

A Unified Presentation of the Dziok-Srivastava and the Owa-Srivastava Linear Operators and Associated with Subclass of Analytic Functions of Complex Order *

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Abstract

Motivated by the success of the familiar Dziok-Srivastava and the Owa-Srivastava linear operators, we introduce here a unified presentation of them. By means of this new linear operator, we then define and investigate a class of analytic functions. Finally, we determine coefficient estimates, sufficient condition in terms of coefficients, maximization theorem concerning of coefficients and radius problem of functions belonging to this class.

Keywords and Phrases: *Dziok-Srivastava Operator, Owa-Srivastava Operator, Subordination, Hadamard product, Generalized hypergeometric function, Complex order, maximization.*

1. Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are *analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the *Hadamard product* (or *convolution*) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad z \in \mathbb{U}.$$

For two analytic functions f and g , we say that the function $f(z)$ is subordinate to $g(z)$ in \mathbb{U} , and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in \mathbb{U} with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in \mathbb{U} , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

See also Duren [8].

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \dots, q$) and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\mathbb{Z}_0^- = \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, s$) the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is defined by

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \frac{z^k}{k!}$$

$$(q \leq s+1, \quad q, s \in \mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}).$$

Here, and in what follows, $(\kappa)_n$ denotes the Pochhammer symbol (or shifted factorial) defined, in terms of the Gamma function Γ , by

$$(\kappa)_n = \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = \begin{cases} 1 & n = 0, \kappa \neq 0 \\ \kappa(\kappa + 1) \dots (\kappa + n - 1) & n \in \mathbb{N}. \end{cases}$$

For the function

$$h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$$

the Dziok-Srivastava linear operator [5] (see also [6] $H_s^q(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$) is defined by the following Hadamard product (or convolution) :

$$\begin{aligned} H_s^q(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) f(z) &= h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \dots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1}} \frac{1}{(k-1)!} a_k z^k. \end{aligned}$$

For notational simplicity, we write

$$H_s^q(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) f(z) = H_s^q(\alpha_1, \beta_1; z) f(z).$$

The fractional derivative of order γ is defined [9], for a function f , by

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\gamma} d\xi \quad (0 \leq \gamma < 1)$$

where the function f is analytic in a simply-connected region of the complex z -plane containing the origin and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Using $D_z^\gamma f$ Owa and Srivastava [10] introduced the operator $\Omega^\gamma : \mathcal{A} \rightarrow \mathcal{A}$, which is known as extension of fractional derivative and fractional integral, as follows

$$\begin{aligned} \Omega^\gamma f(z) &= \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z), \quad \gamma \neq 2, 3, 4, \dots \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k. \end{aligned}$$

Note that $\Omega^0 f(z) = f(z)$.

We define the linear multiplier fractional differential operator $D_\lambda^{m,\gamma}(\alpha_1, \beta) f : \mathcal{A} \rightarrow \mathcal{A}$ by

$$D_\lambda^0(\alpha_1, \beta_1) f(z) = H_s^q(\alpha_1, \beta_1; z) * f(z)$$

$$\begin{aligned} D_\lambda^{1,\gamma}(\alpha_1, \beta_1) f(z) &= (1-\lambda)[\Omega^\gamma(H_s^q(\alpha_1, \beta_1; z) * f(z))] \\ &\quad + \lambda z[\Omega^\gamma(H_s^q(\alpha_1, \beta_1; z) * f(z))]' \end{aligned} \quad (1.2)$$

$$D_\lambda^{2,\gamma}(\alpha_1, \beta_1) f(z) = D_\lambda^\gamma(D_\lambda^{1,\gamma}(\alpha_1, \beta_1) f(z))$$

\vdots

$$D_\lambda^{m,\gamma}(\alpha_1, \beta_1) f(z) = D_\lambda^\gamma(D_\lambda^{m-1,\gamma}(\alpha_1, \beta_1) f(z)), \quad (1.3)$$

where γ and λ aren't zero at the same time and $m \in \mathbb{N}_0$.

