

Volume 29
No. 1
February
2013

Aletheia University
Tamsui, Taipei, Taiwan

## Tamsui Oxford Journal of Information and Mathematical Sciences

Publisher：Aletheia University
Editor－in－Chief：Jing－Ho Yan
Managing Editor：Hsien－Jen Lin ，Ching－Sheng Wang ，Shu－Ying Yeh

## Editorial Board：

## Gerard Jennhwa Chang（張鎮華）

Department of Mathematics
National Taiwan University，Taiwan gjchang＠math．ntu．edu．tw

Meng－Kiat Chuah（蔡孟傑）
Department of Mathematics
National Tsing Hua University，Taiwan
chuah＠math．nthu．edu．tw

## Chih－Yung Chang（張志勇）

Department Computer Science and
Information Engineering
Tamkang University，Taiwan cychang＠cs．tku．edu．tw

## Chin－Chen Chang（張真誠）

Department of Information Engineering and Computer Science Feng Chia University，Taiwan alan3c＠gmail．com

Yue－Cune Chang（張玉坤）
Department of Mathematics
Tamkang University，Taiwan
106300＠mail．tku．edu．tw

## S．S．Dragomir

School of Communication and Informatics
Victoria University of Technology，Australia Sever．Dragomir＠vu．edu．au

## Wen－Tao Huang（黃文濤）

Department of Management Sciences
Tamkang University，Taiwan
akenwt＠yahoo．com．tw

## Tea－Yuan Hwang（黃提源）

Department of Statistics and Actuarial
Science
Aletheia University，Taiwan
hwang＠stat．nthu．edu．tw

## Yau－Hwang Kuo（郭耀煌）

Department of Computer Science and
Information Engineering
National Cheng Kung University，Taiwan kuoyh＠ismp．csie．ncku．edu．tw

## Yao－Tsung Lai（賴耀宗）

Department of Statistics and Actuarial Science
Aletheia University，Taiwan
laiyt＠mail．au．edu．tw

| Ko－Wei Lih（李國偉） | Themistocles M．Rassias |
| :---: | :---: |
| Institute of Mathematics | Department of Mathematics |
| Academia Sinica，Taiwan makwlih＠sinica．edu．tw | National Technical University of Athens Zografou Campus |
| Hong－Yuan Mark Liao（廖弘源） Institute of Information Science | 15780 Athens，Greece trassias＠math．ntua．gr |
| Academia Sinica，Taiwan liao＠iis．sinica．edu．tw | Timothy K．Shih（施國琛） <br> Department of Computer Science |
| Hsien－Jen Lin（林顯仁） <br> Department of Applied Mathematics <br> Aletheia University，Taiwan | National Central University，Taiwan timothykshih＠gmail．com |
| au4409＠mail．au．edu．tw | Kuei－Lin Tseng（曾貴麟） <br> Department of Applied Mathematics |
| Shih－Kuei Lin（林士貴） | Aletheia University，Taiwan |
| Department of Money and Banking National Chengchi University，Taiwan | kltseng＠mail．au．edu．tw |
| square＠nccu．edu．tw | Chung－Shin Wang（王忠信） <br> Department of Computer Science and |
| Yu－Jen Lin（林裕仁） <br> Department of Industrial Engineering and Management <br> ST．John＇s University <br> lyj＠mail．sju．edu．tw | Information Engineering Aletheia University，Taiwan au1089＠mail．au．edu．tw |
|  | Ching－Sheng Wang（王慶生） |
| Josip Pečarić | Department of Computer Science and |
| Faculty of Textile Technology | Information Engineering |
| University of Zagrdb | Aletheia University，Taiwan |
| Prilaz baruna Filipovića 28 a， 10000 Zagreb， Croatia，Europe pecaric＠element．hr | cswang＠mail．au．edu．tw Jing－Ho Yan（顏經和） |
| H．M．Srivastava <br> Department of Mathematics and Statistics University of Victoria，Canada harimsri＠math．uvic．ca | Department of Applied Mathematics Aletheia University，Taiwan jhyan＠email．au．edu．tw |

Gou－Sheng Yang（楊國勝）
Department of Mathematics
Tamkang University，Taiwan
005490＠mail．tku．edu．tw

## Suh－Yuh Yang（楊肅煜）

Department of Mathematics
National Central University，Taiwan
syyang＠math．ncu．edu．tw

Sharon S．Yang（楊曉文）
Department of Finance
National Central University，Taiwan
syang＠ncu．edu．tw

## Gwo－Jong Yu（游國忠）

Department of Computer Science and Information Engineering Aletheia University，Taiwan yugj＠mail．au．edu．tw

## Shu－Ying Yeh（葉淑穎）

Department of Statistics and Actuarial Science
Aletheia University，Taiwan
ginger＠mail．au．edu．tw

Subscription: Tamsui Oxford Journal of Information and Mathematical Sciences (TOJIMS) is published quarterly in February, May, August and November. The current rate of subscription is US $\$ 30.00$ or NT $\$ 900.00$ per year.

Exchange: TOJIMS is intended to be exchanged with other scholarly journals in similar or closely-related fields published by departments, graduate schools or research centers of other universities all over the world.

## Submission of

Manuscripts:
Please read Information for Authors at the end of this issue.

Web site of
TOJIMS:
Full texts of past issues of TOJIMS can be found at http://www1.au.edu.tw/ox_view/edu/tojms/

## Other Information

Tamsui Oxford Journal of Information and Mathematical Sciences is abstracted/ indexed in Science and Technology Information Center, National Science Council of the Republic of China, Zentralblatt fur Mathematik, and MathSciNet.

# A Unified Presentation of the Dziok-Srivastava and the Owa-Srivastava Linear Operators and Associated with Subclass of Analytic Functions of Complex Order * 

Sevtap Sümer Eker ${ }^{\dagger}$<br>Department of Mathematics, Faculty of Science<br>Dicle University TR-21280 Diyarbakur, Turkey<br>Bilal Şeker ${ }^{\ddagger}$<br>Department of Mathematics, Faculty of Science and Letters<br>Batman University TR-72060 Batman, Turkey<br>and<br>Shigeyoshi Owa ${ }^{\S}$<br>Department of Mathematics, Kinki University<br>Higashi-Osaka, Osaka 577-8502, Japan

Received September 9, 2010, Accepted January 18, 2013.

[^0]
#### Abstract

Motivated by the success of the familiar Dziok-Srivastava and the Owa-Srivastava linear operators, we introduce here a unified presentation of them. By means of this new linear operator, we then define and investigate a class of analytic functions. Finally, we determine coefficient estimates, sufficient condition in terms of coefficients, maximization theorem concerning of coefficients and radius problem of functions belonging to this class.


Keywords and Phrases: Dziok-Srivastava Operator, Owa-Srivastava Operator, Subordination, Hadamard product, Generalized hypergeometric function, Complex order, maximization.

## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$
\begin{equation*}
f(z)=z+\sum_{k=2}^{\infty} a_{k} z^{k} \tag{1.1}
\end{equation*}
$$

which are analytic in the open unit disk $\mathbb{U}=\{z \in \mathbb{C}:|z|<1\}$.
Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$
g(z)=z+\sum_{k=2}^{\infty} b_{k} z^{k}
$$

the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$
(f * g)(z)=z+\sum_{k=2}^{\infty} a_{k} b_{k} z^{k}=(g * f)(z), z \in \mathbb{U}
$$

For two analytic functions $f$ and $g$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$
f(z) \prec g(z) \quad(z \in \mathbb{U})
$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ such that

$$
f(z)=g(w(z)) \quad(z \in \mathbb{U})
$$

In particular, if the function $g$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$
f(0)=g(0) \quad \text { and } \quad f(\mathbb{U}) \subset g(\mathbb{U})
$$

See also Duren [8].
For $\alpha_{j} \in \mathbb{C} \quad(j=1,2, \ldots q)$ and $\beta_{j} \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-} ; \mathbb{Z}_{0}^{-}=\{0,-1,-2, \ldots\} \quad(j=$ $1,2, \ldots s)$ the generalized hypergeometric function ${ }_{q} F_{s}\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right)$ is defined by

$$
\begin{aligned}
& { }_{q} F_{s}\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right)=\sum_{k=0}^{\infty} \frac{\left(\alpha_{1}\right)_{k} \ldots\left(\alpha_{q}\right)_{k}}{\left(\beta_{1}\right)_{k} \ldots\left(\beta_{s}\right)_{k}} \frac{z^{k}}{k!} \\
& \left(q \leq s+1, \quad q, s \in \mathbb{N}_{0}=\{0,1,2, \ldots\}=\mathbb{N} \cup\{0\}\right)
\end{aligned}
$$

Here, and in what follows, $(\kappa)_{n}$ denotes the Pochhammer symbol (or shifted factorial) defined, in terms of the Gamma function $\Gamma$, by

$$
(\kappa)_{n}=\frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}= \begin{cases}1 & n=0, \kappa \neq 0 \\ \kappa(\kappa+1) \ldots(\kappa+n-1) & n \in \mathbb{N}\end{cases}
$$

For the function

$$
h\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right)=z_{q} F_{s}\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right)
$$

the Dziok-Srivastava linear operator [5] (see also [6] $H_{s}^{q}\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right)$ is defined by the following Hadamard product (or convolution) :

$$
\begin{gathered}
H_{s}^{q}\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right) f(z)=h\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right) * f(z) \\
=z+\sum_{k=2}^{\infty} \frac{\left(\alpha_{1}\right)_{k-1} \ldots\left(\alpha_{q}\right)_{k-1}}{\left(\beta_{1}\right)_{k-1} \ldots\left(\beta_{s}\right)_{k-1}} \frac{1}{(k-1)!} a_{k} z^{k}
\end{gathered}
$$

For notational simplicity, we write

$$
H_{s}^{q}\left(\alpha_{1}, \ldots, \alpha_{q} ; \beta_{1}, \ldots, \beta_{s} ; z\right) f(z)=H_{s}^{q}\left(\alpha_{1}, \beta_{1} ; z\right) f(z)
$$

The fractional derivative of order $\gamma$ is defined [9], for a function $f$, by

$$
D_{z}^{\gamma} f(z)=\frac{1}{\Gamma(1-\gamma)} \frac{d}{d z} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^{\gamma}} d \xi \quad(0 \leq \gamma<1)
$$

where the function $f$ is analytic in a simply-connected region of the complex $z$-plane containing the origin and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log (z-\xi)$ to be real when $(z-\xi)>0$.

