n+1} (-q; q)_{2n+1} \right] \\
&= \frac{1}{2\sqrt{q}} + q^{\frac{1}{2}} \bar{\varphi}_0(q).
\end{aligned} \tag{10}$$

Thus by applying a left half-shift transformation on $q^{\frac{1}{2}} \bar{\varphi}_0(q)$ we obtain $\bar{\varphi}_1(q) - \frac{1}{2\sqrt{q}}$.

4. Definition of Generalized Mock Theta Functions

We define the generalized mock theta functions as:

$$\bar{\psi}_0(z, \alpha, q) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{2n^2-n+n\alpha}}{(-q; q)_{2n}}, \tag{11}$$

$$\bar{\psi}_1(z, \alpha, q) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{2n^2+n+n\alpha}}{(-q; q)_{2n+1}}, \tag{12}$$

$$\bar{\psi}_2(z, \alpha, q) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{2n^2+n+n\alpha} (q; q^2)_n}{(q^2; q^2)_n (-q; q)_{2n}}, \tag{13}$$

$$\bar{\psi}_3(z, \alpha, q) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2-n+n\alpha} (-q; q)_n^2}{(q^2; q^2)_n (q; q^2)_n}, \tag{14}$$

$$\bar{\varphi}_0(z, \alpha, q) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} (z)_n q^{n+n\alpha} (-q; q)_{2n+1}, \tag{15}$$

and

$$\bar{\varphi}_1(z, \alpha, q) = \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} (z)_n q^{n+n\alpha} (-q; q)_{2n}. \tag{16}$$

For $z=0$ and $\alpha=1$, the generalized functions defined in (11)-(14) reduce to mock theta functions $\bar{\psi}_0(q)$, $\bar{\psi}_1(q)$, $\bar{\psi}_2(q)$ and $\bar{\psi}_3(q)$ respectively. For $z=0$, $\alpha=0$ the generalized functions defined in (15)-(16) reduce to the mock theta functions $\bar{\varphi}_0(q)$ and $\bar{\varphi}_1(q)$, respectively.

5. Generating Functions for Partial Generalized Functions and Partial Mock Theta Functions

I shall be using the following summation identity, which I deduced in [10] to give the generating functions for the generalized functions.

$$\sum_{r=0}^p \alpha_r \beta_r = \beta_{p+1} \sum_{r=0}^p \alpha_r + \sum_{m=0}^p (\beta_m - \beta_{m+1}) \sum_{r=0}^m \alpha_r. \quad (17)$$

Taking $\beta_n = z^n$, $|z| < 1$ in (17), we have

$$\sum_{r=0}^p \alpha_r z^r = z^{p+1} \sum_{r=0}^p \alpha_r + (1-z) \sum_{m=0}^p z^m \sum_{r=0}^m \alpha_r. \quad (18)$$

Letting $p \rightarrow \infty$ in (18)

$$\begin{aligned} \sum_{m=0}^{\infty} z^m \sum_{r=0}^m \alpha_r &= \frac{1}{(1-z)} \sum_{r=0}^{\infty} \alpha_r z^r \\ &= \left[\sum_{r=0}^{\infty} \alpha_r z^r \right] \sum_{n=0}^{\infty} z^n. \end{aligned} \quad (19)$$

Now we define α_r such that $\sum_{r=0}^m \alpha_r$ is a partial generalized function.

(i) Take $\alpha_r = \frac{q^{2r^2-r+r\alpha}}{(-q;q)_{2r}}$ in (19), to have

$$\sum_{m=0}^{\infty} z^m \bar{\psi}_{0,m}(0, \alpha, q) = \left[\sum_{r=0}^{\infty} \frac{q^{2r^2-r+r\alpha}}{(-q;q)_{2r}} z^r \right] \sum_{n=0}^{\infty} z^n$$

$$= {}_1\phi_2 \left[\begin{matrix} q^2 \\ -q, -q^2 \end{matrix}; q^2, zq^{\alpha+1} \right] \sum_{n=0}^{\infty} z^n. \quad (20)$$

Here $\bar{\psi}_{0,m}(0, \alpha, q)$ is the partial generalized function. Thus we have the generating function for $\bar{\psi}_{0,m}(0, \alpha, q)$.