If f is given by (1.1), then by (1.2) and (1.3), we see that

$$\begin{aligned} D_\lambda^{m,\gamma}(\alpha_1, \beta_1) f(z) &= z + \sum_{k=2}^{\infty} \left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} [1 + \lambda(k-1)] \right]^m \frac{(\alpha_1)_{k-1} \dots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1}} \frac{1}{(k-1)!} a_k z^k \quad (1.4) \end{aligned}$$

where $m \in \mathbb{N}_0$, $\lambda \geq 0$ and $0 \leq \gamma < 1$.

Thus, after some calculations we obtain

$$\alpha_1 D_\lambda^{m,\gamma}(\alpha_1, \beta_1) f(z) = z [D_\lambda^{m,\gamma}(\alpha_1, \beta_1) f(z)]' + (\alpha_1 - 1) D_\lambda^{m,\gamma}(\alpha_1, \beta_1) f(z).$$

We note that by specializing $b, A, B, \lambda, \gamma, \alpha, \beta$ and m , we obtain the following subclasses studied by various authors:

(i) For a choice of the parameter $m = 0$, the operator $D_\lambda^{0,\gamma}(\alpha_1, \beta_1)$ reduces to the Dziok-Srivastava operator [5].

(ii) For $\gamma = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, we get the operator introduced by Al-Oboudi [2].

(iii) For $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $\lambda = 0$ and $m = 1$, we get Owa-Srivastava fractional differential operator [10].

(iv) For $\gamma = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$ and $\lambda = 1$, we get Sălăgean differential operator [4].

For various investigations based upon the Dziok-Srivastava linear operator and Owa-Srivastava linear operator, the interested reader may be referred to many recent papers (see [5,6,10], [11,12,13]).

Using the operator $D_\lambda^{m,\gamma}(\alpha_1, \beta_1)$, we define following class. Let $\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b} \left(\frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta_1) f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta_1) f(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{U}, \quad (1.5)$$

where \prec denotes subordination, $b \neq 0$ is any complex number, A and B are arbitrary fixed numbers, $-1 \leq B < A \leq 1$ and $m \in \mathbb{N}_0$.

By specializing $b, A, B, \lambda, \gamma, \alpha, \beta$ and m we obtain several subclasses studied by various authors in earlier papers (see for details [7], [1] ; see also the references cited in each of these recent works).

We use Λ to denote the class of bounded analytic functions $w(z)$ in \mathbb{U} which satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$.

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$ and let*

$$G = \frac{(A - B)^2 |b|^2}{2(\Gamma_k - 1)B(A - B)\operatorname{Re}\{b\} + (1 - B^2)(\Gamma_k - 1)^2}, k = 2, 3, \dots, m - 1$$

$M = [G]$ (Gauss symbol), and $[G]$ is the greatest integer not greater than G .

(a) If $(A - B)^2 |b|^2 > 2(\Gamma_k - 1)B(A - B)\operatorname{Re}\{b\} + (1 - B^2)(\Gamma_k - 1)^2$, then

$$|a_j| \leq \frac{1}{\Psi_j(\Gamma_j)^m} \prod_{k=2}^j \frac{|(A - B)b - B(\Gamma_{k-1} - 1)|}{\Gamma_k - 1}, \quad (2.1)$$

for $j = 2, 3, \dots, M + 2$; and

$$|a_j| \leq \frac{1}{\Psi_j(\Gamma_j)^m} \prod_{k=2}^{M+3} \frac{|(A - B)b - B(\Gamma_{k-1} - 1)|}{\Gamma_k - 1}, \quad (2.2)$$

for $j > M + 2$.

(b) If $(A - B)^2 |b|^2 \leq 2(\Gamma_k - 1)B(A - B)\operatorname{Re}\{b\} + (1 - B^2)(\Gamma_k - 1)^2$, then

$$|a_j| \leq \frac{|(A - B)b|}{\Psi_j(\Gamma_j)^m} \quad j \geq 2, \quad (2.3)$$

where

$$\Gamma_k = \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} [1 + \lambda(k-1)] \quad \text{and} \quad \Psi_k = \frac{(\alpha_1)_{k-1} \dots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1}} \frac{1}{(k-1)!}. \quad (2.4)$$

Proof. Since $f(z) \in \mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$, (1.5) gives

$$\begin{aligned} & D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z) - D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z) \\ = & \left\{ [(A-B)b + B]D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z) - BD_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z) \right\} w(z) \end{aligned} \quad (2.5)$$