Using $D_{z}^{\gamma} f$ Owa and Srivastava [10] introduced the operator $\Omega^{\gamma}: \mathcal{A} \rightarrow \mathcal{A}$, which is known as extension of fractional derivative and fractional integral, as follows

$$
\begin{aligned}
\Omega^{\gamma} f(z) & =\Gamma(2-\gamma) z^{\gamma} D_{z}^{\gamma} f(z), \quad \gamma \neq 2,3,4, \ldots \\
& =z+\sum_{k=2}^{\infty} \frac{\Gamma(k+1) \Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_{k} z^{k}
\end{aligned}
$$

Note that $\Omega^{0} f(z)=f(z)$.
We define the linear multiplier fractional differential operator $D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f$ : $\mathcal{A} \rightarrow \mathcal{A}$ by

$$
\begin{align*}
& D_{\lambda}^{0}\left(\alpha_{1}, \beta_{1}\right) f(z)= H_{s}^{q}\left(\alpha_{1}, \beta_{1} ; z\right) * f(z) \\
& \begin{aligned}
D_{\lambda}^{1, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)= & (1-\lambda)\left[\Omega^{\gamma}\left(H_{s}^{q}\left(\alpha_{1}, \beta_{1} ; z\right) * f(z)\right)\right] \\
& +\lambda z\left[\Omega^{\gamma}\left(H_{s}^{q}\left(\alpha_{1}, \beta_{1} ; z\right) * f(z)\right)\right]^{\prime}
\end{aligned} \\
& \begin{aligned}
D_{\lambda}^{2, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)= & D_{\lambda}^{\gamma}\left(D_{\lambda}^{1, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)\right)
\end{aligned} \tag{1.2}
\end{align*}
$$

$$
\begin{equation*}
D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)=D_{\lambda}^{\gamma}\left(D_{\lambda}^{m-1, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)\right) \tag{1.3}
\end{equation*}
$$

where $\gamma$ and $\lambda$ aren't zero at the same time and $m \in \mathbb{N}_{0}$.
If $f$ is given by (1.1), then by (1.2) and (1.3), we see that

$$
\begin{align*}
& D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z) \\
& \quad=z+\sum_{k=2}^{\infty}\left[\frac{\Gamma(k+1) \Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}[1+\lambda(k-1)]\right]^{m} \frac{\left(\alpha_{1}\right)_{k-1} \ldots\left(\alpha_{q}\right)_{k-1}}{\left(\beta_{1}\right)_{k-1} \ldots\left(\beta_{s}\right)_{k-1}} \frac{1}{(k-1)!} a_{k} z^{k} \tag{1.4}
\end{align*}
$$

where $m \in \mathbb{N}_{0}, \lambda \geq 0$ and $0 \leq \gamma<1$.
Thus, after some calculations we obtain

$$
\alpha_{1} D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)=z\left[D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)\right]^{\prime}+\left(\alpha_{1}-1\right) D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)
$$

We note that by specializing $b, A, B, \lambda, \gamma, \alpha, \beta$ and $m$, we obtain the following subclasses studied by various authors:
(i) For a choice of the parameter $m=0$, the operator $D_{\lambda}^{0, \gamma}\left(\alpha_{1}, \beta_{1}\right)$ reduces to the Dziok-Srivastava operator [5].
(ii) For $\gamma=0, q=2, s=1, \alpha_{1}=\beta_{1}$ and $\alpha_{2}=1$, we get the operator introduced by Al-Oboudi [2].
(iii) For $q=2, s=1, \alpha_{1}=\beta_{1}, \alpha_{2}=1, \lambda=0$ and $m=1$, we get Owa-Srivastava fractional differential operator [10].
(iv) For $\gamma=0, q=2, s=1, \alpha_{1}=\beta_{1}, \alpha_{2}=1$ and $\lambda=1$, we get Sǎlăgean differential operator [4].

For various investigations based upon the Dziok-Srivastava linear operator and Owa-Srivastava linear operator, the interested reader may be refereed to many recent papers (see [5,6,10], [11,12,13]).

Using the operator $D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right)$, we define following class. Let $\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$
\begin{equation*}
1+\frac{1}{b}\left(\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta_{1}\right) f(z)}-1\right) \prec \frac{1+A z}{1+B z}, \quad z \in \mathbb{U} \tag{1.5}
\end{equation*}
$$

where $\prec$ denotes subordination, $b \neq 0$ is any complex number, A and B are arbitrary fixed numbers, $-1 \leq B<A \leq 1$ and $m \in \mathbb{N}_{0}$.

By specializing $b, A, B, \lambda, \gamma, \alpha, \beta$ and $m$ we obtain several subclasses studied by various authors in earlier papers (see for details [7], [1] ; see also the references cited in each of these recent works).

We use $\Lambda$ to denote the class of bounded analytic functions $w(z)$ in $\mathbb{U}$ which satisfy the conditions $w(0)=0$ and $|w(z)|<1$ for $z \in \mathbb{U}$.

## 2. Coefficient Estimates

Theorem 1. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$ and let

$$
G=\frac{(A-B)^{2}|b|^{2}}{2\left(\Gamma_{k}-1\right) B(A-B) \operatorname{Re}\{b\}+\left(1-B^{2}\right)\left(\Gamma_{k}-1\right)^{2}}, k=2,3, \ldots, m-1
$$

$M=[G]$ (Gauss symbol), and $[G]$ is the greatest integer not greater than $G$.
(a) If $(A-B)^{2}|b|^{2}>2\left(\Gamma_{k}-1\right) B(A-B) R e\{b\}+\left(1-B^{2}\right)\left(\Gamma_{k}-1\right)^{2}$, then

$$
\begin{equation*}
\left|a_{j}\right| \leq \frac{1}{\Psi_{j}\left(\Gamma_{j}\right)^{m}} \prod_{k=2}^{j} \frac{\left|(A-B) b-B\left(\Gamma_{k-1}-1\right)\right|}{\Gamma_{k}-1}, \tag{2.1}
\end{equation*}
$$

for $j=2,3, \ldots, M+2$; and

$$
\begin{equation*}
\left|a_{j}\right| \leq \frac{1}{\Psi_{j}\left(\Gamma_{j}\right)^{m}} \prod_{k=2}^{M+3} \frac{\left|(A-B) b-B\left(\Gamma_{k-1}-1\right)\right|}{\Gamma_{k}-1}, \tag{2.2}
\end{equation*}
$$

for $j>M+2$.
(b) If $(A-B)^{2}|b|^{2} \leq 2\left(\Gamma_{k}-1\right) B(A-B) R e\{b\}+\left(1-B^{2}\right)\left(\Gamma_{k}-1\right)^{2}$, then

$$
\begin{equation*}
\left|a_{j}\right| \leq \frac{|(A-B) b|}{\Psi_{j}\left(\Gamma_{j}\right)^{m}} \quad j \geq 2, \tag{2.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{k}=\frac{\Gamma(k+1) \Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}[1+\lambda(k-1)] \quad \text { and } \quad \Psi_{k}=\frac{\left(\alpha_{1}\right)_{k-1} \ldots\left(\alpha_{q}\right)_{k-1}}{\left(\beta_{1}\right)_{k-1} \ldots\left(\beta_{s}\right)_{k-1}} \frac{1}{(k-1)!} . \tag{2.4}
\end{equation*}
$$

Proof. Since $f(z) \in \mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$, (1.5) gives

$$
\begin{align*}
& D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)-D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z) \\
= & \left\{[(A-B) b+B] D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)-B D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)\right\} w(z) \tag{2.5}
\end{align*}
$$

Now (2.5) may be written as

$$
\begin{aligned}
& \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k} \\
= & \left\{(A-B) b z+\sum_{k=2}^{\infty}\left[(A-B) b-B\left(\Gamma_{k}-1\right)\right]\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k}\right\} w(z),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& \sum_{k=2}^{j}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k}+\sum_{k=j+1}^{\infty} d_{k} z^{k} \\
= & \left\{(A-B) b z+\sum_{k=2}^{j-1}\left[(A-B) b-B\left(\Gamma_{k}-1\right)\right]\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k}\right\} w(z),
\end{aligned}
$$

Since $|w(z)|<1$, we have

$$
\begin{align*}
& \left|\sum_{k=2}^{j}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k}+\sum_{k=j+1}^{\infty} d_{k} z^{k}\right| \\
\leq & \left|(A-B) b z+\sum_{k=2}^{j-1}\left[(A-B) b-B\left(\Gamma_{k}-1\right)\right]\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k}\right| \tag{2.6}
\end{align*}
$$

It follows that

$$
\begin{aligned}
& \int_{0}^{2 \pi}\left|\sum_{k=2}^{j}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k}+\sum_{k=j+1}^{\infty} d_{k} z^{k}\right|^{2} d \theta \\
\leq & \int_{0}^{2 \pi}\left|(A-B) b z+\sum_{k=2}^{j-1}\left[(A-B) b-B\left(\Gamma_{k}-1\right)\right]\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k}\right|^{2} d \theta
\end{aligned}
$$

for $z=r e^{i \theta}(0 \leq r<1)$. This gives us that

$$
\begin{aligned}
& \sum_{k=2}^{j}\left(\Gamma_{k}\right)^{2 m}\left(\Gamma_{k}-1\right)^{2}\left(\Psi_{k}\right)^{2}\left|a_{k}\right|^{2} r^{2 k}+\sum_{k=j+1}^{\infty}\left|d_{k}\right|^{2} r^{2 k} \\
\leq & (A-B)^{2}|b|^{2} r^{2}+\sum_{k=2}^{j-1}\left|(A-B) b-B\left(\Gamma_{k}-1\right)\right|^{2}\left(\Gamma_{k}\right)^{2 m}\left(\Psi_{k}\right)^{2}\left|a_{k}\right|^{2} r^{2 k}
\end{aligned}
$$

Let $r \rightarrow 1^{-}$, then on simplification we obtain

$$
\begin{align*}
& \left(\Gamma_{j}\right)^{2 m}\left(\left(\Gamma_{j}\right)^{m}-1\right)^{2}\left(\Psi_{j}\right)^{2}\left|a_{j}\right|^{2} \\
\leq & (A-B)^{2}|b|^{2}+\sum_{k=2}^{j-1}\left\{\left|(A-B) b-B\left(\Gamma_{k}-1\right)\right|^{2}-\left(\Gamma_{k}-1\right)^{2}\right\}\left(\Gamma_{k}\right)^{2 m}\left(\Psi_{k}\right)^{2}\left|a_{k}\right|^{2} \tag{2.7}
\end{align*}
$$

for $j \geq 2$. Now there may be following two cases :
(a) Let $(A-B)^{2}|b|^{2}>2\left(\Gamma_{k}-1\right) B(A-B) R e\{b\}+\left(1-B^{2}\right)\left(\Gamma_{k}-1\right)^{2}$, suppose that $j \leq M+2$, then for $j=2,(2.7)$ gives

$$
\left|a_{2}\right| \leq \frac{(A-B)|b|}{\Gamma_{2}^{m}\left(\Gamma_{2}-1\right) \Psi_{2}}
$$

which gives (2.1) for $j=2$. We establish (2.1) for $j \leq M+2$, from (2.7), by mathematical induction. Suppose (2.1) is valid for $j=2,3, \ldots,(k-1)$. Then it follows from (2.7)

$$
\begin{aligned}
& \left(\Gamma_{j}\right)^{2 m}\left(\Gamma_{j}-1\right)^{2}\left(\Psi_{j}\right)^{2}\left|a_{j}\right|^{2} \\
\leq & (A-B)^{2}|b|^{2}+\sum_{k=2}^{j-1}\left\{\left|(A-B) b-B\left(\Gamma_{k}-1\right)\right|^{2}-\left(\Gamma_{k}-1\right)^{2}\right\}\left(\Gamma_{k}\right)^{2 m}\left(\Psi_{k}\right)^{2}\left|a_{k}\right|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \leq(A-B)^{2}|b|^{2}+\sum_{k=2}^{j-1}\left\{\left|(A-B) b-B\left(\Gamma_{k}-1\right)\right|^{2}-\left(\Gamma_{k}-1\right)^{2}\right\}\left(\Gamma_{k}\right)^{2 m}\left(\Psi_{k}\right)^{2} \\
& \times\left\{\frac{1}{\left(\Psi_{k}\right)^{2}\left(\Gamma_{k}\right)^{2 m}} \prod_{n=2}^{k} \frac{\left|(A-B) b-B\left(\Gamma_{n-1}-1\right)\right|^{2}}{\left(\Gamma_{n}-1\right)^{2}}\right\} \\
&=(A-B)^{2}|b|^{2}+\left\{\left|(A-B) b-B\left(\Gamma_{2}-1\right)\right|^{2}-\left(\Gamma_{2}-1\right)^{2}\right\} \frac{|(A-B) b|^{2}}{\left(\Gamma_{2}-1\right)^{2}} \\
&+\left\{\left|(A-B) b-B\left(\Gamma_{3}-1\right)\right|^{2}-\left(\Gamma_{3}-1\right)^{2}\right\} \frac{|(A-B) b|^{2}}{\left(\Gamma_{2}-1\right)^{2}} \frac{\left|(A-B) b-B\left(\Gamma_{2}-1\right)\right|^{2}}{\left(\Gamma_{3}-1\right)^{2}} \\
&+\ldots \\
&+\left\{\left|(A-B) b-B\left(\Gamma_{j-1}-1\right)\right|^{2}-\left(\Gamma_{j-1}-1\right)^{2}\right\} \prod_{k=2}^{j-1} \frac{\left|(A-B) b-B\left(\Gamma_{k-1}-1\right)\right|^{2}}{\left(\Gamma_{k}-1\right)^{2}}
\end{aligned}
$$