Taking $\alpha = 1$ in (20), we have

$$\sum_{m=0}^{\infty} z^m \bar{\psi}_{0,m}(q) = {}_1\phi_2 \left[\begin{matrix} q^2 \\ -q, -q^2 \end{matrix}; q^2, zq^2 \right] \sum_{n=0}^{\infty} z^n.$$

We list the generating functions for other generalized and partial mock theta functions, omitting calculations, only giving the values of α_r in the parenthesis.

(ii)

$$(1+q) \sum_{m=0}^{\infty} z^m \bar{\psi}_{1,m}(0, \alpha, q) = {}_1\phi_2 \left[\begin{matrix} q^2 \\ -q^2, -q^3 \end{matrix}; q^2, zq^{3+\alpha} \right] \sum_{n=0}^{\infty} z^n. \quad (21)$$

$$(\alpha_r = \frac{q^{2r^2+r+r\alpha}}{(-q; q)_{2r+1}} \text{ in (19)})$$

(iii)

$$(1+q) \sum_{m=0}^{\infty} z^m \bar{\psi}_{1,m}(q) = {}_1\phi_2 \left[\begin{matrix} q^2 \\ -q^2, -q^3 \end{matrix}; q^2, zq^4 \right] \sum_{n=0}^{\infty} z^n. \quad (22)$$

$$(\alpha = 1 \text{ in (21)})$$

(iv)

$$\sum_{m=0}^{\infty} z^m \bar{\psi}_{2,m}(0, \alpha, q) = {}_1\phi_2 \left[\begin{matrix} q \\ -q, -q^2 \end{matrix}; q^2, zq^{3+\alpha} \right] \sum_{n=0}^{\infty} z^n. \quad (23)$$

$$(\alpha_r = \frac{q^{2r^2+r+r\alpha} (q; q^2)_r}{(q^2; q^2)_r (-q; q)_{2r}} \text{ in (19)})$$

(v)

$$\sum_{m=0}^{\infty} z^m \bar{\psi}_{2,m}(q) = {}_1\phi_2 \left[\begin{matrix} q \\ -q, -q^2 \end{matrix}; q^2, zq^4 \right] \sum_{n=0}^{\infty} z^n. \quad (24)$$

$(\alpha = 1 \text{ in (23)})$

(vi)

$$\sum_{m=0}^{\infty} z^m \bar{\psi}_{3,m}(0, \alpha, q) = {}_1\phi_2 \left[\begin{matrix} -q \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}} \end{matrix}; q, zq^{\alpha} \right] \sum_{n=0}^{\infty} z^n. \quad (25)$$

$$(\alpha_r = \frac{q^{r^2-r+r\alpha} (-q; q)_r}{(q; q)_r (q^{\frac{1}{2}}; q)_r (-q^{\frac{1}{2}}; q)_r} \text{ in (19)})$$

(vii)

$$\sum_{m=0}^{\infty} z^m \bar{\psi}_{3,m}(q) = {}_1\phi_2 \left[\begin{matrix} -q \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}} \end{matrix}; q, zq \right] \sum_{n=0}^{\infty} z^n. \quad (26)$$

$(\alpha = 1 \text{ in (25)})$

(viii)

$$\sum_{m=0}^{\infty} z^m \bar{\varphi}_{0,m}(0, \alpha, q) = (1+q) {}_3\phi_2 \left[\begin{matrix} -q^2, -q^3, q^2 \\ 0, 0 \end{matrix}; q^2, zq^{\alpha+1} \right] \sum_{n=0}^{\infty} z^n. \quad (27)$$

$$(\alpha_r = q^{r+r\alpha} (-q; q)_{2r+1} \text{ in (19)})$$

(ix)

$$\sum_{m=0}^{\infty} z^m \bar{\varphi}_{0,m}(q) = (1+q) {}_3\phi_2 \left[\begin{matrix} -q^2, -q^3, q^2 \\ 0, 0 \end{matrix}; q^2, zq \right] \sum_{n=0}^{\infty} z^n. \quad (28)$$

