Tamsui Oxford Journal of Information and Mathematical Sciences **2x(x)** (201x) xxx-xxx
Aletheia University

Weakly Prime Ideals in Near-Rings *

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Received July 6, 2011, Accepted December 18, 2012.

Abstract

In this short note, we introduce the notion of prime ideals in near-ring and obtain equivalent conditions for an ideal to be a weakly prime ideal.

Keywords and Phrases: *Near-ring, Prime ideal, M-system, Weakly prime ideal.*

1. Introduction

Throughout this paper, N denotes a zero-symmetric near-ring not necessarily with identity unless otherwise stated. For $x \in N$, $< x >$ denote the ideal of N generated by x, and $P(N)$ denotes the intersection of all prime ideals of N. In

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[1], D. D. Anderson and E. Smith defined weakly prime ideals in commutative rings, an ideal P of a ring R is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. In this paper we define a notion of weakly prime ideal in near-ring (not necessarily commutative).

A proper ideal P (i.e., an ideal different from N) of N is prime if for ideals A and B of N, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. We define a proper ideal P of N to be weakly prime if $0 \neq AB \subseteq P$, A and B are ideals of N, implies $A \subseteq P$ or $B \subseteq P$. Clearly every prime ideal is weakly prime and $\{0\}$ is always weakly prime ideal of N. The following example shows that a weakly prime ideal need not be a prime ideal in general.

Example 1.1. *Let $N = \{0, a, b, c, d, 1, 2, 3\}$. Define addition and multiplication in N as follows:*

+	0	1	2	3	a	b	c	d
0	0	1	2	3	a	b	c	d
1	1	2	3	0	d	c	a	b
2	2	3	0	1	b	a	d	c
3	3	0	1	2	c	d	b	a
a	a	d	b	c	2	0	1	3
b	b	c	a	d	0	2	3	1
c	c	a	d	b	1	3	0	2
d	d	b	c	a	3	1	2	0

.	0	1	2	3	a	b	c	d
0	0	0	0	0	0	0	0	0
1	0	1	2	3	a	b	c	d
2	0	2	0	2	2	2	0	0
3	0	3	2	1	b	a	c	d
a	0	a	2	b	a	b	c	d
b	0	b	2	a	b	a	c	d
c	0	c	0	c	0	0	0	0
d	0	d	0	d	2	2	0	0

Then $(N, +, .)$ is a near-ring (see [2], Library Nearring (8/2, 857)). Here $\{0, c\}$ is a weakly prime ideal, but not a prime, since $\{0, 2\}^2 \subseteq \{0, c\}$.

For a less trivial example, let M be a unique maximal ideal of a near-ring N with $M^2 = 0$, then every proper ideal of N is easily seen to be weakly prime. Also in $\mathbb{Z}_6$, $\{0\}$ is a weakly prime ideal, but not prime. For basic terminology in near-ring we refer to Pilz [3].

2. Main Results

Theorem 2.1. *Let N be a near-ring and P a weakly prime ideal of N. If P is not a prime, then $P^2 = 0$.*

Proof: Suppose that $P^2 \neq 0$. We show that P is prime. Let A and B be ideals of N such that $AB \subseteq P$. If $AB \neq 0$, then $A \subseteq P$ or $B \subseteq P$. So assume that $AB = 0$. Since $P^2 \neq 0$, there exist $p_0, q_0 \in P$ such that $< p_0 >< q_0 > \neq 0$. Then $(A+ < p_0 >)(B+ < q_0 >) \neq 0$. Suppose $(A+ < p_0 >)(B+ < q_0 >) \nsubseteq P$. Then there exist $a \in A; b \in B$ and $p_0^{'} \in < p_0 >; q_0^{'} \in < q_0 >$ such that $(a+p_0^{'})(b+q_0^{'}) \notin P$ which implies $a(b+q_0^{'}) \notin P$, but $a(b+q_0^{'}) = a(b+q_0^{'}) - ab \in P$ since $AB = 0$, a contradiction. So $0 \neq (A+ < p_0 >)(B+ < q_0 >) \subseteq P$ which implies $A \subseteq P$ or $B \subseteq P$. ∎

Corollary 2.2. *Let N be a near-ring and P an ideal of N. If $P^2 \neq 0$, then P is prime if and only if P is weakly prime.*

Corollary 2.3. *Let P be a weakly prime ideal of N. Then either $P \subseteq P(N)$ or $P(N) \subseteq P$. If $P \subset P(N)$, then P is not prime, while if $P(N) \subset P$, then P is prime.*

It should be noted that a proper ideal P with the property that $P^2 = \{0\}$ need not be weakly prime. Take $N = \mathbb{Z}_8$ and $P = \{\overline{0}, \overline{4}\}$. Clearly $P^2 = \{0\}$, yet P is not weakly prime.