Now (2.5) may be written as

$$\begin{aligned} & \sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^k \\ = & \left\{ (A-B)bz + \sum_{k=2}^{\infty} [(A-B)b - B(\Gamma_k - 1)] (\Gamma_k)^m \Psi_k a_k z^k \right\} w(z), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \sum_{k=2}^j (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k \\ = & \left\{ (A-B)bz + \sum_{k=2}^{j-1} [(A-B)b - B(\Gamma_k - 1)] (\Gamma_k)^m \Psi_k a_k z^k \right\} w(z), \end{aligned}$$

Since $|w(z)| < 1$, we have

$$\begin{aligned} & \left| \sum_{k=2}^j (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k \right| \\ \leq & \left| (A-B)bz + \sum_{k=2}^{j-1} [(A-B)b - B(\Gamma_k - 1)] (\Gamma_k)^m \Psi_k a_k z^k \right|. \end{aligned} \quad (2.6)$$

It follows that

$$\begin{aligned} & \int_0^{2\pi} \left| \sum_{k=2}^j (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k \right|^2 d\theta \\ \leq & \int_0^{2\pi} \left| (A-B)bz + \sum_{k=2}^{j-1} [(A-B)b - B(\Gamma_k - 1)] (\Gamma_k)^m \Psi_k a_k z^k \right|^2 d\theta \end{aligned}$$

for $z = re^{i\theta}$ ($0 \leq r < 1$). This gives us that

$$\begin{aligned} & \sum_{k=2}^j (\Gamma_k)^{2m} (\Gamma_k - 1)^2 (\Psi_k)^2 |a_k|^2 r^{2k} + \sum_{k=j+1}^{\infty} |d_k|^2 r^{2k} \\ & \leq (A - B)^2 |b|^2 r^2 + \sum_{k=2}^{j-1} |(A - B)b - B(\Gamma_k - 1)|^2 (\Gamma_k)^{2m} (\Psi_k)^2 |a_k|^2 r^{2k}. \end{aligned}$$

Let $r \rightarrow 1^-$, then on simplification we obtain

$$\begin{aligned} & (\Gamma_j)^{2m} ((\Gamma_j)^m - 1)^2 (\Psi_j)^2 |a_j|^2 \\ & \leq (A - B)^2 |b|^2 + \sum_{k=2}^{j-1} \left\{ |(A - B)b - B(\Gamma_k - 1)|^2 - (\Gamma_k - 1)^2 \right\} (\Gamma_k)^{2m} (\Psi_k)^2 |a_k|^2, \end{aligned} \tag{2.7}$$

for $j \geq 2$. Now there may be following two cases :

(a) Let $(A - B)^2 |b|^2 > 2(\Gamma_k - 1)B(A - B)Re\{b\} + (1 - B^2)(\Gamma_k - 1)^2$, suppose that $j \leq M + 2$, then for $j = 2$, (2.7) gives

$$|a_2| \leq \frac{(A - B)|b|}{\Gamma_2^m (\Gamma_2 - 1) \Psi_2}$$

which gives (2.1) for $j = 2$. We establish (2.1) for $j \leq M + 2$, from (2.7), by mathematical induction. Suppose (2.1) is valid for $j = 2, 3, \dots, (k - 1)$. Then it follows from (2.7)

$$\begin{aligned} & (\Gamma_j)^{2m} (\Gamma_j - 1)^2 (\Psi_j)^2 |a_j|^2 \\ & \leq (A - B)^2 |b|^2 + \sum_{k=2}^{j-1} \left\{ |(A - B)b - B(\Gamma_k - 1)|^2 - (\Gamma_k - 1)^2 \right\} (\Gamma_k)^{2m} (\Psi_k)^2 |a_k|^2 \end{aligned}$$