$(\alpha = 0 \text{ in (27)})$

(x)

$$\sum_{m=0}^{\infty} z^m \bar{\varphi}_{1,m}(0, \alpha, q) = {}_3\phi_2 \left[\begin{matrix} q^2, -q^2, -q \\ 0, 0 \end{matrix}; q^2, zq^{\alpha+1} \right] \sum_{n=0}^{\infty} z^n. \quad (29)$$

$$(\alpha_r = q^{r+r\alpha} (-q; q)_{2r} \text{ in (19)})$$

(xi)

$$\sum_{m=0}^{\infty} z^m \bar{\varphi}_{1,m}(q) = {}_3\phi_2 \left[\begin{matrix} q^2, -q^2, -q \\ 0, 0 \end{matrix}; q^2, zq \right] \sum_{n=0}^{\infty} z^n. \quad (30)$$

$(\alpha = 0 \text{ in (29)})$

6. Continued Fraction Representation

We shall give the continued fraction representation for generalized functions.

(i) Representation of $\bar{\psi}_0(z, \alpha, q)$ as continued fraction

By definition

$$\bar{\psi}_0(0, \alpha, q) = \sum_{n=0}^{\infty} \frac{q^{2n^2-n+n\alpha}}{(-q; q)_{2n}} \quad (31)$$

Letting $q \rightarrow q^2, \lambda = 0, b = q^2, c = 1/q, a = 1$ in [1, (5.26), p. 97], we have

$$\frac{\sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(-q; q)_{2n}}}{2 \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q; q^2)_n (-q^2; q^2)_{n+1}}} = 1 + \frac{(1 - 1/q^2)q^2}{(1+1)+} \frac{q}{1+} \frac{(1 - 1/q^4)q^4}{2+} \frac{q^3}{1+\dots} \quad (32)$$

Taking $\alpha = 0$ in (31), we have from (32)

$$\frac{\bar{\psi}_0(0, 0, q)}{2 \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q; q^2)_n (-q^2; q^2)_{n+1}}} = 1 + \frac{(1 - 1/q^2)q^2}{(1+1)+} \frac{q}{1+} \frac{(1 - 1/q^4)q^4}{2+} \frac{q^3}{1+\dots} \quad (33)$$

(ii) Representation of $\bar{\psi}_1(z, \alpha, q)$ as continued fraction

By definition

$$\bar{\psi}_1(0, \alpha, q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+n+n\alpha}}{(-q; q)_{2n+1}} = \frac{1}{(1+q)} \sum_{n=0}^{\infty} \frac{q^{2n^2+n+n\alpha}}{(-q^2; q)_{2n}} \quad (34)$$

Letting $q \rightarrow q^2, \lambda = 0, b = q^2, c = q, a = 1$ in [1, (5.26), p. 97], we have

$$\frac{\sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q^2; q)_{2n}}}{2 \sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(-q^3; q^2)_n (-q^2; q^2)_{n+1}}} = 1 + \frac{(1 - 1/q^2)q^2}{(1+1)+} \frac{q^3}{1+} \frac{(1 - 1/q^4)q^4}{2+} \frac{q^5}{1+\dots} \quad (35)$$

Taking $\alpha = 0$ in (34), we have from (35)

$$\frac{\bar{\psi}_1(0, 0, q)}{2 \sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(-q; q^2)_{n+1} (-q^2; q^2)_{n+1}}} = 1 + \frac{(1 - 1/q^2)q^2}{(1 + 1) +} \frac{q^3}{1 +} \frac{(1 - 1/q^4)q^4}{2 +} \frac{q^5}{1 + \dots} \quad (36)$$

(iii) Representation of $\bar{\psi}_2(z, \alpha, q)$ as continued fraction

By definition

$$\bar{\psi}_2(0, \alpha, q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+n+n\alpha} (q; q^2)_n}{(q^2; q^2)_n (-q; q)_{2n}} \quad (37)$$

Letting $q \rightarrow q^2, \lambda = 0, b = q, c = 1/q, a = 1$ in [1, (5.26), p. 97], we have

$$\frac{\sum_{n=0}^{\infty} \frac{q^{2n^2} (q; q^2)_n}{(q^2; q^2)_n (-q; q^2)_n (-q^2; q^2)_n}}{(1+q) \sum_{n=0}^{\infty} \frac{q^{2n^2+2n} (q; q^2)_n}{(q^2; q^2)_n (-q; q^2)_n (-q^2; q^2)_{n+1}}} = 1 + \frac{(1 - 1/q)q^2}{(1 + q) +} \frac{q}{1 +} \frac{(1 - 1/q^3)q^4}{1 + q + \dots} \quad (38)$$

