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Variants of Chebyshev's Method with Optimal Order of Convergence*

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Abstract

In this paper, we derive a one-parameter family of Chebyshev's method for finding simple roots of nonlinear equations. Further, we present a new fourth-order variant of Chebyshev's method from this family without adding any functional evaluation to the previously used three functional evaluations. Chebyshev-Halley type methods are seen as the special cases of the proposed family. New classes of higher (third and fourth) order multipoint iterative methods free from second-order derivative are also derived by semi-discrete modifications of cubically convergent methods. Fourth-order multipoint iterative methods are optimal, since they require three functional evaluations per step. The new methods are tested and compared with other well-known methods on the number of problems.

Keywords and Phrases: *Nonlinear equations, Newton's method, Chebyshev's method, Chebyshev-Halley type methods, Multipoint methods, Traub-Ostrowski's method, Jarratt's method, Optimal order of convergence.*

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1. Introduction

One of the most important and challenging problem in computational mathematics is to compute approximate solutions of the nonlinear equation

$$f(x) = 0. \tag{1}$$

Newton's method for multiple roots appears in the work of Schröder [1], which is given as

$$x_{n+1} = x_n - \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}. \tag{2}$$

This method has quadratic convergence, including the case of simple root. Another well-known third-order modification of Newton's method is the classical Chebyshev's method [2, 3], given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\{f(x_n)\}^2 f''(x_n)}{\{f'(x_n)\}^3}. \tag{3}$$

In this paper, we obtain a new general family of Chebyshev's method for finding simple roots of nonlinear equations numerically. The classical Chebyshev-Halley methods [4] are obtained as the particular cases of our proposed scheme. Further, we have also developed a new fourth order variant of Chebyshev's method. The beauty of this method is that it uses the same number of functional evaluations as that of classical Chebyshev's method. Therefore, the efficiency of our proposed fourth-order method in terms of functional evaluations is better than the existing classical Chebyshev's method. Furthermore, we also develop two new optimal fourth-order multipoint methods free from second-order derivative.

2. Family of Chebyshev's Method and Convergence Analysis

Let r be the required root of equation (1) and $x = x_0$ be the initial guess known for the required root. Assume

$$x_1 = x_0 + h, \quad |h| << 1, \tag{4}$$

be the first approximation to the root. Therefore

$$f(x_1) = 0. \tag{5}$$

Expanding the function $f(x_1)$ by Taylor's theorem about x_0 and retaining the terms up to $O(h^2)$, we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) = 0. \tag{6}$$

Further simplifying, we get

$$h = -\frac{f(x_0)}{f'(x_0)} - \frac{h^2}{2}\frac{f''(x_0)}{f'(x_0)}. \tag{7}$$

Approximating h on the right-hand side of equation (7) by the correction term $-\frac{f(x_n)f''(x_n)}{\{f'(x)\}^2 - af(x_n)f''(x_n)}$, $a \in \mathbb{R}$ (free disposable parameter) given in formula (2), we obtain

$$h = -\frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{\{f(x_0)\}^2 f'(x_0) f''(x_0)}{\left[\{f'(x_0)\}^2 - af(x_0)f''(x_0)\right]^2}. \tag{8}$$

Thus the first approximation to the required root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{\{f(x_0)\}^2 f'(x_0) f''(x_0)}{\left[\{f'(x_0)\}^2 - af(x_0)f''(x_0)\right]^2}. \tag{9}$$

Therefore, the general formula for successive approximations can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\{f(x_n)\}^2 f'(x_n) f''(x_n)}{\left[\{f'(x_n)\}^2 - af(x_n)f''(x_n)\right]^2},$$

or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\{f(x_n)\}^2 f'(x_n) f''(x_n)}{\{f'(x_n)\}^4 + a^2\{f(x_n)\}^2\{f''(x_n)\}^2 - 2af(x_n)\{f'(x_n)\}^2 f''(x_n)}. \tag{10}$$

This formula looks like a Chebyshev's formula and describes the one-parameter family of Chebyshev's method. Note that for $a = 0$ in (10), one can immediately recover the classical Chebyshev's formula.