$$\begin{aligned}
&\leq (A-B)^2|b|^2 + \sum_{k=2}^{j-1} \left\{ |(A-B)b - B(\Gamma_k - 1)|^2 - (\Gamma_k - 1)^2 \right\} (\Gamma_k)^{2m} (\Psi_k)^2 \\
&\quad \times \left\{ \frac{1}{(\Psi_k)^2 (\Gamma_k)^{2m}} \prod_{n=2}^k \frac{|(A-B)b - B(\Gamma_{n-1} - 1)|^2}{(\Gamma_n - 1)^2} \right\} \\
&= (A-B)^2|b|^2 + \left\{ |(A-B)b - B(\Gamma_2 - 1)|^2 - (\Gamma_2 - 1)^2 \right\} \frac{|(A-B)b|^2}{(\Gamma_2 - 1)^2} \\
&\quad + \left\{ |(A-B)b - B(\Gamma_3 - 1)|^2 - (\Gamma_3 - 1)^2 \right\} \frac{|(A-B)b|^2}{(\Gamma_2 - 1)^2} \frac{|(A-B)b - B(\Gamma_2 - 1)|^2}{(\Gamma_3 - 1)^2} \\
&\quad + \dots \\
&\quad + \left\{ |(A-B)b - B(\Gamma_{j-1} - 1)|^2 - (\Gamma_{j-1} - 1)^2 \right\} \prod_{k=2}^{j-1} \frac{|(A-B)b - B(\Gamma_{k-1} - 1)|^2}{(\Gamma_k - 1)^2}
\end{aligned}$$

Thus, we get

$$|a_j| \leq \frac{1}{\Psi_j(\Gamma_j)^m} \prod_{k=2}^j \frac{|(A-B)b - B(\Gamma_{k-1} - 1)|}{\Gamma_k - 1}$$

which completes the proof of (2.1).

Next, we suppose $j > M + 2$. Then (2.7) gives

$$\begin{aligned}
&(\Gamma_j)^{2m} (\Gamma_j - 1)^2 (\Psi_j)^2 |a_j|^2 \leq (A-B)^2 |b|^2 \\
&\quad + \sum_{k=2}^{M+2} \left\{ |(A-B)b - B(\Gamma_k - 1)|^2 - (\Gamma_k - 1)^2 \right\} (\Gamma_k)^{2m} (\Psi_k)^2 |a_k|^2 \\
&\quad + \sum_{k=M+3}^{j-1} \left\{ |(A-B)b - B(\Gamma_k - 1)|^2 - (\Gamma_k - 1)^2 \right\} (\Gamma_k)^{2m} (\Psi_k)^2 |a_k|^2
\end{aligned}$$

On substituting upper estimates for a_2, a_3, \dots, a_{M+2} obtained above, and simplifying, we obtain (2.2).

(b) Let $(A - B)^2|b|^2 \leq 2(\Gamma_k - 1)B(A - B)Re\{b\} + (1 - B^2)(\Gamma_k - 1)^2$, then it follows from (2.7)

$$(\Gamma_j)^{2m}((\Gamma_j)^m - 1)^2(\Psi_j)^2|a_j|^2 \leq (A - B)^2|b|^2(j \geq 2),$$

which proves (2.3).

The bounds in (2.1) are sharp for the functions $f(z)$ given by

$$D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z) = \begin{cases} z(1 + Bz)^{\frac{(A-B)b}{B\lambda}}; & B \neq 0 \\ \exp\left(\frac{Abz}{\lambda}\right); & B = 0 \end{cases}.$$

Also, the bounds in (2.3) are sharp for the functions $f_k(z)$ given by

$$D_\lambda^{m,\gamma}(\alpha_1, \beta)f_k(z) = \begin{cases} z(1 + Bz)^{\frac{(A-B)b}{B\lambda(k-1)}}; & B \neq 0 \\ \exp\left(\frac{Ab}{\lambda(k-1)}z^{k-1}\right); & B = 0 \end{cases}.$$

□

3. A Sufficient Condition for a Function to be in $\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$

Theorem 2. *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{k=2}^{\infty} (\Gamma_k)^m \{(1 - B)(\Gamma_k - 1) + (A - B)|b|\} \Psi_k |a_k| \leq (A - B)|b| \quad (3.1)$$

holds where Γ_k and Ψ_k are given by (2.4), then $f(z)$ belongs to

$$\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B).$$

Poorf. Suppose that the inequality (3.1) holds. Then we have for $z \in \mathbb{U}$

$$\begin{aligned}
& \left| D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f(z) - D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z) \right| \\
& \quad - \left| (A-B)bD_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z) - B[D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f(z) - D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z)] \right| \\
& = \left| \sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^k \right| \\
& \quad - \left| (A-B)bz + (A-B)b \sum_{k=2}^{\infty} (\Gamma_k)^m \Psi_k a_k z^k - B \sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^k \right| \\
& \leq \sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k |a_k| r^k - (A-B)|b|r + (A-B)|b| \sum_{k=2}^{\infty} (\Gamma_k)^m \Psi_k |a_k| r^k \\
& \quad + B \sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k |a_k| r^k
\end{aligned}$$