Taking $\alpha = -1$ in (37), we have from (38)

$$\frac{\bar{\psi}_2(0, -1, q)}{(1+q) \sum_{n=0}^{\infty} \frac{q^{2n^2+2n} (q; q^2)_n}{(q^2; q^2)_n (-q; q^2)_n (-q^2; q^2)_{n+1}}} = 1 + \frac{(1 - 1/q)q^2}{(1 + q) +} \frac{q}{1 +} \frac{(1 - 1/q^3)q^4}{1 + q + \dots} \quad (39)$$

(iv) Representation of $\bar{\psi}_3(z, \alpha, q)$ as continued fraction

By definition

$$\bar{\psi}_3(0, \alpha, q) = \sum_{n=0}^{\infty} \frac{q^{n^2-n+n\alpha} (-q; q)_n^2}{(q; q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^{n^2-n+n\alpha} (-q; q)_n}{(q; q)_n \left(q^{\frac{1}{2}}; q\right)_n \left(-q^{\frac{1}{2}}; q\right)_n} \quad (40)$$

Letting $\lambda = 0, b = -q, c = 1/q^{\frac{1}{2}}, a = -1/q^{\frac{1}{2}}$ in [1, (5.26), p. 97], we have

$$\frac{\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-q;q)_n}{(q;q)_n(q^{\frac{1}{2}};q)_n(-q^{\frac{1}{2}};q)_n}}{\left(1+1/q^{\frac{1}{2}}\right) \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(q;q)_n(q^{\frac{1}{2}};q)_{n+1}(-q^{\frac{1}{2}};q)_n}} = 1 + \frac{(1+1/q)(-q^{\frac{1}{2}})}{\left(1+q^{\frac{1}{2}}\right)} \frac{q^{\frac{1}{2}}}{1+} \frac{(1+1/q^2)(-q^{\frac{3}{2}})}{\left(1+q^{\frac{1}{2}}\right)} \dots \quad (41)$$

Taking $\alpha = 0$ in (40), we have from (41)

$$\frac{\bar{\psi}_3(0,0,q)}{\left(1+1/q^{\frac{1}{2}}\right) \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(q;q)_n(q^{\frac{1}{2}};q)_{n+1}(-q^{\frac{1}{2}};q)_n}} = 1 + \frac{(1+1/q)(-q^{\frac{1}{2}})}{\left(1+q^{\frac{1}{2}}\right)} \frac{q^{\frac{1}{2}}}{1+} \frac{(1+1/q^2)(-q^{\frac{3}{2}})}{\left(1+q^{\frac{1}{2}}\right)} \dots \quad (42)$$

7. Expansions of the Mock Theta Functions

We have general expansion formula [5, p 70],[8, p 56]

$$\sum_{k=0}^{\infty} \frac{(1-aq^{2k})(a,b,c,a/bc)_k q^k}{(1-a)(q,aq/b,aq/c,bcq)_k} \sum_{m=0}^{\infty} \alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(aq,bq,cq,aq/bc)_m}{(q,aq/b,aq/c,bcq)_m} \alpha_m. \quad (43)$$

Letting $q \rightarrow q^2$ and $b,c \rightarrow \infty$ in (43), we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k (1-aq^{4k}) (a;q^2)_k q^{k^2-k}}{(1-a)(q^2;q^2)_k} \sum_{m=0}^{\infty} \alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(-1)^m (aq^2;q^2)_m q^{m^2+m}}{(q^2;q^2)_m} \alpha_m. \quad (44)$$

Putting $a = 0$ in (44)

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2-k}}{(q^2;q^2)_k} \sum_{m=0}^{\infty} \alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m}}{(q^2;q^2)_m} \alpha_m. \quad (45)$$

Putting $a = 1$ in (44)

$$\sum_{k=0}^{\infty} (-1)^k (1+q^{2k}) q^{k^2-k} \sum_{m=0}^{\infty} \alpha_{m+k} = \sum_{m=0}^{\infty} (-1)^m q^{m^2+m} \alpha_m. \quad (46)$$

