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An Inequality of Ostrowski's Type for Preinvex Functions with Applications *

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Abstract

An inequality of Ostrowski's type for preinvex functions is introduced. Applications to some special means are considered.

Keywords and Phrases: *Preinvex functions, Invex functions, Ostrowski's inequality, Means.*

1. Introduction

Let K be a nonempty closed set in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions. Let $x \in K$. Then the set K is said to be invex at x with respect to $\eta(\cdot, \cdot)$, if

$$x + t \cdot \eta(y, x) \in K, \ \ \forall x, y \in K, \quad t \in [0, 1].$$

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K is said to be an invex set with respect to η, if K is invex at each $x \in K$. The invex set K is also called a η-connected set. For the sake of simplicity, we always assume that $K = [a, a + \eta(b, a)]$, unless otherwise specified.

Definition 1. *[11]* The function f on the invex set K is said to be preinvex with respect to

$$f(x + t \cdot \eta(y, x)) \leq (1 - t) f(x) + t f(y), \quad \forall x, y \in K, \quad t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It may be noted that every convex function is a preinvex function, but the converse is not true [2].

Definition 2. *[11]* The differentiable function f on the invex set K is said to be an invex function with respect to $\eta(y, x)$, if

$$f(y) - f(x) \geq \langle f'(x), \eta(y, x) \rangle, \quad \forall x, y \in K,$$

where $f'(x)$ is the differential of f at x, and we denote $\langle \cdot, \cdot \rangle$ to the inner product.

It is clear that the differentiable preinvex functions are invex and the converse is also true under certain conditions, see [11, 12]. Also, it is true that every convex set is invex with respect to $\eta(y, x) = y - x$, but the converse may not be true [7]. Extensive work has been reported in the literature on generalized convex functions see [2, 3, 6, 10].

In the recent paper, Noor [8] has obtained the following Hermite-Hadamard inequalities for the preinvex.

Theorem 1. *Let $f : K := [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then,*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)\, dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}$$

Other results connected with (1.1) were established by Noor [9], where he introduce a new inequalities involving two log-preinvex.

The main concern of this paper is to establish an Ostrowski's type inequality for differentiable functions whose derivative in absolute value is preinvex functions.

2. Ostrowski Type Inequalities

In 1938, Ostrowski established a very interesting inequality for differentiable functions with bounded derivatives, as follows: Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \le M$. Then,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \le M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \tag{2.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller ones. For recent results and generalizations concerning Ostrowski's inequality we refer the reader to the comprehensive book [5].

In order to prove our main inequality which is of Ostrowski's type, we need the following lemma:

Lemma 1. *[1] Let $f : K := [a, a + \eta(b,a)] \to \mathbb{R}$ be a differentiable function on K° (the interior of K) and $a, b \in K^{\circ}$ with $a < a + \eta(b,a)$. If f' is preinvex and $f' \in L[a,b]$. Then,*

$$\begin{aligned} & f(2a - x + \eta(b,a)) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx \\ & = \eta(b,a) \cdot \int_0^1 p(t)\, f'(a + t \cdot \eta(b,a))\, dt, \end{aligned} \tag{2.2}$$

where,

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{a+\eta(b,a)-x}{\eta(b,a)}\right] \\ t-1, & t \in \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}, 1\right] \end{cases},$$

for each $x \in K$.

Proof. Let $I = \int_0^1 p(t)\, f(ta + t \cdot \eta(b,a))\, dt$. Using integration by parts, we get