Letting $r \rightarrow 1^-$, then we have

$$\begin{aligned}
& \left| D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f(z) - D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z) \right| \\
& \quad - \left| (A-B)bD_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z) - B[D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f(z) - D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z)] \right| \\
& \leq \sum_{k=2}^{\infty} (\Gamma_k)^m [(1-B)(\Gamma_k - 1) + (A-B)|b|] \Psi_k |a_k| - (A-B)|b| \leq 0
\end{aligned}$$

Hence it follows that

$$\left| \frac{\frac{D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z)} - 1}{(A-B)b - B \left[\frac{D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f(z)} - 1 \right]} \right| < 1, \quad z \in \mathbb{U}$$

Letting

$$w(z) = \frac{\frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} - 1}{(A - B)b - B \left[\frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} - 1 \right]},$$

then $w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence we have

$$\frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} = \frac{1 + [B + b(A - B)]w(z)}{1 + Bw(z)},$$

which shows that $f(z)$ belong to $\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$. \square

4. Maximization of $|a_3 - \mu a_2^2|$

We shall need in our discussion the following lemma:

Lemma 1. ([3]) *Let $w(z) = \sum_{k=1}^{\infty} c_k z^k$ be analytic with $w(0) = 0$ and $|w(z)| < 1$ in \mathbb{U} . If μ is any complex number, then*

$$|c_2 - \mu c_1^2| \leq \max \{1, |\mu|\}. \quad (4.1)$$

Equality in (4.1) may be attained with the function $w(z) = z^2$ and $w(z) = z$ for $|\mu| < 1$ and $|\mu| \geq 1$, respectively.

Theorem 3. *If a function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{(A - B)|b|}{(\Gamma_2 - 1)(\Gamma_3 - 1)(\Gamma_3)^m \Psi_3} \max \{1, |d|\}, \quad (4.2)$$

where

$$d = (A - B)b\mu \frac{(\Gamma_3)^m}{(\Gamma_2)^{2m}} \frac{\Gamma_3 - 1}{(\Gamma_2 - 1)^2} \frac{\Psi_3}{(\Psi_2)^2} - \frac{(A - B)b - B(\Gamma_2 - 1)}{\Gamma_2 - 1}.$$

The result is sharp.

Poorf. Since $f(z) \in \mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$, we have

$$1 + \frac{1}{b} \left(\frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (4.3)$$

where $w(z) = \sum_{k=1}^{\infty} c_k z^k$ is analytic in \mathbb{U} satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. From we have (4.3)

$$\begin{aligned} w(z) &= \frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z) - D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)}{[(A-B)b + B] D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z) - B D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)} \\ &= \frac{\sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^{k-1}}{(A-B)b + \sum_{k=2}^{\infty} [(A-B)b - B(\Gamma_k - 1)] (\Gamma_k)^m \Psi_k a_k z^{k-1}} \\ &= \frac{\sum_{k=2}^{\infty} (\Gamma_k)^m (\Gamma_k - 1) \Psi_k a_k z^{k-1}}{(A-B)b} \left[1 + \frac{\sum_{k=2}^{\infty} [(A-B)b - B(\Gamma_k - 1)] (\Gamma_k)^m \Psi_k a_k z^{k-1}}{(A-B)b} \right]^{-1} \end{aligned}$$

and then comparing the coefficients of z and z^2 on both sides, we have

$$a_2 = \frac{(A-B)b}{(\Gamma_2)^m (\Gamma_2 - 1) \Psi_2} c_1$$

and

$$a_3 = \frac{(A-B)b}{(\Gamma_3 - 1)(\Gamma_3)^m \Psi_3} \left\{ c_2 + \frac{[(A-B)b - B(\Gamma_2 - 1)]}{(\Gamma_2 - 1)} c_1^2 \right\}.$$