Writing the inner sum of the left side of (43) as the difference of two series, we have

$$\begin{aligned} & \frac{(aq, bq, cq, aq/bc)_\infty}{(q, aq/b, aq/c, bcq)_\infty} \sum_{m=0}^{\infty} \alpha_m - \sum_{k=0}^{\infty} \frac{(1 - aq^{2k+2}) (a, b, c, a/bc)_{k+1} q^{k+1}}{(1-a)(q, aq/b, aq/c, bcq)_{k+1}} \sum_{m=0}^k \alpha_m \\ &= \sum_{m=0}^{\infty} \frac{(aq, bq, cq, aq/bc)_m}{(q, aq/b, aq/c, bcq)_m} \alpha_m. \end{aligned} \quad (47)$$

Putting $a = 0$ in (47), we have

$$\begin{aligned} & \frac{(bq, cq)_\infty}{(q, bcq)_\infty} \sum_{m=0}^{\infty} \alpha_m - \sum_{k=0}^{\infty} \frac{(b, c)_{k+1} q^{k+1}}{(q, bcq)_{k+1}} \sum_{m=0}^k \alpha_m \\ &= \sum_{m=0}^{\infty} \frac{(bq, cq)_m}{(q, bcq)_m} \alpha_m. \end{aligned} \quad (48)$$

Taking $b = c = 0$ in (48), we have

$$\frac{1}{(q)_\infty} \sum_{m=0}^{\infty} \alpha_m - \sum_{k=0}^{\infty} \frac{q^{k+1}}{(q)_{k+1}} \sum_{m=0}^k \alpha_m = \sum_{m=0}^{\infty} \frac{1}{(q)_m} \alpha_m. \quad (49)$$

(a) Expansions for $\bar{\psi}_0(q)$

(i) Taking $\alpha_m = \frac{(-1)^m q^{m^2-m} (q^2; q^2)_m}{(-q; q)_{2m}}$ in (45), we have

$$\begin{aligned} \bar{\psi}_0(q) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k^2-2k}}{(q^2; q^2)_k} \sum_{m=0}^{\infty} \frac{(-1)^{m+k} q^{m^2-m+2mk} (q^2; q^2)_{m+k}}{(-q; q)_{2m+2k}} \\ &= \sum_{k=0}^{\infty} \frac{q^{2k^2-2k}}{(-q; q)_{2k}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2-m+2mk} (q^{2k+2}; q^2)_m}{(-q^{2k+1}; q)_{2m}} \\ &= \sum_{k=0}^{\infty} \frac{q^{2k^2-2k}}{(-q; q)_{2k}} {}_2\phi_2 \left[\begin{matrix} q^2, q^{2k+2}, \\ -q^{2k+1}, -q^{2k+2} \end{matrix}; q^2, q^{2k} \right]. \end{aligned} \quad (50)$$

(ii)

$$\begin{aligned} \bar{\psi}_0(q) &= \sum_{k=0}^{\infty} \frac{(1+q^{2k}) q^{2k^2-2k}}{(-q;q)_{2k}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2-m+2mk}}{(-q^{2k+1};q)_{2m}} \\ &= \sum_{k=0}^{\infty} \frac{(1+q^{2k}) q^{2k^2-2k}}{(-q;q)_{2k}} {}_2\phi_2 \left[\begin{matrix} q^2, 0 \\ -q^{2k+1}, -q^{2k+2} \end{matrix} ; q^2, q^{2k} \right]. \quad (51) \\ &\left(\alpha_m = \frac{(-1)^m q^{m^2-m}}{(-q;q)_{2m}} \text{ in (46)} \right) \end{aligned}$$

(iii)

$$\begin{aligned} \frac{(-q, -q^2; q^2)_\infty}{(q, q^2; q^2)_\infty} \bar{\psi}_0(q) - 2(1 + 1/q) \sum_{k=0}^{\infty} \frac{(-q, -q^2; q^2)_k q^{k+1}}{(q, q^2; q^2)_{k+1}} \bar{\psi}_{0,k}(q) \\ = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q; q^2)_m (q^2; q^2)_m} = {}_0\phi_1 \left[\begin{matrix} \text{--} \\ q, \end{matrix} ; q^2, q^2 \right]. \quad (52) \\ \left(q \rightarrow q^2, b = -1, c = -1/q, \alpha_m = \frac{q^{2m^2}}{(-q; q)_{2m}} \text{ in (48)} \right) \end{aligned}$$