Thus, we get

$$
\left|a_{j}\right| \leq \frac{1}{\Psi_{j}\left(\Gamma_{j}\right)^{m}} \prod_{k=2}^{j} \frac{\left|(A-B) b-B\left(\Gamma_{k-1}-1\right)\right|}{\Gamma_{k}-1}
$$

which completes the proof of (2.1).
Next, we suppose $j>M+2$. Then (2.7) gives

$$
\left(\Gamma_{j}\right)^{2 m}\left(\Gamma_{j}-1\right)^{2}\left(\Psi_{j}\right)^{2}\left|a_{j}\right|^{2} \leq(A-B)^{2}|b|^{2}
$$

$$
\begin{aligned}
& +\sum_{k=2}^{M+2}\left\{\left|(A-B) b-B\left(\Gamma_{k}-1\right)\right|^{2}-\left(\Gamma_{k}-1\right)^{2}\right\}\left(\Gamma_{k}\right)^{2 m}\left(\Psi_{k}\right)^{2}\left|a_{k}\right|^{2} \\
& +\sum_{k=M+3}^{j-1}\left\{\left|(A-B) b-B\left(\Gamma_{k}-1\right)\right|^{2}-\left(\Gamma_{k}-1\right)^{2}\right\}\left(\Gamma_{k}\right)^{2 m}\left(\Psi_{k}\right)^{2}\left|a_{k}\right|^{2}
\end{aligned}
$$

On substituting upper estimates for $a_{2}, a_{3}, \ldots, a_{M+2}$ obtained above, and simplifying, we obtain (2.2).
(b) Let $(A-B)^{2}|b|^{2} \leq 2\left(\Gamma_{k}-1\right) B(A-B) R e\{b\}+\left(1-B^{2}\right)\left(\Gamma_{k}-1\right)^{2}$, then it follows from (2.7)

$$
\left(\Gamma_{j}\right)^{2 m}\left(\left(\Gamma_{j}\right)^{m}-1\right)^{2}\left(\Psi_{j}\right)^{2}\left|a_{j}\right|^{2} \leq(A-B)^{2}|b|^{2}(j \geq 2)
$$

which proves (2.3).
The bounds in (2.1) are sharp for the functions $f(z)$ given by

$$
D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)=\left\{\begin{array}{cc}
z(1+B z)^{\frac{(A-B) b}{B \lambda}} ; & B \neq 0 \\
z \exp \left(\frac{A b z}{\lambda}\right) ; & B=0
\end{array}\right\}
$$

Also, the bounds in (2.3) are sharp for the functions $f_{k}(z)$ given by

$$
D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f_{k}(z)=\left\{\begin{array}{rc}
z(1+B z)^{\frac{(A-B) b}{B \lambda(k-1)}} ; & B \neq 0 \\
z \exp \left(\frac{A b}{\lambda(k-1)} z^{k-1}\right) ; & B=0
\end{array}\right\}
$$

## 3. A Sufficient Condition for a Function to be in $\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$

Theorem 2. Let the function $f(z)$ defined by (1.1) and let

$$
\begin{equation*}
\sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left\{(1-B)\left(\Gamma_{k}-1\right)+(A-B)|b|\right\} \Psi_{k}\left|a_{k}\right| \leq(A-B)|b| \tag{3.1}
\end{equation*}
$$

holds where $\Gamma_{k}$ and $\Psi_{k}$ are given by (2.4), then $f(z)$ belongs to

$$
\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)
$$

Poorf. Suppose that the inequality (3.1) holds. Then we have for $z \in \mathbb{U}$

$$
\begin{aligned}
& \left|D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)-D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)\right| \\
& \quad-\left|(A-B) b D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)-B\left[D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)-D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)\right]\right| \\
& =\left|\sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k}\right| \\
& -\left|(A-B) b z+(A-B) b \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k}-B \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k}\right| \\
& \leq \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k}\left|a_{k}\right| r^{k}-(A-B)|b| r+(A-B)|b| \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m} \Psi_{k}\left|a_{k}\right| r^{k} \\
& \quad+B \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k}\left|a_{k}\right| r^{k}
\end{aligned}
$$

Letting $r \rightarrow 1^{-}$, then we have

$$
\begin{aligned}
& \left|D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)-D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)\right| \\
& \quad-\left|(A-B) b D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)-B\left[D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)-D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)\right]\right| \\
& \leq \sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left[(1-B)\left(\Gamma_{k}-1\right)+(A-B)|b|\right] \Psi_{k}\left|a_{k}\right|-(A-B)|b| \leq 0
\end{aligned}
$$

Hence it follows that

$$
\left|\frac{\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-1}{(A-B) b-B\left[\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-1\right]}\right|<1, \quad z \in \mathbb{U}
$$

Letting

$$
w(z)=\frac{\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-1}{(A-B) b-B\left[\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-1\right]}
$$

then $w(0)=0, w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Hence we have

$$
\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}=\frac{1+[B+b(A-B)] w(z)}{1+B w(z)}
$$

which shows that $f(z)$ belong to $\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$.

## 4. Maximization of $\left|a_{3}-\mu a_{2}^{2}\right|$

We shall need in our discussion the following lemma:
Lemma 1. ([3]) Let $w(z)=\sum_{k=1}^{\infty} c_{k} z^{k}$ be analytic with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{U}$. If $\mu$ is any complex number, then

$$
\begin{equation*}
\left|c_{2}-\mu c_{1}^{2}\right| \leq \max \{1,|\mu|\} \tag{4.1}
\end{equation*}
$$

Equailty in (4.1) may be attained with the function $w(z)=z^{2}$ and $w(z)=z$ for $|\mu|<1$ and $|\mu| \geq 1$, respectively.

Theorem 3. If a function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$ and $\mu$ is any complex number, then

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{(A-B)|b|}{\left(\Gamma_{2}-1\right)\left(\Gamma_{3}-1\right)\left(\Gamma_{3}\right)^{m} \Psi_{3}} \max \{1,|d|\} \tag{4.2}
\end{equation*}
$$

where

$$
d=(A-B) b \mu \frac{\left(\Gamma_{3}\right)^{m}}{\left(\Gamma_{2}\right)^{2 m}} \frac{\Gamma_{3}-1}{\left(\Gamma_{2}-1\right)^{2}} \frac{\Psi_{3}}{\left(\Psi_{2}\right)^{2}}-\frac{(A-B) b-B\left(\Gamma_{2}-1\right)}{\Gamma_{2}-1}
$$

The result is sharp.

Poorf. Since $f(z) \in \mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$, we have

$$
\begin{equation*}
1+\frac{1}{b}\left(\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-1\right)=\frac{1+A w(z)}{1+B w(z)} \tag{4.3}
\end{equation*}
$$

where $w(z)=\sum_{k=1}^{\infty} c_{k} z^{k}$ is analytic in $\mathbb{U}$ satisfies the condition $w(0)=0$ and $|w(z)|<1$ for $z \in \mathbb{U}$. From we have (4.3)

$$
\begin{gathered}
w(z)=\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)-D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{[(A-B) b+B] D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)-B D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)} \\
=\frac{\sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k-1}}{(A-B) b+\sum_{k=2}^{\infty}\left[(A-B) b-B\left(\Gamma_{k}-1\right)\right]\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k-1}} \\
=\frac{\sum_{k=2}^{\infty}\left(\Gamma_{k}\right)^{m}\left(\Gamma_{k}-1\right) \Psi_{k} a_{k} z^{k-1}}{(A-B) b}\left[1+\frac{\sum_{k=2}^{\infty}\left[(A-B) b-B\left(\Gamma_{k}-1\right)\right]\left(\Gamma_{k}\right)^{m} \Psi_{k} a_{k} z^{k-1}}{(A-B) b}\right]^{-1}
\end{gathered}
$$

and then comparing the coefficients of $z$ and $z^{2}$ on both sides, we have

$$
a_{2}=\frac{(A-B) b}{\left(\Gamma_{2}\right)^{m}\left(\Gamma_{2}-1\right) \Psi_{2}} c_{1}
$$

and

$$
a_{3}=\frac{(A-B) b}{\left(\Gamma_{3}-1\right)\left(\Gamma_{3}\right)^{m} \Psi_{3}}\left\{c_{2}+\frac{\left[(A-B) b-B\left(\Gamma_{2}-1\right)\right]}{\left(\Gamma_{2}-1\right)} c_{1}^{2}\right\}
$$

Hence,

$$
\begin{equation*}
a_{3}-\mu a_{2}^{2}=\frac{(A-B) b}{\left(\Gamma_{2}-1\right)\left(\Gamma_{3}-1\right)\left(\Gamma_{3}\right)^{m} \Psi_{3}} \max \left\{c_{2}-d c_{1}^{2}\right\} \tag{4.4}
\end{equation*}
$$

where

$$
d=(A-B) b \mu \frac{\left(\Gamma_{3}\right)^{m}}{\left(\Gamma_{2}\right)^{2 m}} \frac{\Gamma_{3}-1}{\left(\Gamma_{2}-1\right)^{2}} \frac{\Psi_{3}}{\left(\Psi_{2}\right)^{2}}-\frac{\left[(A-B) b-B\left(\Gamma_{2}-1\right)\right]}{\left(\Gamma_{2}-1\right)}
$$

Taking modulus both sides in (4.4), we have

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{(A-B)|b|}{\left(\Gamma_{2}-1\right)\left(\Gamma_{3}-1\right)\left(\Gamma_{3}\right)^{m} \Psi_{3}}\left|c_{2}-d c_{1}^{2}\right| \tag{4.5}
\end{equation*}
$$

Using Lemma 1 in (4.5), we have

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{(A-B)|b|}{\left(\Gamma_{2}-1\right)\left(\Gamma_{3}-1\right)\left(\Gamma_{3}\right)^{m} \Psi_{3}} \max \{1,|d|\}
$$

which is (4.2) of Theorem 3.

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of Lemma 1 is sharp.

## 5. Radius Theorem

Theorem 4 . Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$. Then

$$
\operatorname{Re}\left\{\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}\right\}>0 \quad \text { for } \quad|z|<r_{m}
$$

where

$$
\begin{equation*}
r_{m}=\frac{2}{|b|(A-B)+\left[|b|^{2}(A-B)^{2}+4 B\{(A-B) R e(b)+B\}\right]^{\frac{1}{2}}}, \tag{5.1}
\end{equation*}
$$

such that

$$
\begin{equation*}
|b|^{2}(A-B)^{2}+4 B\{(A-B) R e(b)+B\} \geq 0 \tag{5.2}
\end{equation*}
$$

The result is sharp.