Special cases

I. *Chebyshev-Halley type methods*

If we remove the term $\left(a^2\{f(x_n)\}^2\{f''(x_n)\}^2\right)$ from the denominator: $\{f'(x)\}^4 + a^2\{f(x_n)\}^2\{f''(x_n)\}^2 - 2af(x_n)\{f'(x_n)\}^2 f''(x_n)$ in formula (10), we obtain a family of methods defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\{f(x_n)\}^2 f''(x_n)}{\{f'(x)\}^3 - 2af(x_n)f'(x_n)f''(x_n)}.$$

This family resembles with the well-known cubically convergent family of Chebyshev-Halley type methods [4].

II. *Another new cubically convergent family of methods*

If we remove the term $\left(-2af(x_n)\{f'(x_n)\}^2 f''(x_n)\right)$ from the denominator: $\{f'(x)\}^4 + a^2\{f(x_n)\}^2\{f''(x_n)\}^2 - 2af(x_n)\{f'(x_n)\}^2 f''(x_n)$ in formula (10), we obtain a family of methods defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\{f(x_n)\}^2 f'(x_n) f''(x_n)}{\{f'(x)\}^4 + a^2\{f(x_n)\}^2\{f''(x_n)\}^2}.$$

It is investigated that this family is also cubically convergent for all $a \in \mathbb{R}$.

Theorem 2.1 *Assume that $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $r \in D$. Let $f(x)$ be sufficently smooth in the neighborhood of the root r, then the order of convergence of the methods defined by family* (10) *is three for every value of $a \in \mathbb{R}$.*

Proof. Let e_n be the error at the n^{th} iteration, then $e_n = x_n - r$. Expanding $f(x_n)$ and $f'(x_n)$ about r and using the fact that $f(r) = 0$, $f'(r) \neq 0$, we have

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5 + O(e_n^5)], \tag{11}$$

where $c_k = \frac{1}{k!}\frac{f^k(r)}{f'(r)},\ k = 2, 3, \ldots$ Furthermore, we have

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)], \tag{12}$$

and

$$f''(x_n) = f'(r)[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + O(e_n^4)]. \tag{13}$$

Then

$$\frac{f(x_n)}{f'(x_n)} = [e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(-4c_2^3 + 7c_2c_3 - 3c_4\right)e_n^4 + O(e_n^5)], \tag{14}$$

and

$$\left\{\frac{f(x_n)}{f'(x_n)}\right\}^2 \frac{f''(x_n)}{f'(x_n)} = [2c_2e_n^2 + \big(6c_3 - 8c_2^2\big)e_n^3 + O(e_n^4)]. \tag{15}$$

Using (14) and (15) in equation (10) and simplifying, we get

$$e_{n+1} = \big\{2c_2^2(1-2a) - c_3\big\}e_n^3 + O(e_n^4). \tag{16}$$

Therefore, it can be concluded that for all $a \in \mathbb{R}$, the family (10) converges cubically. For $a = \frac{1}{2}$, error equation (16) reduces to

$$e_{n+1} = -c_3e_n^3 + O(e_n^4). \tag{17}$$

3. Fourth-order Variant of a Chebyshev's Method and Convergence Analysis

Here we intend to develop a new optimal fourth-order variant of Chebyshev's method. This method is very interesting because it has very higher order of convergence and computational efficiency unlike Chebyshev's method.
Considering the Newton-like iterative method with a parameter $\alpha \in \mathbb{R}$

$$y_n = x_n - \alpha\frac{f(x_n)}{f'(x_n)}. \tag{18}$$

We now modify family (10) of Chebyshev's method by using the second-order derivative at y_n instead of x_n and obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\{f(x_n)\}^2 f'(x_n)f''(y_n)}{\big[\{f'(x)\}^2 - af(x_n)f''(y_n)\big]^2}. \tag{19}$$

Obviously, when we take $(a,\ \alpha) = (0,\ 0)$, we get classical Chebyshev's method.

Theorem 3.1 *Assume that $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $r \in D$. Let $f(x)$ be sufficently smooth in the neighborhood of the root r, then the order of convergence of the method defined by formula* (19) *is of order four if $(a,\ \alpha) = \big(\frac{1}{2},\ \frac{1}{3}\big)$.*

Proof. The proof of said convergence of method (19) can be proved on similar lines as in the Theorem (2.1). Expanding $f''(y_n) = f\left(x_n - \alpha\frac{f(x_n)}{f'(x_n)}\right)$ about $x = r$, we have

$$\begin{aligned} f''(y_n) =& f'(r)[2c_2 + 6c_3(1-\alpha)e_n + (6\alpha c_2c_3 + 12c_4(1-\alpha)^2e_n^2) + (12\alpha c_3(c_3 - c_2^2) \\ &+ 24c_2c_4\alpha(1-\alpha) + 20c_5(1-\alpha)^3)e_n^3 + O(e_n^4)]. \end{aligned} \tag{20}$$