(iv)

$$\begin{aligned} \bar{\psi}_0(q) - (q^2; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2k+2}}{(q^2; q^2)_{k+1}} \bar{\psi}_{0,k}(q) &= (q^2; q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(-q; q)_{2m} (q^2; q^2)_m} \\ &= (q^2; q^2)_\infty {}_1\phi_2 \left[\begin{matrix} 0, \\ -q, -q^2 \end{matrix} ; q^2, q^2 \right]. \quad (53) \\ \left(q \rightarrow q^2, \alpha_m = \frac{q^{2m^2}}{(-q; q)_{2m}} \text{ in (49)} \right) \end{aligned}$$

(b) Expansions for $\bar{\psi}_1(q)$

(i)

$$\begin{aligned} \bar{\psi}_1(q) &= \sum_{k=0}^{\infty} \frac{q^{2k^2}}{(-q; q)_{2k+1}} \\ &\times {}_2\phi_2 \left[\begin{matrix} q^2, q^{2k+2} \\ -q^{2k+2}, -q^{2k+3} \end{matrix}; q^2, q^{2k+2} \right]. \end{aligned} \quad (54)$$

$$\left(\alpha_m = \frac{(-1)^m q^{m^2+m} (q^2; q)_m}{(-q^2; q)_{2m}} \text{ in (45)} \right)$$

(ii)

$$\begin{aligned} \bar{\psi}_1(q) &= \sum_{k=0}^{\infty} \frac{(1+q^{2k}) q^{2k^2}}{(-q; q)_{2k+1}} {}_2\phi_2 \left[\begin{matrix} q^2, 0 \\ -q^{2k+2}, -q^{2k+3} \end{matrix}; q^2, q^{2k+2} \right]. \end{aligned} \quad (55)$$

$$\left(\alpha_m = \frac{(-1)^m q^{m^2+m}}{(-q; q)_{2m+1}} \text{ in (46)} \right)$$

(iii)

$$\begin{aligned} \frac{(-q, -q^2; q^2)_{\infty}}{(q^2, q^3; q^2)_{\infty}} \bar{\psi}_1(q) - 2(1+q)^2 \sum_{k=0}^{\infty} \frac{(-q^2, -q^3; q^2)_k q^{2k+2}}{(q^2, q^3; q^2)_{k+1}} \bar{\psi}_{1,k}(q) \\ = {}_0\phi_1 \left[\begin{matrix} -, \\ q^3, \end{matrix}; q^2, q^4 \right]. \end{aligned} \quad (56)$$

$$\left(q \rightarrow q^2, b = -1, c = -q, \alpha_m = \frac{q^{2m^2+2m}}{(-q; q)_{2m+1}} \text{ in (48)} \right)$$

(iv)

$$\bar{\psi}_1(q) - (q^2; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2k+2}}{(q^2; q^2)_{k+1}} \bar{\psi}_{1,k}(q) = \frac{(q^2; q^2)_\infty}{(1+q)} {}_1\phi_2 \left[\begin{matrix} 0, \\ -q^2, -q^3 \end{matrix}; q^2, q^4 \right]. \quad (57)$$

$$\left(q \rightarrow q^2, \alpha_m = \frac{q^{2m^2+2m}}{(-q; q)_{2m+1}} \text{ in (49)} \right)$$

(c) Expansions for $\bar{\psi}_2(q)$

(i)

$$\bar{\psi}_2(q) = \sum_{k=0}^{\infty} \frac{q^{2k^2} (q; q^2)_k}{(q^2; q^2)_k (-q; q)_{2k}} {}_2\phi_2 \left[\begin{matrix} q^2, q^{2k+1} \\ -q^{2k+1}, -q^{2k+2} \end{matrix}; q^2, q^{2k+2} \right]. \quad (58)$$

$$\left(\alpha_m = \frac{(-1)^m q^{m^2+m} (q; q^2)_m}{(-q; q)_{2m}} \text{ in (45)} \right)$$