Hence,

$$a_3 - \mu a_2^2 = \frac{(A-B)b}{(\Gamma_2 - 1)(\Gamma_3 - 1)(\Gamma_3)^m \Psi_3} \max \{c_2 - d c_1^2\}, \quad (4.4)$$

where

$$d = (A-B)b\mu \frac{(\Gamma_3)^m}{(\Gamma_2)^{2m}} \frac{\Gamma_3 - 1}{(\Gamma_2 - 1)^2} \frac{\Psi_3}{(\Psi_2)^2} - \frac{[(A-B)b - B(\Gamma_2 - 1)]}{(\Gamma_2 - 1)}$$

Taking modulus both sides in (4.4), we have

$$|a_3 - \mu a_2^2| \leq \frac{(A - B)|b|}{(\Gamma_2 - 1)(\Gamma_3 - 1)(\Gamma_3)^m \Psi_3} |c_2 - dc_1^2|. \quad (4.5)$$

Using Lemma 1 in (4.5), we have

$$|a_3 - \mu a_2^2| \leq \frac{(A - B)|b|}{(\Gamma_2 - 1)(\Gamma_3 - 1)(\Gamma_3)^m \Psi_3} \max \{1, |d|\},$$

which is (4.2) of Theorem 3. \square

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of Lemma 1 is sharp.

5. Radius Theorem

Theorem 4 . *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$. Then*

$$\operatorname{Re} \left\{ \frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} \right\} > 0 \quad \text{for } |z| < r_m,$$

where

$$r_m = \frac{2}{|b|(A - B) + [|b|^2(A - B)^2 + 4B\{(A - B)\operatorname{Re}(b) + B\}]^{\frac{1}{2}}}, \quad (5.1)$$

such that

$$|b|^2(A - B)^2 + 4B\{(A - B)\operatorname{Re}(b) + B\} \geq 0. \quad (5.2)$$

The result is sharp.

Poorf. Since $f(z) \in \mathcal{H}_\lambda^{m,\gamma}(b, \alpha, \beta; A, B)$, we have

$$1 + \frac{1}{b} \left(\frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)} = P(z). \quad (5.3)$$

It is well know that every function in the class $P(A, B)$ is subordinate to $\frac{1+Az}{1+Bz}$ and the transformation

$$P(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

maps the circle $|w(z)| \leq 1$ onto the circle

$$\left| P(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2}. \quad (5.4)$$

Equations (5.3) and (5.4) yield

$$\left| \frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} - \frac{1 - B[B + b(A - B)]r^2}{1 - B^2r^2} \right| \leq \frac{|b|(A - B)r}{1 - B^2r^2}. \quad (5.5)$$

Therefore, from (5.5) we have

$$Re \left\{ \frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} \right\} \geq \frac{1 - |b|(A - B)r - B[(A - B)Re(b) + B]r^2}{1 - B^2r^2}. \quad (5.6)$$

Hence $Re \left\{ \frac{D_\lambda^{m+1,\gamma}(\alpha_1, \beta)f(z)}{D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z)} \right\} > 0$ for $|z| < r_m$ defined by (5.1).

To show (5.1) is sharp, we let

$$D_\lambda^{m,\gamma}(\alpha_1, \beta)f(z) = \begin{cases} z(1 + Bz)^{\frac{(A-B)b}{B\lambda}}; & B \neq 0 \\ z \exp\left(\frac{Abz}{\lambda}\right); & B = 0 \end{cases}$$

and

$$t = \frac{-r \left(Br + \left(\frac{\bar{b}}{b} \right)^{\frac{1}{2}} \right)}{1 + Br \left(\frac{\bar{b}}{b} \right)^{\frac{1}{2}}}$$

and obtain

$$\frac{D_{\lambda}^{m+1,\gamma}(\alpha_1, \beta)f_0(t)}{D_{\lambda}^{m,\gamma}(\alpha_1, \beta)f_0(t)} = \frac{1 - |b|(A - B)r - B[B + b(A - B)]r^2}{1 - B^2r^2}. \quad (5.7)$$

□

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