Lemma 2.4. *Let N be a near-ring and P an ideal of N. Then the following are equivalent:*

i) For any $a, b, c \in N$ with $0 \neq a(< b > + < c >) \subseteq P$, we have $a \in P$ or b and c in P

ii) For $x \in N \backslash P$, we have $(P :< x > + < y >) = P \cup (0 :< x > + < y >)$ for any $y \in N$.

iii) For $x \in N \backslash P$, we have $(P :< x > + < y >) = P$ or $(P :< x > + < y >) = (0 :< x > + < y >)$ for any $y \in N$.

iv) P is weakly prime

Proof: $(i) \Rightarrow (ii)$ Let $t \in (P :< x > + < y >)$ for any $x \in N \backslash P$ and $y \in N$. Then $t(< x > + < y >) \subseteq P$. If $t(< x > + < y >) = 0$, then $t \in (0 :< x > + < y >)$. Otherwise $0 \neq t(< x > + < y >) \subseteq P$. Then $t \in P$ by hypothesis. $(ii) \Rightarrow (iii)$ follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them. $(iii) \Rightarrow (iv)$ Let A and B be ideals of N such that $AB \subseteq P$ and suppose $A \nsubseteq P$ and $B \nsubseteq P$. Then there exist $a \in A$ and $b \in B$ with $a, b \notin P$. Now we claim that $AB = 0$.

Let $b_1 \in B$. Then $A(< b > + < b_1 >) \subseteq P$ which implies $A \subseteq (P :< b > + < b_1 >)$. Then by assumption, $A(< b > + < b_1 >) = 0$ which gives $Ab_1 = 0$. Thus $AB = 0$ and hence P is weakly prime ideal of N. $(iv) \Rightarrow (i)$ is clear. ∎

Theorem 2.5. *Let N be a near-ring and P an ideal of N. Then*
i) P is weakly prime
ii) For any ideals I, J of N with $P \subset I$ and $P \subset J$, we have either $IJ = 0$ or $IJ \nsubseteq P$.
iii) For any ideals I, J of N with $I \nsubseteq P$ and $J \nsubseteq P$, we have either $IJ = 0$ or $IJ \nsubseteq P$.

Proof: $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear. $(ii) \Rightarrow (iii)$. Let I, J be ideals of N with $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $i_1 \in I$ and $j_1 \in J$ such that $i_1, j_1 \notin P$.

Suppose that $< i >< j > \neq 0$ for some $i \in I$ and some $j \in J$. Then $(P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \neq 0$ and $P \subset P+ < i > + < i_1 >; P \subset P+ < j > + < j_1 >$. By hypothesis, $(P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \nsubseteq P$ which implies $< i > (P+ < j > + < j_1 >)+ < i_1 > (P+ < j > + < j_1 >) \nsubseteq P$. So there exist $i^{'} \in< i >; i_1^{'} \in< i_1 >; j^{'}, j^{''} \in< j >; j_1^{'}, j_1^{''} \in< j_1 >$ and $p_1, p_2 \in P$ such that $i^{'}(p_1 + j^{'} + j_1^{'}) + i_1^{'}(p_2 + j^{''} + j_1^{''}) \notin P$. Therefore $i^{'}(p_1+j^{'}+j_1^{'})-i^{'}(j^{'}+j_1^{'})+i^{'}(j^{'}+j_1^{'})+i_1^{'}(p_2+j^{''}+j_1^{''})-i_1^{'}(j^{''}+j_1^{''})+i_1^{'}(j^{''}+j_1^{''}) \notin P$. But since $i^{'}(p_1+j^{'}+j_1^{'})-i^{'}(j^{'}+j_1^{'}) \in P$ and $i_1^{'}(p_2+j^{''}+j_1^{''})-i_1^{'}(j^{''}+j_1^{''}) \in P$, we have P does not contain either $i^{'}(j^{'} + j_1^{'})$ or $i_1^{'}(j^{''} + j_1^{''})$ which shows that $IJ \nsubseteq P$. ∎