(ii)

$$\bar{\psi}_2(q) = \sum_{k=0}^{\infty} \frac{(q; q^2)_k (1+q^{2k}) q^{2k^2}}{(q^2; q^2)_k (-q; q)_{2k}} {}_3\phi_3 \left[\begin{matrix} q^2, q^{2k+1}, 0 \\ q^{2k+2}, -q^{2k+1}, -q^{2k+2} \end{matrix}; q^2, q^{2k+2} \right]. \quad (59)$$

$$\left(\alpha_m = \frac{(-1)^m q^{m^2+m} (q; q^2)_m}{(q^2; q^2)_m (-q; q)_{2m}} \text{ in (46)} \right)$$

(iii)

$$\begin{aligned} & \frac{(-q, -q^2; q^2)_\infty}{(q, q^2; q^2)_\infty} \bar{\psi}_2(q) - \frac{2(1+q)}{q} \sum_{k=0}^{\infty} \frac{(-q, -q^2; q^2)_k q^{2k+2}}{(q, q^2; q^2)_{k+1}} \bar{\psi}_{2,k}(q) \\ &= {}_1\phi_2 \left[\begin{matrix} 0, \\ q, q^2 \end{matrix}; q^2, q^4 \right]. \end{aligned} \quad (60)$$

$$\left(q \rightarrow q^2, b = -1/q, c = -1, \alpha_m = \frac{q^{2m^2+2m} (q; q^2)_m}{(q^2; q^2)_m (-q; q)_{2m}} \text{ in (48)} \right)$$

(iv)

$$\begin{aligned} \bar{\psi}_2(q) - (q^2; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2k+2}}{(q^2; q^2)_{k+1}} \bar{\psi}_{2,k}(q) \\ = (q^2; q^2)_\infty {}_2\phi_3 \left[\begin{matrix} q, 0, \\ q^2, -q, -q^2 \end{matrix}; q^2, q^4 \right]. \end{aligned} \quad (61)$$

$$\left(q \rightarrow q^2, \alpha_m = \frac{q^{2m^2+2m} (q; q^2)_m}{(q^2; q^2)_m (-q; q)_{2m}} \text{ in (49)} \right)$$

(d) Expansions for $\bar{\psi}_3(q)$

(i)

$$\begin{aligned} \bar{\psi}_3(q) &= \sum_{k=0}^{\infty} \frac{q^{k^2} (-q; q)_k}{(q; q)_k \left(q^{\frac{1}{2}}; q \right)_k \left(-q^{\frac{1}{2}}; q \right)_k} \\ &\times {}_2\phi_2 \left[\begin{matrix} q, -q^{k+1} \\ q^{k+\frac{1}{2}}, -q^{k+\frac{1}{2}} \end{matrix}; q, q^k \right]. \end{aligned} \quad (62)$$

$$\left(q \rightarrow q^2, \alpha_m = \frac{(-1)^m q^{\frac{m^2-m}{2}} (-q; q)_m}{\left(q^{\frac{1}{2}}; q \right)_m \left(-q^{\frac{1}{2}}; q \right)_m} \text{ in (45)} \right)$$

(ii)

$$\bar{\psi}_3(q) = \sum_{k=0}^{\infty} \frac{(-q; q)_k (1 + q^k) q^{k^2-k}}{(q; q)_k \left(q^{\frac{1}{2}}; q \right)_k \left(-q^{\frac{1}{2}}; q \right)_k} {}_3\phi_3 \left[\begin{matrix} q, -q^{k+1}, 0 \\ q^{k+1}, q^{k+\frac{1}{2}}, -q^{k+\frac{1}{2}} \end{matrix}; q, q^k \right]. \quad (63)$$

$$\left(q \rightarrow q^2, \alpha_m = \frac{(-1)^m q^{\frac{m^2-m}{2}} (-q; q)_m}{(q; q)_m \left(q^{\frac{1}{2}}; q \right)_m \left(-q^{\frac{1}{2}}; q \right)_m} \text{ in (46)} \right)$$