Using (11), (12) and (20) in formula (19) and simplifying, we get the final error equation as

$$\begin{aligned} e_{n+1} =& \{(1-2a)2c_2^2 - (1-3\alpha)c_3\}e_n^3 + \{(28a - 12a^2 - 9)c_2^3 \\ &+ (12 - 24a - 15\alpha + 24a\alpha)c_2c_3 - (3 - 12\alpha + 6\alpha^2)c_4\}e_n^4 + O(e_n^5). \end{aligned} \tag{21}$$

For the method to be of fourth-order convergence, we must have

$$1 - 2a = 0 \text{ and } 1 - 3\alpha = 0,$$

which implies

$$a = \frac{1}{2} \text{ and } \alpha = \frac{1}{3}. \tag{22}$$

Using (22) in equation (21), we obtain the following error equation for fourth-order variant as

$$e_{n+1} = \left(2c_2^3 - c_2c_3 + \frac{1}{3}c_4\right)e_n^4 + O(e_n^5). \tag{23}$$

The efficiency index [2] of the present method is equal to $\sqrt[3]{4} \cong 1.587$, which is better than the ones of classical Chebyshev's method $\sqrt[3]{3} \cong 1.442$ and Newton's method $\sqrt[2]{2} \cong 1.414$ respectively. Therefore, this method is very interesting because it has higher order of convergence and computational efficiency than Chebyshev's method.

4. Families of Multipoint Iteration Methods and their Convergence Analysis

The practical difficulty associated with the above mentioned methods given by (10) or (19) may be the evaluation of second-order derivative. Recently, some new variants of Newton's method free from second-order derivative have been developed in [2, 3, 5, 6, 7, 8, 9] and the references cited theirin by discretization of second-order derivative or by predictor-corrector approach or by considering different quadrature formulae for the computation of integral arising from Newton's theorem. These multipoint methods are of great practical importance since they overcome the limitations of one-point methods regarding the convergence order and computational efficiency. According to Kung-Traub conjecture [9], the order of convergence of any multipoint method without memory consuming function evaluations per iteration, can not exceed the bound (called optimal order). Thus, the optimal order for a method with three functional evaluations per step would be four. Traub-Ostrowski's method [2, 3], Jarratt's method [5], King's method [6] and Maheswari's method [7] etc. are famous optimal fourth order methods, because they require three functions evaluations per step. Nowadays, obtaining new optimal methods of order four is still important, because they have very high efficiency index.

Here, we also intend to develop new fourth-order multipoint methods free from second-order derivative. The main idea of proposed methods lies in the discretization of second-order derivative involved in family (10) of Chebyshev's method.

a. First family

Expanding the function $f(x_n - \beta u),\ \beta \neq 0 \in \mathbb{R}$ but finite, about the point $x = x_n$ with $f(x_n) \neq 0$, we have

$$f(x_n - \beta u) = f(x_n) - \beta u f'(x_n) + \frac{\beta^2 u^2}{2!} f''(x_n) + O(e_n^3). \tag{24}$$

Let us take $u = \frac{f(x_n)}{f'(x_n)}$, and inserting this into (24), we obtain

$$f(x_n) f''(x_n) \approx \frac{2\{f'(x_n)\}^2}{\beta^2 f(x_n)} \{f(x_n - \beta u) - (1-\beta) f(x_n)\}. \tag{25}$$

Using the approximate value of $f(x_n)f''(x_n)$ into formula (10), we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left[1 + \beta^2 f(x_n) \frac{\{f(x_n - \beta u) - (1-\beta) f(x_n)\}}{\{(\beta^2 + 2a(1-\beta)) f(x_n) - 2a f(x_n - \beta u)\}^2}\right]. \tag{26}$$

Special cases

For different specific values of parameters a and α, the following various

multipoint methods can be deduced from (26), e.g.
i. For $(a,\ \beta) = \left(-\frac{1}{2},\ 1\right)$, we get the new formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(x_n)f(x_n-u)}{\{f(x_n)+f(x_n-u)\}^2}\right]. \tag{27}$$

ii. For $(a,\ \beta) = (-1,\ 1)$, we get the new formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(x_n)f(x_n-u)}{\{f(x_n)+2f(x_n-u)\}^2}\right]. \tag{28}$$

iii. For $(a,\ \beta) = \left(\frac{1}{2},\ 1\right)$, we get the new formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(x_n)f(x_n-u)}{\{f(x_n)-f(x_n-u)\}^2}\right]. \tag{29}$$