From [3], a subset M of N is called m-system if $a, b \in M$, then there exist $a_1 \in< a >$ and $b_1 \in< b >$ such that $a_1 b_1 \in M$. A subset M of N is called weakly m-system if $M \cap A \neq \phi$ and $M \cap B \neq \phi$ for any ideals A, B of N, then either $AB \cap M \neq \phi$ or $AB = 0$. Clearly every m-system is a weakly m-system, but a weakly m-system need not be a m-system, since in Example 1.1, $M = \{1, 2, 3, a, b, d\}$ is a weakly m-system, but not a m-system since $x_1 x_2 \notin M$ for all $x_1, x_2 \in< 2 >$. It is clear that, an ideal P of N is weakly prime if and only if $N \backslash P$ is weakly m- system. A well known result that, if M is a non-void m-system of N and I is an ideal of N with $I \cap M = \phi$, then there exist a prime ideal $P \neq N$ containing I with $P \cap M = \phi$. A similar result does hold for weakly m-system.

Theorem 2.6. *Let $M \subseteq N$ be a non-void weakly m-system in N and I an ideal of N with $I \cap M = \phi$. Then I is contained in a weakly prime ideal $P \neq N$ with $P \cap M = \phi$.*

Proof: Let $\mathbb{A} = \{J : J$ is an ideal of N with $J \cap M = \phi\}$. Clearly $I \in \mathbb{A}$. Then by Zorn's Lemma, $\mathbb{A}$ contains a maximal element (say) P with $P \cap M = \phi$. We show that P is weakly prime ideal of N. Let A and B be ideals of N with

$P \subset A$ and $P \subset B$. Then by maximality of $\mathbb{A}$, $A \cap M \neq \phi$ and $B \cap M \neq \phi$. Since M is weakly m-system, we have $AB = 0$ or $AB \cap M \neq \phi$; that is $AB = 0$ or $AB \nsubseteq P$ since $P \cap M = \phi$. So by Theorem 2.5, P is weakly prime ideal of N and also containing I. ■

Theorem 2.7. *Let N be a decomposable near-ring with identity. If P is a weakly prime ideal of N, then either $P = 0$ or P is prime.*

Proof: Suppose that $N = N_1 \times N_2$ and let $P = P_1 \times P_2$ be a weakly prime ideal of N. We may assume that $P \neq 0$. Now, let A be a non-zero ideal of N_1 and B be a non-zero ideal of N_2 such that $0 \neq (A, B) \subseteq P$. Then $0 \neq (A, N_2)(N_1, B) \subseteq P$ which implies $(A, N_2) \subseteq P$ or $(N_1, B) \subseteq P$. Suppose that $(A, N_2) \subseteq P$. Then $(0, N_2) \subseteq P$ and so $P = P_1 \times N_2$. We show that P_1 is a prime ideal of N_1. Let A_1 and B_1 be ideals of N_1 such that $A_1B_1 \subseteq P_1$. Then $(0, 0) \neq (A_1, N_2)(B_1, N_2) = (A_1B_1, N_2) \subseteq P$, so $(A_1, N_2) \subseteq P$ or $(B_1, N_2) \subseteq P$ and hence $A_1 \subseteq P_1$ or $B_1 \subseteq P_1$. So P is prime ideal of N. The case where $(N_1, B) \subseteq P$ is similar. ■

References

[1] D. D. Anderson and E. Smith, Weakly prime ideals, *Houston J. Math.*, **29** no. 4 (2003), 831-840.

[2] F. Binder and C. Nöbauer, *Table of All Nearrings with Identity Up to Order 15.*, http://verdi.algebra.uni-linz.ac.at/Sonata/encyclo/ (14 June 2003).

[3] G. Pilz, *Near-Rings*, North-Holland, Amsterdam, 1983.