(iii)

$$\begin{aligned} & \frac{\left(q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q\right)_{\infty}}{(q, -1; q)_{\infty}} \bar{\psi}_3(q) - \sum_{k=0}^{\infty} \frac{\left(q^{-\frac{1}{2}}, -q^{-\frac{1}{2}}; q\right)_{k+1} q^{k+1}}{(q, -1; q)_{k+1}} \bar{\psi}_{3,k}(q) \\ & = {}_1\phi_2 \left[\begin{matrix} -q, \\ -1, q \end{matrix}; q, q \right]. \end{aligned} \quad (64)$$

$$\left(b = 1/q^{\frac{1}{2}}, c = -1/q^{\frac{1}{2}}, \alpha_m = \frac{q^{m^2} (-q; q)_m}{(q; q)_m \left(q^{\frac{1}{2}}; q\right)_m \left(-q^{\frac{1}{2}}; q\right)_m} \text{ in (48)} \right)$$

(iv)

$$\bar{\psi}_3(q) - (q; q)_{\infty} \sum_{k=0}^{\infty} \frac{q^{k+1}}{(q; q)_{k+1}} \bar{\psi}_{3,k}(q) = (q; q)_{\infty} {}_3\phi_3 \left[\begin{matrix} -q, 0, 0 \\ q, q^{\frac{1}{2}}, -q^{\frac{1}{2}} \end{matrix}; q, q \right]. \quad (65)$$

$$\left(\alpha_m = \frac{q^{m^2} (-q; q)_m}{(q; q)_m \left(q^{\frac{1}{2}}; q\right)_m \left(-q^{\frac{1}{2}}; q\right)_m} \text{ in (49)} \right)$$

(e) Expansions for $\bar{\varphi}_0(q)$

(i)

$$\begin{aligned} & \frac{(-q^2, q^3; q^2)_{\infty}}{(q^2, -q^3; q^2)_{\infty}} \bar{\varphi}_0(q) - 2(1-q) \sum_{k=0}^{\infty} \frac{(-q^2, q^3; q^2)_k q^{2k+2}}{(q^2, -q^3; q^2)_{k+1}} \bar{\varphi}_{0,k}(q) \\ & = (1+q) {}_3\phi_2 \left[\begin{matrix} -q^2, -q^2, q^3, \\ 0, 0, \end{matrix}; q^2, q \right]. \end{aligned} \quad (66)$$

$$(q \rightarrow q^2, b = -1, c = q, \alpha_m = q^m (-q; q)_{2m+1} \text{ in (48)})$$

(ii)

$$\begin{aligned} & \bar{\varphi}_0(q) - (q^2; q^2)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2k+2}}{(q^2; q^2)_{k+1}} \bar{\varphi}_{0,k}(q) \\ & = (1+q) (q^2; q^2)_{\infty} {}_2\phi_1 \left[\begin{matrix} -q^2, -q^3, \\ 0, \end{matrix}; q^2, q \right] \end{aligned} \quad (67)$$

$$(q \rightarrow q^2, \alpha_m = q^m (-q; q)_{2m+1} \text{ in (49)})$$

(f) Expansions for $\bar{\varphi}_1(q)$

(i)

$$\begin{aligned} & \frac{(-q^2, q^3; q^2)_\infty}{(q^2, -q^3; q^2)_\infty} \bar{\varphi}_1(q) - \sum_{k=0}^{\infty} \frac{(-1, q; q^2)_{k+1} q^{2k+2}}{(q^2, -q^3; q^2)_{k+1}} \bar{\varphi}_{1,k}(q) \\ &= {}_4\phi_3 \left[\begin{matrix} -q, -q^2, -q^2, q^3, \\ -q^3, 0, 0, \end{matrix}; q^2, q \right]. \end{aligned} \quad (68)$$

$$(q \rightarrow q^2, b = -1, c = q, \alpha_m = q^m (-q; q)_{2m} \text{ in (48)})$$

(ii)

$$\bar{\varphi}_1(q) - (q^2; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2k+2}}{(q^2; q^2)_{k+1}} \bar{\varphi}_{1,k}(q) = (q^2; q^2)_\infty {}_2\phi_1 \left[\begin{matrix} -q, -q^2, \\ 0, \end{matrix}; q^2, q \right] \quad (69)$$

$$(q \rightarrow q^2, \alpha_m = q^m (-q; q)_{2m} \text{ in (49)})$$

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