Poorf. Since $f(z) \in \mathcal{H}_{\lambda}^{m, \gamma}(b, \alpha, \beta ; A, B)$, we have

$$
\begin{align*}
1+\frac{1}{b}\left(\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-1\right) & =\frac{1+A w(z)}{1+B w(z)} \\
& =P(z) \tag{5.3}
\end{align*}
$$

It is well know that every function in the class $P(A, B)$ is subordinate to $\frac{1+A z}{1+B z}$ and the transformation

$$
P(z)=\frac{1+A w(z)}{1+B w(z)}
$$

maps the circle $|w(z)| \leq 1$ onto the circle

$$
\begin{equation*}
\left|P(z)-\frac{1-A B r^{2}}{1-B^{2} r^{2}}\right| \leq \frac{(A-B) r}{1-B^{2} r^{2}} \tag{5.4}
\end{equation*}
$$

Equations (5.3) and (5.4) yield

$$
\begin{equation*}
\left|\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}-\frac{1-B[B+b(A-B)] r^{2}}{1-B^{2} r^{2}}\right| \leq \frac{|b|(A-B) r}{1-B^{2} r^{2}} \tag{5.5}
\end{equation*}
$$

Therefore, from (5.5) we have

$$
\begin{equation*}
\operatorname{Re}\left\{\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}\right\} \geq \frac{1-|b|(A-B) r-B[(A-B) R e(b)+B] r^{2}}{1-B^{2} r^{2}} \tag{5.6}
\end{equation*}
$$

Hence $\operatorname{Re}\left\{\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f(z)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)}\right\}>0$ for $|z|<r_{m}$ defined by (5.1).
To show (5.1) is sharp, we let

$$
D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f(z)=\left\{\begin{array}{cc}
z(1+B z)^{\frac{(A-B) b}{B \lambda} ;} & B \neq 0 \\
z \exp \left(\frac{A b z}{\lambda}\right) ; & B=0
\end{array}\right\}
$$

and

$$
t=\frac{-r\left(B r+\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right)}{1+B r\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}}
$$

and obtain

$$
\begin{equation*}
\frac{D_{\lambda}^{m+1, \gamma}\left(\alpha_{1}, \beta\right) f_{0}(t)}{D_{\lambda}^{m, \gamma}\left(\alpha_{1}, \beta\right) f_{0}(t)}=\frac{1-|b|(A-B) r-B[B+b(A-B)] r^{2}}{1-B^{2} r^{2}} . \tag{5.7}
\end{equation*}
$$

## References

[1] A. A. Attiya, On a generalization class of bounded starlike functions of complex order, Appl.Math.And.Comp., 187 (2007), 62-67.
[2] F. M. Al-Oboudi, On univalent functions defined by a generalized Sǎlăgean operator, Int.J.Math.Sci., 2004 (2004), 1429-1436.
[3] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20 no. 1 (1969), 8-12.
[4] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Math. Springer-Verlag, 1013 (1983), 362-372.
[5] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric functions, Appl. Math. Comput., 103 no. 1 (1999), 1-13.
[6] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric functions, Integral Transform. Spec. Funct., 14 (2003), 7-18.
[7] M. K. Aouf, B. Şeker, and S. Sümer Eker, On a generalization class of bounded starlike functions of complex order, in Proceed. of the Int. Symp. on Geometric Funct. Theo. and Appl.: GFTA 2007 Proceed., Istanbul, Turkey, August, 20 24, 2007, S. Owa and Y. Polatoğlu, eds., TC. Istanbul Kültür Univ. Publ., Istanbul, Turkey, Vol. 91, TC Istanbul Kültür University (2008), 251-257.
[8] P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
[9] S. Owa, On the distortion theorem I, Kyungpook Math. J., 18 (1978), 53-58.
[10] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Cand. J. Math., 39 (1987), 1057-1077.
[11] M. K. Aouf, A. Shamandy, A. O. Mostafa, and E. A. Adwan, Partial Sums Of Certain Classes Of Analytic Functions Defined By Dziok-Srivastava Operator, Acta Universitatis Apulensis, 30 (2012), 65-76.
[12] M. K. Aouf and T. M. Seoudy, Inclusion Properties For Certain kUniformly Subclasses Of Analytic Functions Associated with DziokSrivastava Operator, Acta Universitatis Apulensis, 29 (2012), 17-29.
[13] H. M. Srivastava, M. Darus and R. W. Ibrahim, Classes of analytic functions with fractional powers defined by means of a certain linear operator, Integral Transforms and Special Functions, 22:1 (2011), 17-28.

# Sharma-Mittal Entropy and Coding Theorem * 

Satish Kumar and Arun Choudhary ${ }^{\dagger}$<br>Department of Mathematics, Geeta Institute of Management \& Technology, Kanipla-136131, Kurukshetra, Haryana, India

Received September 9, 2010, Accepted January 8, 2013.


#### Abstract

A relation between Shannon entropy and Kerridge inaccuracy, which is known as Shannon inequality, is well known in information theory. In this communication, first we generalized Shannon inequality and then given its application in coding theory and discuss some particular cases.


Keywords and Phrases: Shannon inequality, Codeword length, Holder's inequality, Kraft inequality, Optimal code length.

## 1. Introduction

Let $\Delta_{n}=\left\{P=\left(p_{1}, p_{2}, \ldots, p_{n}\right) ; p_{k} \geq 0, \quad \sum_{k=1}^{n} p_{k}=1\right\}, n \geq 2$ be a set of ncomplete probability distributions.

For $\mathrm{P} \in \Delta_{n}$, Shannon's measure of information [9] is defined as

$$
\begin{equation*}
H(P)=-\sum_{k=1}^{n} p_{k} \log _{D} p_{k} \tag{1.1}
\end{equation*}
$$

The measure (1.1) has been generalized by various authors and has found applications in various disciplines such as economics, accounting, crime and physics etc.

[^1]Sharma and Mittal [10] generalized (1.1) in the following form:

$$
\begin{equation*}
H(P ; \alpha, \beta)=\frac{1}{2^{1-\beta}-1}\left[\left(\sum_{k=1}^{n} p_{k}^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}}-1\right] \tag{1.2}
\end{equation*}
$$

where $\alpha, \beta>0, \alpha \neq \beta, \alpha \neq 1 \neq \beta$.
For $P, Q \in \Delta_{n}$, Kerridge [6] introduced a quantity known as inaccuracy defined as:

$$
\begin{equation*}
H(P, Q)=-\sum_{k=1}^{n} p_{k} \log _{D} q_{k} \tag{1.3}
\end{equation*}
$$

There is well known relation between $H(P)$ and $H(P, Q)$ which is given by

$$
\begin{equation*}
\mathrm{H}(\mathrm{P}) \leq \mathrm{H}(\mathrm{P}, \mathrm{Q}) \tag{1.4}
\end{equation*}
$$

The relation (1.4) is known as Shannon inequality and its importance is well known in coding theory.

In the literature of information theory, there are many approaches to extend the relation (1.4) for other measures. Nath and Mittal [7] extended the relation (1.4) in the case of entropy of type $\beta$.

Using the method of Nath and Mittal [7], Lubbe [14] generalized (1.4) in the case of Renyi's entropy. On the other hand, using the method of Campbell, Lubbe [14] generalized (1.4) for the case of entropy of type $\beta$. Using these generalizations, coding theorems are proved by these authors for these measures.

The objective of this communication is to generalize (1.4) for (1.2) and give its application in coding theory.

## 2. Generalization of Shannon Inequality

For $P, Q \in \Delta_{n}$, Sharma and Gupta [4] defined a measure of inaccuracy, denoted by $H(P, Q ; \alpha, \beta)$ as

$$
\begin{equation*}
H(P, Q ; \alpha, \beta)=\frac{1}{2^{1-\beta}-1}\left[\left(\sum_{k=1}^{n} p_{k} q_{k}^{(\alpha-1)}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)}-1\right] \tag{2.1}
\end{equation*}
$$

where $\alpha, \beta>0, \alpha \neq \beta, \alpha \neq 1 \neq \beta$.
Since $H(P, Q ; \alpha, \beta) \neq H(P ; \alpha, \beta)$, we will not interpret (2.1) as a measure of inaccuracy. But $H(P, Q ; \alpha, \beta)$ is a generalization of the measure of inaccuracy defined in (1.2). In spite of the fact that $H(P, Q ; \alpha, \beta)$ is not a measure of inaccuracy in its usual sense, its study is justified because it leads to meaningful new measures of length. In the following theorem, we will determine a relation between (1.2) and (2.1) of the type (1.4).

Since (2.1) is not a measure of inaccuracy in its usual sense, we will call the generalized relation as pseudo-generalization of the Shannon inequality.

Application of Holder's Inequality
Theorem 1. If $P, Q \in \Delta_{n}$ then it holds that

$$
\begin{equation*}
H(P ; \alpha, \beta) \leq H(P, Q ; \alpha, \beta) \tag{2.2}
\end{equation*}
$$

under the condition

$$
\begin{equation*}
\sum_{k=1}^{n} q_{k}^{\alpha} \leq \sum_{k=1}^{n} p_{k}^{\alpha} \tag{2.3}
\end{equation*}
$$

and equality holds if

$$
q_{k}=p_{k} ; \quad k=1,2, \ldots, n
$$

Proof: (a) If $0<\alpha<1<\beta$. By Holder's inequality [11]

$$
\begin{equation*}
\left(\sum_{k=1}^{n} x_{k}^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n} y_{k}^{q}\right)^{\frac{1}{q}} \leq \sum_{k=1}^{n} x_{k} y_{k} \tag{2.4}
\end{equation*}
$$

for all $x_{k}, y_{k}>0, i=1,2, \ldots, n$ and $\frac{1}{p}+\frac{1}{q}=1, p<1(\neq 0), q<0$ or $q<$ $1(\neq 0), p<0$. We see that equality holds if and only if there exists a positive constant c such that

$$
x_{k}^{p}=c y_{k}^{q} .
$$

Making the substitutions

$$
p=\frac{\alpha-1}{\alpha}, \quad q=1-\alpha
$$

$$
x_{k}=p_{k}^{\frac{\alpha}{\alpha-1}} q_{k}^{\alpha}, \quad y_{k}=p_{k}^{\frac{\alpha}{1-\alpha}}
$$

in (2.4), we get

$$
\left(\sum_{k=1}^{n} p_{k} q_{k}^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}\left(\sum_{k=1}^{n} p_{k}^{\alpha}\right)^{\frac{1}{1-\alpha}} \leq \sum_{k=1}^{n} q_{k}^{\alpha} ; \quad \alpha>0, \alpha \neq 1
$$

Using the condition (2.3), we get

$$
\begin{equation*}
\left(\sum_{k=1}^{n} p_{k} q_{k}^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}\left(\sum_{k=1}^{n} p_{k}^{\alpha}\right)^{\frac{1}{1-\alpha}} \leq \sum_{k=1}^{n} p_{k}^{\alpha} ; \quad \alpha>0, \alpha \neq 1 \tag{2.5}
\end{equation*}
$$

Since $0<\alpha<1<\beta$, (2.5) becomes

$$
\begin{equation*}
\left(\sum_{k=1}^{n} p_{k} q_{k}^{\alpha-1}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)} \leq\left(\sum_{k=1}^{n} p_{k}^{\alpha}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)} \tag{2.6}
\end{equation*}
$$

using (2.6) and the fact that $\beta>1$, we get (2.2).
(b) If $\alpha>1, \beta>1 ; 0<\alpha<1 ; \quad \beta>1(\alpha<\beta$ or $\beta<\alpha) ; \quad 0<\beta<1<$ $\alpha$.

The proof follows on the similar lines.

## Application in Coding Theory.

We will now give an application of Theorem 1 in coding theory. Let a finite set of n-input symbols with probabilities $p_{1}, p_{2}, \ldots, p_{n}$ be encoded in terms of symbols taken from the alphabet $\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$.