$$
\begin{aligned}
I &= \int_0^1 p(t)\, f'(a+t\cdot\eta(b,a))\,dt \\
&= \int_0^{\frac{a+\eta(b,a)-x}{\eta(b,a)}} t f'(a+t\cdot\eta(b,a))\,dt + \int_{\frac{a+\eta(b,a)-x}{\eta(b,a)}}^1 (t-1) f'(a+t\cdot\eta(b,a))\,dt \\
&= \frac{1}{\eta(b,a)}\cdot\frac{a+\eta(b,a)-x}{\eta(b,a)}\cdot f\left(a+\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\cdot\eta(b,a)\right) \\
&\quad - \frac{1}{\eta(b,a)}\int_0^{\frac{a+\eta(b,a)-x}{\eta(b,a)}} f(a+t\cdot\eta(b,a))\,dt \\
&\quad + \frac{1}{\eta(b,a)}\cdot\frac{x-a}{\eta(b,a)}\cdot f\left(a+\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\cdot\eta(b,a)\right) \\
&\quad - \frac{1}{\eta(b,a)}\int_{\frac{a+\eta(b,a)-x}{\eta(b,a)}}^1 f(a+t\cdot\eta(b,a))\,dt \\
&= \frac{1}{\eta(b,a)} f(2a-x+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_0^1 f(a+t\cdot\eta(b,a))\,dt,
\end{aligned}
$$

i.e.,

$$
\eta(b,a)\cdot I = f(2a-x+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx,
$$

which completes the proof. □

The following theorem gives an Ostrowski type inequality for differentiable functions whose derivative in absolute value is preinvex.

Theorem 2. *Let $f : K := [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable function on K° (the interior of K) and $a, b \in K^\circ$ with $a < a+\eta(b,a)$. If $|f'|$ is a preinvex. Then,*

$$\left|f\left(2a-x+\eta\left(b,a\right)\right)-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(u\right)du\right|$$
$$\leq\left|\eta\left(b,a\right)\right|\cdot\left\{\left[\frac{3}{2}\cdot\left(\frac{a-x}{\eta\left(b,a\right)}+1\right)^{2}-\frac{2}{3}\cdot\left(1+\frac{a-x}{\eta\left(b,a\right)}\right)^{3}+\frac{x-a}{\eta\left(b,a\right)}-\frac{2}{3}\right]\left|f'\left(a\right)\right|\right.$$
$$\left.+\left[\frac{1}{6}-\frac{1}{2}\left(\frac{a-x}{\eta\left(b,a\right)}+1\right)^{2}+\frac{2}{3}\left(\frac{a-x}{\eta\left(b,a\right)}+1\right)^{3}\right]\left|f'\left(b\right)\right|\right\},\quad(2.3)$$

for each $x\in K$.

Proof. Form Lemma 1 and preinvexity of $|f'|$ we have

$$\left|f\left(2a-x+\eta\left(b,a\right)\right)-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(u\right)du\right|$$
$$=\left|\eta\left(b,a\right)\cdot\int_{0}^{1}p\left(t\right)f'\left(a+t\cdot\eta\left(b,a\right)\right)dt\right|$$
$$\leq\left|\eta\left(b,a\right)\right|\cdot\left[\int_{0}^{\frac{a+\eta(b,a)-x}{\eta(b,a)}}t\left|f'\left(a+t\cdot\eta\left(b,a\right)\right)\right|dt\right.$$
$$\left.+\int_{\frac{a+\eta(b,a)-x}{\eta(b,a)}}^{1}\left(1-t\right)\left|f'\left(a+t\cdot\eta\left(b,a\right)\right)\right|dt\right]$$
$$\leq\left|\eta\left(b,a\right)\right|\cdot\left\{\int_{0}^{\frac{a+\eta(b,a)-x}{\eta(b,a)}}t\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt\right.$$
$$\left.+\int_{\frac{a+\eta(b,a)-x}{\eta(b,a)}}^{1}\left(1-t\right)\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt\right\}$$
$$=\left|\eta\left(b,a\right)\right|\cdot\left\{\left[\frac{3}{2}\cdot\left(\frac{a-x}{\eta\left(b,a\right)}+1\right)^{2}-\frac{2}{3}\cdot\left(1+\frac{a-x}{\eta\left(b,a\right)}\right)^{3}+\frac{x-a}{\eta\left(b,a\right)}-\frac{2}{3}\right]\left|f'\left(a\right)\right|\right.$$
$$\left.+\left[\frac{1}{6}-\frac{1}{2}\left(\frac{a-x}{\eta\left(b,a\right)}+1\right)^{2}+\frac{2}{3}\left(\frac{a-x}{\eta\left(b,a\right)}+1\right)^{3}\right]\left|f'\left(b\right)\right|\right\},$$

which completes the proof. □

Corollary 1. *In Theorem 2, for*

(1) $x = a$, *we have:*

$$\left| f\left(a+\eta\left(b,a\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(u\right) du \right| \leq \frac{\left|\eta\left(b,a\right)\right|}{6} \cdot \left(2\left|f'\left(b\right)\right| + \left|f'\left(a\right)\right|\right), \quad (2.4)$$