Note that the family (26) can produce many more new multipoint methods by choosing different values of the parameters.

b. Second family

Replacing the second-order derivative in (10) by the following definition

$$f''(x_n) \approx \frac{f'(x_n) - f'(x_n - \beta u)}{\beta u},\ \ \beta \neq 0 \in \mathbb{R},$$

we get the following new family as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{1}{2}\frac{\beta f'(x_n)\{f'(x_n)-f'(x_n-\beta u)\}}{\{(\beta-a)f'(x_n)+af'(x_n-\beta u)\}^2}\right]. \tag{30}$$

Special cases

For different specific values of parameters a and β, the following various multipoint methods can be obtained from (30), e.g.
i. For $(a,\ \beta) = (1,\ 1)$, we get the new formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f'(x_n)\{f'(x_n)-f'(x_n-u)\}}{2\{f'(x_n-u)\}^2}\right]. \tag{31}$$

ii. For $(a,\ \beta) = \left(\frac{1}{2},\ \frac{2}{3}\right)$, we get the new formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{12f'(x_n)\left\{f'(x_n)-f'\left(x_n-\frac{2}{3}u\right)\right\}}{\left\{f'(x_n)+3f'\left(x_n-\frac{2}{3}u\right)\right\}^2}\right]. \tag{32}$$

Other modifications can be obtained from formula (10) by replacing the second-order derivative by other finite difference approximations.

The order of convergence of family (26) and (30) will be studied in Theorem 4.1 in the subsequent section.

Theorem 4.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous and sufficiently differentiable function defined in D. If $f(x)$ has a simple root $r \in D$, then for sufficiently close initial guess x_0 to r,*

(i) the family (26) has 3^{rd} order of convergence, for

$$a \neq \frac{1}{2} \ \&\ \beta = 1, \quad a = \frac{1}{2} \ \&\ \beta \neq 1, \quad a \neq \frac{1}{2} \ \&\ \beta \neq 1,$$

and 4^{th} order of convergence for $a = \frac{1}{2}$ & $\beta = 1$.

(ii) the family (30) has 3^{rd} order of convergence, for

$$a \neq \frac{1}{2} \ \&\ \beta = \frac{2}{3}, \quad a = \frac{1}{2} \ \&\ \beta \neq \frac{2}{3}, \quad a \neq \frac{1}{2} \ \&\ \beta \neq \frac{2}{3},$$

and 4^{th} order of convergence for $a = \frac{1}{2}$ & $\beta = \frac{2}{3}$.

Proof. Since $f(x)$ is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ about $x = r$ by Taylor's expansion, we have

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^5)], \tag{33}$$

and

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)], \tag{34}$$

where c_k and e_n are defined earlier.

$$f(x_n-\beta u) = (1-\beta)e_n+\{(1-\beta+\beta^2)c_2\}e_n^2-\{2\beta^2c_2^2-((1-\beta)^3+2\beta)c_3\}e_n^3+O(e_n^4). \tag{35}$$

Using symbolic computation in the programming package Mathematica, we get the following error equation for the family (26):

$$\begin{aligned} e_{n+1} =& \{2(1-2a)c_2^2 - (1-\beta)c_3\}e_n^3 + \{(28a - 12a^2 - 9)c_2^3 \\ &+ (12 - 24a - 5\beta + 8a\beta)c_2c_3 - (3 - 4\beta + \beta^2)c_4\}e_n^4 + O(e_n^5). \end{aligned} \tag{36}$$

For $a = \frac{1}{2}$ and $\beta = 1$, in equation (36), we get

$$e_{n+1} = (2c_2^3 - c_2c_3)e_n^4 + O(e_n^5). \tag{37}$$

Similarly for scheme (30), we have the following error equation

$$e_{n+1} = \left\{2(1-2a)c_2^2 - \left(1 - \frac{3\beta}{2}\right)c_3\right\}e_n^3 + \left\{(28a - 12a^2 - 9)c_2^3 + \left(12 - 24a - \frac{15\beta}{2} + 12a\beta\right)c_2c_3 - (3 - 6\beta + 2\beta^2)c_4\right\}e_n^4 + O(e_n^5). \tag{38}$$