Then it is known Feinstein [3] that there always exist a uniquely decipherable code with lengths $N_{1}, N_{2}, \ldots, N_{n}$ iff

$$
\begin{equation*}
\sum_{k=1}^{n} D^{-N_{k}} \leq 1 \tag{2.7}
\end{equation*}
$$

If $L=\sum_{k=1}^{n} p_{k} N_{k}$ is the average codeword length, then for a code which satisfies (2.7), it has been shown that Feinstein [3],

$$
\begin{equation*}
L \geq H(P) \tag{2.8}
\end{equation*}
$$

with equality iff $N_{k}=-\log _{D} p_{k} ; \quad k=1,2, \ldots, n$
and that by suitable encoded into words of long sequences, the average length can be made arbitrary close to $H(P)$. This is Shannon's noiseless coding theorem. By considering Renyi's [8] entropy, a coding theorem and analogous to the above noiseless coding theorem has been established by Campbell [1] and the authors obtained bounds for it in terms of $H_{\alpha}(P)=\frac{1}{1-\alpha} \log _{D} \sum_{k=1}^{n} p_{k}^{\alpha} ; \alpha \neq$ $1, \alpha>0$. It may be seen that the mean codeword length $L=\sum_{k=1}^{n} p_{k} N_{k}$ had been generalized parametrically and their bounds had been studied in terms of generalized measures of entropies.

We define the measure of length $L(\alpha, \beta)$ by

$$
\begin{equation*}
L(\alpha, \beta)=\frac{1}{2^{1-\beta}-1}\left[\left(\sum_{k=1}^{n} p_{k} D^{N_{k}(1-\alpha)}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)}-1\right] \tag{2.9}
\end{equation*}
$$

where $\alpha, \beta>0, \alpha \neq \beta, \alpha \neq 1 \neq \beta$.
Also, we have used the condition

$$
\begin{equation*}
\sum_{k=1}^{n} D^{-N_{k} \alpha} \leq \sum_{k=1}^{n} p_{k}^{\alpha}, \quad \alpha>0 \tag{2.10}
\end{equation*}
$$

to find the bounds. It may be seen that in the case when $\alpha=1$, then (2.10) reduces to Kraft Inequality (2.7).

Theorem 2. If $N_{k}, k=1,2, \ldots, n$ are the lengths of codewords satisfying (2.10), then

$$
\begin{equation*}
H(P ; \alpha, \beta) \leq L(\alpha, \beta)<D^{1-\beta} H(P ; \alpha, \beta)+\frac{1-D^{1-\beta}}{1-2^{1-\beta}} \tag{2.11}
\end{equation*}
$$

Proof: In (2.2) choose $Q=\left(q_{1}, q_{2}, \ldots, q_{n}\right)$ where

$$
\begin{equation*}
q_{k}=D^{-N_{k}} \tag{2.12}
\end{equation*}
$$

with choice of $\mathrm{Q},(2.2)$ becomes

$$
H(P ; \alpha, \beta) \leq \frac{1}{2^{1-\beta}-1}\left[\left(\sum_{k=1}^{n} p_{k} D^{N_{k}(1-\alpha)}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)}-1\right]
$$

i.e., $H(P ; \alpha, \beta) \leq L(\alpha, \beta)$ which proves the first part of (2.11).

The equality holds iff $D^{-N_{k}}=p_{k}, k=1,2, \ldots, n$ which is equivalent to

$$
\begin{equation*}
N_{k}=-\log _{D} p_{k} ; \quad k=1,2, \ldots, n \tag{2.13}
\end{equation*}
$$

Choose all $N_{k}$ such that

$$
-\log _{D} p_{k} \leq N_{k}<-\log _{D} p_{k}+1
$$

Using the above relation, it follows that

$$
\begin{equation*}
D^{-N_{k}}>p_{k} D^{-1} \tag{2.14}
\end{equation*}
$$

We now have two possibilities:

1) If $\alpha>1$, (2.14) gives us

$$
\begin{equation*}
\left(\sum_{k=1}^{n} p_{k} D^{N_{k}(1-\alpha)}\right)>\sum_{k=1}^{n} p_{k}^{\alpha} D^{1-\alpha} . \tag{2.15}
\end{equation*}
$$

Now consider two cases:
i) let $0<\beta<1$. Raising both sides of (2.15) with $(\beta-1) /(\alpha-1)$, we get

$$
\begin{equation*}
\left(\sum_{k=1}^{n} p_{k} D^{N_{k}(1-\alpha)}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)}<\left(\sum_{k=1}^{n} p_{k}^{\alpha}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)} D^{1-\beta} \tag{2.16}
\end{equation*}
$$

Since $2^{1-\beta}-1>0$ for $\beta<1$, we get from (2.16) the right hand side in (2.11).
ii) Let $\beta>1$. The proof follows similarly.
2) If $0<\alpha<1$, The proof follows on the same lines.

## Particular's cases:

(1) Since $D \geq 2$, we have

$$
\frac{1-D^{1-\beta}}{1-2^{1-\beta}} \geq 1
$$

It follows then the upper bound of $L(\alpha, \beta)$ in $(2.11)$ is greater than unity.
(2) If $\beta=\alpha$, then (2.11) becomes

$$
H(P ; \alpha) \leq L(\alpha)<D^{1-\alpha} H(P ; \alpha)+\frac{1-D^{1-\alpha}}{1-2^{1-\alpha}}
$$

where

$$
H(P ; \alpha)=\frac{1}{2^{1-\alpha}-1}\left[\sum_{k=1}^{n} p_{k}^{\alpha}-1\right], \quad \alpha>0, \alpha \neq 1
$$

be the Havrda-Charvat [5] Entropy and later on it studied by Vajda [13], Daroczy [2] and Tsallis [12].

$$
L(\alpha)=\frac{1}{2^{1-\alpha}-1}\left[\left(\sum_{k=1}^{n} p_{k} D^{-N_{k}(\alpha-1)}\right)-1\right], \quad \alpha>0, \alpha \neq 1
$$

be the new mean codeword length.
(3) If $\beta \rightarrow 1$ then (2.11) becomes

$$
H(P ; \alpha) \leq L(\alpha)<H(P ; \alpha)+\log D
$$

Where $H_{\alpha}(P)=\frac{1}{1-\alpha} \log _{D} \sum_{k=1}^{n} p_{k}^{\alpha}, \alpha>0, \alpha \neq 1$ be the Renyi's [8] Entropy and $L(\alpha)=\frac{1}{1-\alpha} \log _{D} \sum_{k=1}^{n} p_{k} D^{-N_{k}(\alpha-1)}, \alpha>0, \alpha \neq 1$ be the new mean codeword length.
(4) If $\beta=\alpha$ and $\alpha \rightarrow 1$ then (2.11) becomes

$$
\frac{H(P)}{\log D} \leq L<\frac{H(P)}{\log D}+1
$$

Which is the Shannon [9] classical noiseless coding theorem.

## Conclusion:

We know that optimal code is that code for which the value $L(\alpha, \beta)$ is equal to its lower bound. From the result of the theorem 2, it can be seen that the mean codeword length of the optimal code is dependent on two parameters $\alpha$ and $\beta$, while in the case of Shannon's theorem it does not depend on any parameter. So it can be reduced significantly by taking suitable values of parameters.

## References

[1] L. L. Campbell, A coding theorem and Renyi's entropy, Information and Control, 8 (1965), 423-429.
[2] Z. Daroczy, Generalized Information Functions, Information and Control, 16 (1970), 36-51.
[3] A. Feinstein, Foundations of Information Theory, McGraw-Hill, New York, 1958.
[4] H. C. Gupta and Bhu Dev Sharma, On non-additive measures of inaccuracy, Czechoslovak Mathematical Journal, 26 no. 4 (1976), 584-595.
[5] J. F. Havrda and F. Charvat, Quantification Methods of Classification Process, The Concept of structural $\alpha$-entropy, Kybernetika, 3 (1967), 3035.
[6] D. F. Kerridge, Inaccuracy and inference, J. Roy. Statist Soc. Sec. B, $\mathbf{2 3}$ (1961), 184-194.
[7] P. Nath and D. P. Mittal, A generalization of Shannon's inequality and its application in coding theory, Inform. and Control, 23 (1973), 439-445.
[8] A. Renyi, On Measure of entropy and information, Proc. 4th Berkeley Symp. Maths. Stat. Prob., 1 (1961), 547-561.
[9] C. E. Shannon, A mathematical theory of information, Bell System Techn. J., 27 (1948), 378-423, 623-656.
[10] B. D. Sharma and D. P. Mittal, New non-additive measures of entropy for discrete Probability distributions, J. Math. Sci., 10 (1975), 28-40.
[11] O. Shisha, Inequalities, Academic Press, New York, 1967.
[12] C. Tsallis, Possible Generalization of Boltzmann Gibbs Statistics, J. Stat. Phy., 52 (1988), 479.
[13] I. Vajda, On measure of entropy and information, Proc. Fourth Berk. Symp. in Math, Stat. and Prob., 1 (1961), 547-561.
[14] J. C. A. van der Lubbe, On certain coding theorems for the information of order $\alpha$ and of type $\beta$, In: Trans. Eighth Prague Conf. Inform. Theory, Statist. Dec. Functions, Random Processes, Academia, Prague, C (1978), 253-266.

# An Inequality of Ostrowski's Type for Preinvex Functions with Applications * 

Mohammad W. Alomari ${ }^{\dagger}$<br>Department of Mathematics, Faculty of Science, Jerash University, 26150 Jerash, Jordan<br>and<br>Sabir Hussain ${ }^{\ddagger}$<br>Department of Mathematics, University of Engineering and Technology, Lahore

Received November 22, 2010, Accepted November 9, 2012.


#### Abstract

An inequality of Ostrowski's type for preinvex functions is introduced. Applications to some special means are considered.


Keywords and Phrases: Preinvex functions, Invex functions, Ostrowski's inequality, Means.

## 1. Introduction

Let $K$ be a nonempty closed set in $\mathbb{R}^{n}$. Let $f: K \rightarrow \mathbb{R}$ and $\eta: K \times K \rightarrow \mathbb{R}$ be continuous functions. Let $x \in K$. Then the set $K$ is said to be invex at $x$ with respect to $\eta(\cdot, \cdot)$, if

$$
x+t \cdot \eta(y, x) \in K, \quad \forall x, y \in K, \quad t \in[0,1] .
$$

[^2]$K$ is said to be an invex set with respect to $\eta$, if $K$ is invex at each $x \in K$. The invex set $K$ is also called a $\eta$-connected set. For the sake of simplicity, we always assume that $K=[a, a+\eta(b, a)]$, unless otherwise specified.
Definition 1. [11] The function $f$ on the invex set $K$ is said to be preinvex with respect to
$$
f(x+t \cdot \eta(y, x)) \leq(1-t) f(x)+t f(y), \quad \forall x, y \in K, \quad t \in[0,1]
$$

The function $f$ is said to be preconcave if and only if $-f$ is preinvex.
It may be noted that every convex function is a preinvex function, but the converse is not true [2].
Definition 2. [11] The differentiable function $f$ on the invex set $K$ is said to be an invex function with respect to $\eta(y, x)$, if

$$
f(y)-f(x) \geq\left\langle f^{\prime}(x), \eta(y, x)\right\rangle, \quad \forall x, y \in K
$$

where $f^{\prime}(x)$ is the differential of $f$ at $x$, and we denote $\langle\cdot, \cdot\rangle$ to the inner product.