(2) $x = \frac{2a+\eta(b,a)}{2}$, *we have:*

$$\left| f\left(\frac{2a+\eta\left(b,a\right)}{2}\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(u\right) du \right| \leq \frac{\left|\eta\left(b,a\right)\right|}{8} \cdot \left(\left|f'\left(a\right)\right| + \left|f'\left(b\right)\right|\right), \quad (2.5)$$

(3) $x = \eta\left(b,a\right)$, *we have:*

$$\begin{aligned} &\left| f\left(2a\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(u\right) du \right| \\ &\leq \left|\eta\left(b,a\right)\right| \cdot \left\{ \left[\frac{3}{2} \cdot \frac{a^2}{\eta^2\left(b,a\right)} - \frac{2}{3} \cdot \frac{a^3}{\eta^3\left(b,a\right)} + \frac{1}{3} - \frac{a}{\eta\left(b,a\right)}\right] \left|f'\left(a\right)\right| \right. \\ &\quad \left. + \left[\frac{1}{6} - \frac{1}{2} \cdot \frac{a^2}{\eta^2\left(b,a\right)} + \frac{2}{3} \cdot \frac{a^3}{\eta^3\left(b,a\right)}\right] \left|f'\left(b\right)\right| \right\}. \quad (2.6) \end{aligned}$$

3. Applications to Special Means

In the following we study certain generalizations of some notions for a positive-valued function of a positive variable.

Definition 3. *([4])* A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M\left(ax, ay\right) = aM\left(x, y\right)$, for all $a > 0$,

(2) Symmetry : $M\left(x, y\right) = M\left(y, x\right)$,

(3) Reflexivity : $M(x,x) = x$,

(4) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) \leq M(x',y')$,

(5) Internality: $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.

We shall consider some means for arbitrary positive real numbers α, β $(\alpha \neq \beta)$ [4].

(1) The arithmetic mean :

$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}.$$

(2) The geometric mean :

$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}.$$

(3) The harmonic mean :

$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$

(4) The power mean :

$$P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

(5) The identric mean:

$$I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean :

$$L := L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \ |\alpha| \neq |\beta|.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \; p \in \mathbb{R}\backslash\{-1,0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \le G \le L \le I \le A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function

$$M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to \mathbb{R}_+,$$

which is one of the above mentioned means, therefore one can obtained variant inequalities for these means as follows:

Setting $\eta(b,a) = M(b,a)$ and choose $x = 2a$ in (2.3), provided that $2a \lneqq \eta(b,a) = M(b,a)$, one can obtain the following interesting inequality involving means:

$$\begin{aligned}
&\left| f(M(b,a)) - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(u)\,du \right| \\
&\le |M(b,a)| \cdot \left\{ \left[\frac{3}{2} \cdot \left(1-\frac{a}{M(b,a)}\right)^2 - \frac{2}{3} \cdot \left(1-\frac{a}{M(b,a)}\right)^3 + \frac{a}{M(b,a)} - \frac{2}{3} \right] |f'(a)| \right. \\
&\qquad \left. + \left[\frac{1}{6} - \frac{1}{2}\left(1-\frac{a}{M(b,a)}\right)^2 + \frac{2}{3}\left(1-\frac{a}{M(b,a)}\right)^3 \right] |f'(b)| \right\}. \quad (3.1)
\end{aligned}$$

Letting $M := A, G, H, P_r, I, L, L_p$, we get the required inequalities, and the details are left to the interested reader.

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