For $a = \frac{1}{2}$ and $\beta = \frac{2}{3}$, in equation (36), we get

$$e_{n+1} = \left(2c_2^3 - c_2c_3 + \frac{1}{9}c_4\right)e_n^4 + O(e_n^5). \tag{39}$$

5. Numerical Results

In this section, we shall present the numerical results obtained by employing the methods namely Newton's method (NM), Chebyshev's method (CM), cubically convergent variant of Chebyshev's method (10) for $a = 1$ (CVCM) and quartically convergent variant of Chebyshev's method (19) (QVCM) respectively to solve the nonlinear equations given in Table 1. The results are summarized in Table 2. We also compare Newton's method (NM), Traub-Ostrowski's method (TOM), Jarratt's method (JM), Maheswari's method (MM) with our optimal multipoint methods (29) (MTOM) and (32) (MJM) introduced in this contribution. The results are summarized in Table 3. Computations have been performed using C^{++} in double precision arithmetic. We use $\epsilon = 10^{-15}$. The following stopping criteria are used for computer programs:

$$(i)\ |x_{n+1} - x_n| < \epsilon, \quad (ii)\ |f(x_{n+1})| < \epsilon.$$

Table 1: Test Problems

No	Problems	[a, b]	Initial guess	Root (r)
1.	$e^x - 4x^2 = 0$	[0.5, 2]	0.5	0.714805901050568
			2.0	
2.	$x^3 + 4x^2 - 10 = 0$	[1, 2]	1.0	1.3652300134140969
			2.0	
3.	$\cos x - x = 0$	[0, 2]	0.0	0.7390851332151600
			2.0	
4.	$x^2 - e^x - 3x + 2 = 0$	[0, 1]	0.0	0.000000000000000
			1.0	
5.	$xe^{x^2} - \sin x^2 + 3\cos x + 5 = 0$	$[-1.5, \ -0.5]$	-1.5	1.207647800445557
			-0.5	
6.	$\sin^2 x - x^2 + 1 = 0$	[1, 3]	1.0	1.404491662979126
			3.0	
7.	$e^{x^2+7x-30} - 1 = 0$	[2.9, 3.5]	2.9	3.000000000000000
			3.5	

Table 2: Results of problems (Number of Iterations)

Problem	Initial guess	NM	CM	CVCM	QVCM
1.	0.5	4	3	3	2
	2.0	5	4	4	3
2.	1.0	4	3	3	2
	2.0	4	3	3	2
3.	0.0	4	3	3	3
	2.0	3	3	3	2
4.	0.0	3	2	2	2
	1.0	3	3	2	2
5.	-1.5	5	3	4	3
	-0.5	9	Divergent	7	5
6.	1.0	5	4	4	3
	3.0	5	4	4	3
7.	2.9	6	Divergent	5	3
	3.5	11	7	9	6

Table 3: Results of problems (D below-stands for divergent)

Number of iterations

Problem	Initial guess	NM	TOM	JM	MM	MTOM	MJM
1.	0.5	4	2	2	3	2	3
	2.0	5	3	3	3	3	3
2.	1.0	4	2	2	3	2	3
	2.0	4	2	2	3	2	3
3.	0.0	4	2	3	3	3	3
	2.0	3	2	2	2	2	2
4.	0.0	3	2	2	3	2	2
	1.0	3	2	2	2	2	2
5.	-1.5	5	2	3	3	3	3
	-0.5	9	4	3	D	6	7
6.	1.0	5	3	3	4	3	3
	3.0	5	3	3	3	3	3
7.	2.9	6	3	3	36	3	4
	3.5	11	5	5	6	6	6

6 . Conclusions

In this paper, we obtained a new simple and elegant root-finding family of Chebyshev's method. Chebyshev-Halley type methods are seen as the special cases of our proposed family. Furthermore, we presented a new fourth-order variant of Chebyshev's method. Then we introduced two new multipoint optimal methods of order four. The additional advantage of the presented multipoint methods is similar to that of Traub-Ostrowski's method, Jarratt's method etc. because they do not require the computation of second-order derivative to reach such a high convergence order. Finally, we provide numerical tests showing that these methods are equally competitive to other methods available in literature for finding simple roots of nonlinear equations.

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