It is clear that the differentiable preinvex functions are invex and the converse is also true under certain conditions, see [11, 12]. Also, it is true that every convex set is invex with respect to $\eta(y, x)=y-x$, but the converse may not be true [7]. Extensive work has been reported in the literature on generalized convex functions see $[2,3,6,10]$.

In the recent paper, Noor [8] has obtained the following Hermite-Hadamard inequalities for the preinvex.
Theorem 1. Let $f: K:=[a, a+\eta(b, a)] \rightarrow(0, \infty)$ be a preinvex function on the interval of real numbers $K^{\circ}$ (the interior of $K$ ) and $a, b \in K^{\circ}$ with $a<a+\eta(b, a)$. Then,

$$
\begin{equation*}
f\left(\frac{2 a+\eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(x) d x \leq \frac{f(a)+f(b)}{2} \tag{1.1}
\end{equation*}
$$

Other results connected with (1.1) were established by Noor [9], where he introduce a new inequalities involving two log-preinvex.

The main concern of this paper is to establish an Ostrowski's type inequality for differentiable functions whose derivative in absolute value is preinvex functions.

## 2. Ostrowski Type Inequalities

In 1938, Ostrowski established a very interesting inequality for differentiable functions with bounded derivatives, as follows: Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on $I^{\circ}$, the interior of the interval $I$, such that $f^{\prime} \in L[a, b]$, where $a, b \in I$ with $a<b$. If $\left|f^{\prime}(x)\right| \leq M$. Then,

$$
\begin{equation*}
\left|f(x)-\frac{1}{b-a} \int_{a}^{b} f(u) d u\right| \leq M(b-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right] \tag{2.1}
\end{equation*}
$$

holds for all $x \in[a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller ones. For recent results and generalizations concerning Ostrowski's inequality we refer the reader to the comprehensive book [5].

In order to prove our main inequality which is of Ostrowski's type, we need the following lemma:

Lemma 1. [1] Let $f: K:=[a, a+\eta(b, a)] \rightarrow \mathbb{R}$ be a differentiable function on $K^{\circ}$ (the interior of $K$ ) and $a, b \in K^{\circ}$ with $a<a+\eta(b, a)$. If $f^{\prime}$ is preinvex and $f^{\prime} \in L[a, b]$. Then,

$$
\begin{align*}
& f(2 a-x+\eta(b, a))-\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(x) d x \\
& =\eta(b, a) \cdot \int_{0}^{1} p(t) f^{\prime}(a+t \cdot \eta(b, a)) d t \tag{2.2}
\end{align*}
$$

where,

$$
p(t)= \begin{cases}t, & t \in\left[0, \frac{a+\eta(b, a)-x}{\eta(b, a)}\right] \\ t-1, & t \in\left(\frac{a+\eta(b, a)-x}{\eta(b, a)}, 1\right]\end{cases}
$$

for each $x \in K$.
Proof. Let $I=\int_{0}^{1} p(t) f(t a+t \cdot \eta(b, a)) d t$. Using integration by parts, we
get

$$
\begin{aligned}
I= & \int_{0}^{1} p(t) f^{\prime}(a+t \cdot \eta(b, a)) d t \\
= & \int_{0}^{\frac{a+\eta(b, a)-x}{\eta(b, a)}} t f^{\prime}(a+t \cdot \eta(b, a)) d t+\int_{\frac{a+\eta(b, a)-x}{\eta(b, a)}}^{1}(t-1) f^{\prime}(a+t \cdot \eta(b, a)) d t \\
= & \frac{1}{\eta(b, a)} \cdot \frac{a+\eta(b, a)-x}{\eta(b, a)} \cdot f\left(a+\frac{a+\eta(b, a)-x}{\eta(b, a)} \cdot \eta(b, a)\right) \\
& \quad-\frac{1}{\eta(b, a)} \int_{0}^{\frac{a+\eta(b, a)-x}{\eta(b, a)}} f(a+t \cdot \eta(b, a)) d t \\
& +\frac{1}{\eta(b, a)} \cdot \frac{x-a}{\eta(b, a)} \cdot f\left(a+\frac{a+\eta(b, a)-x}{\eta(b, a)} \cdot \eta(b, a)\right) \\
& \quad-\frac{1}{\eta(b, a)} \int_{\frac{a+\eta(b, a)-x}{\eta(b, a)}}^{1} f(a+t \cdot \eta(b, a)) d t \\
= & \frac{1}{\eta(b, a)} f(2 a-x+\eta(b, a))-\frac{1}{\eta(b, a)} \int_{0}^{1} f(a+t \cdot \eta(b, a)) d t
\end{aligned}
$$

i.e.,

$$
\eta(b, a) \cdot I=f(2 a-x+\eta(b, a))-\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(x) d x
$$

which completes the proof.
The following theorem gives an Ostrowski type inequality for differentiable functions whose derivative in absolute value is preinvex.

Theorem 2. Let $f: K:=[a, a+\eta(b, a)] \rightarrow \mathbb{R}$ be a differentiable function on $K^{\circ}$ (the interior of $K$ ) and $a, b \in K^{\circ}$ with $a<a+\eta(b, a)$. If $\left|f^{\prime}\right|$ is a preinvex. Then,

$$
\begin{align*}
& \left|f(2 a-x+\eta(b, a))-\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(u) d u\right| \\
& \leq|\eta(b, a)| \cdot\left\{\left[\frac{3}{2} \cdot\left(\frac{a-x}{\eta(b, a)}+1\right)^{2}-\frac{2}{3} \cdot\left(1+\frac{a-x}{\eta(b, a)}\right)^{3}+\frac{x-a}{\eta(b, a)}-\frac{2}{3}\right]\left|f^{\prime}(a)\right|\right. \\
& \left.\quad+\left[\frac{1}{6}-\frac{1}{2}\left(\frac{a-x}{\eta(b, a)}+1\right)^{2}+\frac{2}{3}\left(\frac{a-x}{\eta(b, a)}+1\right)^{3}\right]\left|f^{\prime}(b)\right|\right\}, \tag{2.3}
\end{align*}
$$

for each $x \in K$.
Proof. Form Lemma 1 and preinvexity of $\left|f^{\prime}\right|$ we have

$$
\begin{aligned}
& \left|f(2 a-x+\eta(b, a))-\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(u) d u\right| \\
& =\left|\eta(b, a) \cdot \int_{0}^{1} p(t) f^{\prime}(a+t \cdot \eta(b, a)) d t\right| \\
& \leq|\eta(b, a)| \cdot\left[\int_{0}^{\frac{a+\eta(b, a)-x}{\eta(b, a)}} t\left|f^{\prime}(a+t \cdot \eta(b, a))\right| d t\right. \\
& \left.+\int_{\frac{a+\eta(b, a)-x}{\eta(b, a)}}^{1}(1-t)\left|f^{\prime}(a+t \cdot \eta(b, a))\right| d t\right] \\
& \leq|\eta(b, a)| \cdot\left\{\int_{0}^{\frac{a+\eta(b, a)-x}{\eta(b, a)}} t\left[(1-t)\left|f^{\prime}(a)\right|+t\left|f^{\prime}(b)\right|\right] d t\right. \\
& \left.+\int_{\frac{a+\eta(b, a)-x}{\eta(b, a)}}^{1}(1-t)\left[(1-t)\left|f^{\prime}(a)\right|+t\left|f^{\prime}(b)\right|\right] d t\right\} \\
& =|\eta(b, a)| \cdot\left\{\left[\frac{3}{2} \cdot\left(\frac{a-x}{\eta(b, a)}+1\right)^{2}-\frac{2}{3} \cdot\left(1+\frac{a-x}{\eta(b, a)}\right)^{3}+\frac{x-a}{\eta(b, a)}-\frac{2}{3}\right]\left|f^{\prime}(a)\right|\right. \\
& \left.+\left[\frac{1}{6}-\frac{1}{2}\left(\frac{a-x}{\eta(b, a)}+1\right)^{2}+\frac{2}{3}\left(\frac{a-x}{\eta(b, a)}+1\right)^{3}\right]\left|f^{\prime}(b)\right|\right\},
\end{aligned}
$$

which completes the proof.

Corollary 1. In Theorem 2, for
(1) $x=a$, we have:

$$
\begin{align*}
&\left|f(a+\eta(b, a))-\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(u) d u\right| \\
& \leq \frac{|\eta(b, a)|}{6} \cdot\left(2\left|f^{\prime}(b)\right|+\left|f^{\prime}(a)\right|\right) \tag{2.4}
\end{align*}
$$

(2) $x=\frac{2 a+\eta(b, a)}{2}$, we have:

$$
\begin{align*}
&\left|f\left(\frac{2 a+\eta(b, a)}{2}\right)-\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(u) d u\right| \\
& \leq \frac{|\eta(b, a)|}{8} \cdot\left(\left|f^{\prime}(a)\right|+\left|f^{\prime}(b)\right|\right) \tag{2.5}
\end{align*}
$$

(3) $x=\eta(b, a)$, we have:

$$
\begin{align*}
\left\lvert\, f(2 a)-\frac{1}{\eta(b, a)}\right. & \int_{a}^{a+\eta(b, a)} f(u) d u \mid \\
\leq|\eta(b, a)| \cdot\{ & \left\{\left[\frac{3}{2} \cdot \frac{a^{2}}{\eta^{2}(b, a)}-\frac{2}{3} \cdot \frac{a^{3}}{\eta^{3}(b, a)}+\frac{1}{3}-\frac{a}{\eta(b, a)}\right]\left|f^{\prime}(a)\right|\right. \\
& \left.+\left[\frac{1}{6}-\frac{1}{2} \cdot \frac{a^{2}}{\eta^{2}(b, a)}+\frac{2}{3} \cdot \frac{a^{3}}{\eta^{3}(b, a)}\right]\left|f^{\prime}(b)\right|\right\} . \tag{2.6}
\end{align*}
$$

## 3. Applications to Special Means

In the following we study certain generalizations of some notions for a positivevalued function of a positive variable.

Definition 3. ([4]) A function $M: \mathbb{R}_{+}^{2} \rightarrow \mathbb{R}_{+}$, is called a Mean function if it has the following properties:
(1) Homogeneity: $M(a x, a y)=a M(x, y)$, for all $a>0$,
(2) Symmetry: $M(x, y)=M(y, x)$,
(3) Reflexivity : $M(x, x)=x$,
(4) Monotonicity: If $x \leq x^{\prime}$ and $y \leq y^{\prime}$, then $M(x, y) \leq M\left(x^{\prime}, y^{\prime}\right)$,
(5) Internality: $\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}$.

We shall consider some means for arbitrary positive real numbers $\alpha, \beta$ $(\alpha \neq \beta)[4]$.
(1) The arithmetic mean:

$$
A:=A(\alpha, \beta)=\frac{\alpha+\beta}{2}
$$

(2) The geometric mean :

$$
G:=G(\alpha, \beta)=\sqrt{\alpha \beta} .
$$

(3) The harmonic mean :

$$
H:=H(\alpha, \beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}
$$

(4) The power mean :

$$
P_{r}(\alpha, \beta)=\left(\frac{\alpha^{r}+\beta^{r}}{2}\right)^{\frac{1}{r}}, \quad r \geq 1
$$

(5) The identric mean:

$$
I(\alpha, \beta)= \begin{cases}\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha=\beta\end{cases}
$$

(6) The logarithmic mean :

$$
L:=L(\alpha, \beta)=\frac{\alpha-\beta}{\ln |\alpha|-\ln |\beta|},|\alpha| \neq|\beta| .
$$

(7) The generalized log-mean:

$$
L_{p}:=L_{p}(\alpha, \beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, p \in \mathbb{R} \backslash\{-1,0\}
$$

It is well known that $L_{p}$ is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1}:=L$ and $L_{0}:=I$. In particular, we have the following inequality $H \leq G \leq L \leq$ $I \leq A$.

Now, let $a$ and $b$ be positive real numbers such that $a<b$. Consider the function

$$
M:=M(a, b):[a, a+\eta(b, a)] \times[a, a+\eta(b, a)] \rightarrow \mathbb{R}_{+}
$$

which is one of the above mentioned means, therefore one can obtained variant inequalities for these means as follows:

Setting $\eta(b, a)=M(b, a)$ and choose $x=2 a$ in (2.3), provided that $2 a \lesseqgtr$ $\eta(b, a)=M(b, a)$, one can obtain the following interesting inequality involving means:

$$
\begin{align*}
\mid f(M(b, a))- & \left.\frac{1}{M(b, a)} \int_{a}^{a+M(b, a)} f(u) d u \right\rvert\, \\
\leq|M(b, a)| \cdot\{[ & {\left[\frac{3}{2} \cdot\left(1-\frac{a}{M(b, a)}\right)^{2}-\frac{2}{3} \cdot\left(1-\frac{a}{M(b, a)}\right)^{3}+\frac{a}{M(b, a)}-\frac{2}{3}\right]\left|f^{\prime}(a)\right| } \\
& \left.+\left[\frac{1}{6}-\frac{1}{2}\left(1-\frac{a}{M(b, a)}\right)^{2}+\frac{2}{3}\left(1-\frac{a}{M(b, a)}\right)^{3}\right]\left|f^{\prime}(b)\right|\right\} . \tag{3.1}
\end{align*}
$$

Letting $M:=A, G, H, P_{r}, I, L, L_{p}$, we get the required inequalities, and the details are left to the interested reader.

Acknowledgment. The authors would like to thank the anonymous referee for the valuable comments that have been implemented in the final version of the paper.

## References

[1] M. Alomari and M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA, 13 no. 2 (2010) article No. 3. Preprint.
[2] T. Antczak, Mean value in invexity analysis, Nonl. Anal.,, 60 (2005), 14731484.
[3] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc. Ser. B, 28 (1986), 1-9.
[4] P. S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.
[5] S. S. Dragomir and Tt. M. Rassias, (Eds) Ostrowski Type Inequalities and Applications in Numerical Integration, Kluwer Academic Publishers, Dordrecht, 2002.
[6] M. A. Hanson, On sufficiency of the Kuhn Tucker conditions, J. Math. Anal. Appl., 80 (1981), 545-550.
[7] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl., 189 (1995), 901-908.
[8] M. Aslam Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, Preprint, 2007.
[9] M. Aslam Noor, On Hadamard integral inequalities involving two log-preinvex functions, JIPAM, 8 no. 3 (2007), Article No. 75.
[10] R. Pini, Invexity and generalized convexity, Optimization, 22 (1991), 513-525.
[11] T. Weir and B. Mond, Preinvex functions in multiobjective optimization, $J$. Math. Anal. Appl., 136 (1988), 29-38.
[12] X. M. Yang, X .Q. Yang, and K. L. Teo, Generalized invexity and generalized invariant monotonicity, J. Optim. Theory Appl., 117 (2003), 607-625.

# Variants of Chebyshev's Method with Optimal Order of Convergence* 

Ramandeep Behl<br>Department of Mathematics, Panjab University, Chandigarh-160 014, India<br>and<br>V. Kanwar $^{\dagger}$<br>University Institute of Engineering and Technology, Panjab University, Chandigarh-160 014, India

Received December 29, 2010, Accepted December 27, 2012.


#### Abstract

In this paper, we derive a one-parameter family of Chebyshev's method for finding simple roots of nonlinear equations. Further, we present a new fourth-order variant of Chebyshev's method from this family without adding any functional evaluation to the previously used three functional evaluations. Chebyshev-Halley type methods are seen as the special cases of the proposed family. New classes of higher (third and fourth) order multipoint iterative methods free from second-order derivative are also derived by semi-discrete modifications of cubically convergent methods. Fourth-order multipoint iterative methods are optimal, since they require three functional evaluations per step. The new methods are tested and compared with other well-known methods on the number of problems.


Keywords and Phrases: Nonlinear equations, Newton's method, Chebyshev's method, Chebyshev-Halley type methods, Multipoint methods, TraubOstrowski's method, Jarratt's method, Optimal order of convergence.

[^3]
## 1. Introduction

One of the most important and challenging problem in computational mathematics is to compute approximate solutions of the nonlinear equation

$$
\begin{equation*}
f(x)=0 \tag{1}
\end{equation*}
$$

Newton's method for multiple roots appears in the work of Schröder [1], which is given as

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}{\left\{f^{\prime}\left(x_{n}\right)\right\}^{2}-f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)} \tag{2}
\end{equation*}
$$

This method has quadratic convergence, including the case of simple root.
Another well-known third-order modification of Newton's method is the classical Chebyshev's method [2, 3], given by

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{n}\right)\right\}^{2} f^{\prime \prime}\left(x_{n}\right)}{\left\{f^{\prime}\left(x_{n}\right)\right\}^{3}} . \tag{3}
\end{equation*}
$$

In this paper, we obtain a new general family of Chebyshev's method for finding simple roots of nonlinear equations numerically. The classical Chebyshev-Halley methods [4] are obtained as the particular cases of our proposed scheme. Further, we have also developed a new fourth order variant of Chebyshev's method. The beauty of this method is that it uses the same number of functional evaluations as that of classical Chebyshev's method. Therefore, the efficiency of our proposed fourth-order method in terms of functional evaluations is better than the existing classical Chebyshev's method. Furthermore, we also develop two new optimal fourth-order multipoint methods free from second-order derivative.

## 2. Family of Chebyshev's Method and Convergence Analysis

Let r be the required root of equation (1) and $x=x_{0}$ be the initial guess known for the required root. Assume

$$
\begin{equation*}
x_{1}=x_{0}+h, \quad|h| \ll 1, \tag{4}
\end{equation*}
$$

be the first approximation to the root. Therefore

$$
\begin{equation*}
f\left(x_{1}\right)=0 \tag{5}
\end{equation*}
$$

Expanding the function $f\left(x_{1}\right)$ by Taylor's theorem about $x_{0}$ and retaining the terms up to $O\left(h^{2}\right)$, we get

$$
\begin{equation*}
f\left(x_{0}\right)+h f^{\prime}\left(x_{0}\right)+\frac{h^{2}}{2} f^{\prime \prime}\left(x_{0}\right)=0 \tag{6}
\end{equation*}
$$

Further simplifying, we get

$$
\begin{equation*}
h=-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}-\frac{h^{2}}{2} \frac{f^{\prime \prime}\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)} \tag{7}
\end{equation*}
$$

Approximating $h$ on the right-hand side of equation (7) by the correction term $-\frac{f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}{\left\{f^{\prime}(x)\right\}^{2}-a f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}, a \in \mathbb{R}$ (free disposable parameter) given in formula (2), we obtain

$$
\begin{equation*}
h=-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{0}\right)\right\}^{2} f^{\prime}\left(x_{0}\right) f^{\prime \prime}\left(x_{0}\right)}{\left[\left\{f^{\prime}\left(x_{0}\right)\right\}^{2}-a f\left(x_{0}\right) f^{\prime \prime}\left(x_{0}\right)\right]^{2}} \tag{8}
\end{equation*}
$$

Thus the first approximation to the required root is given by

$$
\begin{equation*}
x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{0}\right)\right\}^{2} f^{\prime}\left(x_{0}\right) f^{\prime \prime}\left(x_{0}\right)}{\left[\left\{f^{\prime}\left(x_{0}\right)\right\}^{2}-a f\left(x_{0}\right) f^{\prime \prime}\left(x_{0}\right)\right]^{2}} \tag{9}
\end{equation*}
$$

Therefore, the general formula for successive approximations can be written as
$x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{n}\right)\right\}^{2} f^{\prime}\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}{\left[\left\{f^{\prime}\left(x_{n}\right)\right\}^{2}-a f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)\right]^{2}}$,
or
$x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{n}\right)\right\}^{2} f^{\prime}\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}{\left\{f^{\prime}\left(x_{n}\right)\right\}^{4}+a^{2}\left\{f\left(x_{n}\right)\right\}^{2}\left\{f^{\prime \prime}\left(x_{n}\right)\right\}^{2}-2 a f\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)\right\}^{2} f^{\prime \prime}\left(x_{n}\right)}$.
This formula looks like a Chebyshev's formula and describes the oneparameter family of Chebyshev's method. Note that for $a=0$ in (10), one can immediately recover the classical Chebyshev's formula.

## Special cases

I. Chebyshev-Halley type methods

If we remove the term $\left(a^{2}\left\{f\left(x_{n}\right)\right\}^{2}\left\{f^{\prime \prime}\left(x_{n}\right)\right\}^{2}\right)$ from the denominator: $\left\{f^{\prime}(x)\right\}^{4}+$ $a^{2}\left\{f\left(x_{n}\right)\right\}^{2}\left\{f^{\prime \prime}\left(x_{n}\right)\right\}^{2}-2 a f\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)\right\}^{2} f^{\prime \prime}\left(x_{n}\right)$ in formula (10), we obtain a family of methods defined by

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{n}\right)\right\}^{2} f^{\prime \prime}\left(x_{n}\right)}{\left\{f^{\prime}(x)\right\}^{3}-2 a f\left(x_{n}\right) f^{\prime}\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}
$$

This family resembles with the well-known cubically convergent family of Chebyshev-Halley type methods [4].
II. Another new cubically convergent family of methods

If we remove the term $\left(-2 a f\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)\right\}^{2} f^{\prime \prime}\left(x_{n}\right)\right)$ from the denominator: $\left\{f^{\prime}(x)\right\}^{4}+a^{2}\left\{f\left(x_{n}\right)\right\}^{2}\left\{f^{\prime \prime}\left(x_{n}\right)\right\}^{2}-2 a f\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)\right\}^{2} f^{\prime \prime}\left(x_{n}\right)$ in formula (10), we obtain a family of methods defined by

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{n}\right)\right\}^{2} f^{\prime}\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)}{\left\{f^{\prime}(x)\right\}^{4}+a^{2}\left\{f\left(x_{n}\right)\right\}^{2}\left\{f^{\prime \prime}\left(x_{n}\right)\right\}^{2}}
$$

It is investigated that this family is also cubically convergent for all $a \in \mathbb{R}$.
Theorem 2.1 Assume that $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval $D$ has a simple root $r \in D$. Let $f(x)$ be sufficently smooth in the neighborhood of the root $r$, then the order of convergence of the methods defined by family (10) is three for every value of $a \in \mathbb{R}$.

Proof. Let $e_{n}$ be the error at the $n^{t h}$ iteration, then $e_{n}=x_{n}-r$. Expanding $f\left(x_{n}\right)$ and $f^{\prime}\left(x_{n}\right)$ about $r$ and using the fact that $f(r)=0, f^{\prime}(r) \neq 0$, we have

$$
\begin{equation*}
f\left(x_{n}\right)=f^{\prime}(r)\left[e_{n}+c_{2} e_{n}^{2}+c_{3} e_{n}^{3}+c_{4} e_{n}^{4}+c_{5}+O\left(e_{n}^{5}\right)\right] \tag{11}
\end{equation*}
$$

where $c_{k}=\frac{1}{k!} \frac{f^{k}(r)}{f^{\prime}(r)}, k=2,3, \ldots$ Furthermore, we have

$$
\begin{equation*}
f^{\prime}\left(x_{n}\right)=f^{\prime}(r)\left[1+2 c_{2} e_{n}+3 c_{3} e_{n}^{2}+4 c_{4} e_{n}^{3}+5 c_{5} e_{n}^{4}+O\left(e_{n}^{5}\right)\right] \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{\prime \prime}\left(x_{n}\right)=f^{\prime}(r)\left[2 c_{2}+6 c_{3} e_{n}+12 c_{4} e_{n}^{2}+20 c_{5} e_{n}^{3}+O\left(e_{n}^{4}\right)\right] \tag{13}
\end{equation*}
$$

Then

$$
\begin{equation*}
\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=\left[e_{n}-c_{2} e_{n}^{2}+2\left(c_{2}^{2}-c_{3}\right) e_{n}^{3}+\left(-4 c_{2}^{3}+7 c_{2} c_{3}-3 c_{4}\right) e_{n}^{4}+O\left(e_{n}^{5}\right)\right] \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\right\}^{2} \frac{f^{\prime \prime}\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=\left[2 c_{2} e_{n}^{2}+\left(6 c_{3}-8 c_{2}^{2}\right) e_{n}^{3}+O\left(e_{n}^{4}\right)\right] \tag{15}
\end{equation*}
$$

Using (14) and (15) in equation (10) and simplifying, we get

$$
\begin{equation*}
e_{n+1}=\left\{2 c_{2}^{2}(1-2 a)-c_{3}\right\} e_{n}^{3}+O\left(e_{n}^{4}\right) \tag{16}
\end{equation*}
$$

Therefore, it can be concluded that for all $a \in \mathbb{R}$, the family (10) converges cubically. For $a=\frac{1}{2}$, error equation (16) reduces to

$$
\begin{equation*}
e_{n+1}=-c_{3} e_{n}^{3}+O\left(e_{n}^{4}\right) \tag{17}
\end{equation*}
$$

## 3. Fourth-order Variant of a Chebyshev's Method and Convergence Analysis

Here we intend to develop a new optimal fourth-order variant of Chebyshev's method. This method is very interesting because it has very higher order of convergence and computational efficiency unlike Chebyshev's method.
Considering the Newton-like iterative method with a parameter $\alpha \in \mathbb{R}$

$$
\begin{equation*}
y_{n}=x_{n}-\alpha \frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \tag{18}
\end{equation*}
$$

We now modify family (10) of Chebyshev's method by using the second-order derivative at $y_{n}$ instead of $x_{n}$ and obtain

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}-\frac{1}{2} \frac{\left\{f\left(x_{n}\right)\right\}^{2} f^{\prime}\left(x_{n}\right) f^{\prime \prime}\left(y_{n}\right)}{\left[\left\{f^{\prime}(x)\right\}^{2}-a f\left(x_{n}\right) f^{\prime \prime}\left(y_{n}\right)\right]^{2}} \tag{19}
\end{equation*}
$$

Obviously, when we take $(a, \alpha)=(0,0)$, we get classical Chebyshev's method.

Theorem 3.1 Assume that $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval $D$ has a simple root $r \in D$. Let $f(x)$ be sufficently smooth in the neighborhood of the root $r$, then the order of convergence of the method defined by formula (19) is of order four if $(a, \alpha)=\left(\frac{1}{2}, \frac{1}{3}\right)$.

Proof. The proof of said convergence of method (19) can be proved on similar lines as in the Theorem (2.1). Expanding $f^{\prime \prime}\left(y_{n}\right)=f\left(x_{n}-\alpha \frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\right)$ about $x=r$, we have

$$
\begin{align*}
f^{\prime \prime}\left(y_{n}\right)= & f^{\prime}(r)\left[2 c_{2}+6 c_{3}(1-\alpha) e_{n}+\left(6 \alpha c_{2} c_{3}+12 c_{4}(1-\alpha)^{2} e_{n}^{2}\right)+\left(12 \alpha c_{3}\left(c_{3}-c_{2}^{2}\right)\right.\right. \\
& \left.\left.+24 c_{2} c_{4} \alpha(1-\alpha)+20 c_{5}(1-\alpha)^{3}\right) e_{n}^{3}+O\left(e_{n}^{4}\right)\right] \tag{20}
\end{align*}
$$

Using (11), (12) and (20) in formula (19) and simplifying, we get the final error equation as

$$
\begin{align*}
e_{n+1}= & \left\{(1-2 a) 2 c_{2}^{2}-(1-3 \alpha) c_{3}\right\} e_{n}^{3}+\left\{\left(28 a-12 a^{2}-9\right) c_{2}^{3}\right. \\
& \left.+(12-24 a-15 \alpha+24 a \alpha) c_{2} c_{3}-\left(3-12 \alpha+6 \alpha^{2}\right) c_{4}\right\} e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{21}
\end{align*}
$$

For the method to be of fourth-order convergence, we must have

$$
1-2 a=0 \text { and } 1-3 \alpha=0
$$

which implies

$$
\begin{equation*}
a=\frac{1}{2} \text { and } \alpha=\frac{1}{3} \tag{22}
\end{equation*}
$$

Using (22) in equation (21), we obtain the following error equation for fourthorder variant as

$$
\begin{equation*}
e_{n+1}=\left(2 c_{2}^{3}-c_{2} c_{3}+\frac{1}{3} c_{4}\right) e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{23}
\end{equation*}
$$

The efficiency index [2] of the present method is equal to $\sqrt[3]{4} \cong 1.587$, which is better than the ones of classical Chebyshev's method $\sqrt[3]{3} \cong 1.442$ and Newton's method $\sqrt[2]{2} \cong 1.414$ respectively. Therefore, this method is very interesting because it has higher order of convergence and computational efficiency than Chebyshev's method.

## 4. Families of Multipoint Iteration Methods and their Convergence Analysis

The practical difficulty associated with the above mentioned methods given by (10) or (19) may be the evaluation of second-order derivative. Recently,
some new variants of Newton's method free from second-order derivative have been developed in $[2,3,5,6,7,8,9]$ and the references cited theirin by discretization of second-order derivative or by predictor-corrector approach or by considering different quadrature formulae for the computation of integral arising from Newton's theorem. These multipoint methods are of great practical importance since they overcome the limitations of one-point methods regarding the convergence order and computational efficiency. According to Kung-Traub conjecture [9], the order of convergence of any multipoint method without memory consuming function evaluations per iteration, can not exceed the bound (called optimal order). Thus, the optimal order for a method with three functional evaluations per step would be four. Traub-Ostrowski's method [2, 3], Jarratt's method [5], King's method [6] and Maheswari's method [7] etc. are famous optimal fourth order methods, because they require three functions evaluations per step. Nowadays, obtaining new optimal methods of order four is still important, because they have very high efficiency index.

Here, we also intend to develop new fourth-order multipoint methods free from second-order derivative. The main idea of proposed methods lies in the discretization of second-order derivative involved in family (10) of Chebyshev's method.

## a. First family

Expanding the function $f\left(x_{n}-\beta u\right), \beta \neq 0 \in \mathbb{R}$ but finite, about the point $x=x_{n}$ with $f\left(x_{n}\right) \neq 0$, we have

$$
\begin{equation*}
f\left(x_{n}-\beta u\right)=f\left(x_{n}\right)-\beta u f^{\prime}\left(x_{n}\right)+\frac{\beta^{2} u^{2}}{2!} f^{\prime \prime}\left(x_{n}\right)+O\left(e_{n}^{3}\right) \tag{24}
\end{equation*}
$$

Let us take $u=\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}$, and inserting this into (24), we obtain

$$
\begin{equation*}
f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right) \approx \frac{2\left\{f^{\prime}\left(x_{n}\right)\right\}^{2}}{\beta^{2} f\left(x_{n}\right)}\left\{f\left(x_{n}-\beta u\right)-(1-\beta) f\left(x_{n}\right)\right\} \tag{25}
\end{equation*}
$$

Using the approximate value of $f\left(x_{n}\right) f^{\prime \prime}\left(x_{n}\right)$ into formula (10), we have

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\beta^{2} f\left(x_{n}\right) \frac{\left\{f\left(x_{n}-\beta u\right)-(1-\beta) f\left(x_{n}\right)\right\}}{\left\{\left(\beta^{2}+2 a(1-\beta)\right) f\left(x_{n}\right)-2 a f\left(x_{n}-\beta u\right)\right\}^{2}}\right] \tag{26}
\end{equation*}
$$

## Special cases

For different specific values of parameters $a$ and $\alpha$, the following various
multipoint methods can be deduced from (26), e.g.
i. For $(a, \beta)=\left(-\frac{1}{2}, 1\right)$, we get the new formula

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\frac{f\left(x_{n}\right) f\left(x_{n}-u\right)}{\left\{f\left(x_{n}\right)+f\left(x_{n}-u\right)\right\}^{2}}\right] . \tag{27}
\end{equation*}
$$

ii. For $(a, \beta)=(-1,1)$, we get the new formula

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\frac{f\left(x_{n}\right) f\left(x_{n}-u\right)}{\left\{f\left(x_{n}\right)+2 f\left(x_{n}-u\right)\right\}^{2}}\right] . \tag{28}
\end{equation*}
$$

iii. For $(a, \beta)=\left(\frac{1}{2}, 1\right)$, we get the new formula

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\frac{f\left(x_{n}\right) f\left(x_{n}-u\right)}{\left\{f\left(x_{n}\right)-f\left(x_{n}-u\right)\right\}^{2}}\right] . \tag{29}
\end{equation*}
$$

Note that the family (26) can produce many more new multipoint methods by choosing different values of the parameters.

## b. Second family

Replacing the second-order derivative in (10) by the following definition

$$
f^{\prime \prime}\left(x_{n}\right) \approx \frac{f^{\prime}\left(x_{n}\right)-f^{\prime}\left(x_{n}-\beta u\right)}{\beta u}, \beta \neq 0 \in \mathbb{R}
$$

we get the following new family as

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\frac{1}{2} \frac{\beta f^{\prime}\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)-f^{\prime}\left(x_{n}-\beta u\right)\right\}}{\left\{(\beta-a) f^{\prime}\left(x_{n}\right)+a f^{\prime}\left(x_{n}-\beta u\right)\right\}^{2}}\right] . \tag{30}
\end{equation*}
$$

## Special cases

For different specific values of parameters $a$ and $\beta$, the following various multipoint methods can be obtained from (30), e.g.
i. For $(a, \beta)=(1,1)$, we get the new formula

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\frac{f^{\prime}\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)-f^{\prime}\left(x_{n}-u\right)\right\}}{2\left\{f^{\prime}\left(x_{n}-u\right)\right\}^{2}}\right] . \tag{31}
\end{equation*}
$$

ii. For $(a, \beta)=\left(\frac{1}{2}, \frac{2}{3}\right)$, we get the new formula

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}\left[1+\frac{12 f^{\prime}\left(x_{n}\right)\left\{f^{\prime}\left(x_{n}\right)-f^{\prime}\left(x_{n}-\frac{2}{3} u\right)\right\}}{\left\{f^{\prime}\left(x_{n}\right)+3 f^{\prime}\left(x_{n}-\frac{2}{3} u\right)\right\}^{2}}\right] . \tag{32}
\end{equation*}
$$

Other modifications can be obtained from formula (10) by replacing the secondorder derivative by other finite difference approximations.

The order of convergence of family (26) and (30) will be studied in Theorem 4.1 in the subsequent section.

Theorem 4.1 Let $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and sufficiently differentiable function defined in $D$. If $f(x)$ has a simple root $r \in D$, then for sufficiently close initial guess $x_{0}$ to $r$,
(i) the family (26) has $3^{\text {rd }}$ order of convergence, for
$a \neq \frac{1}{2} \& \beta=1, \quad a=\frac{1}{2} \& \beta \neq 1, \quad a \neq \frac{1}{2} \& \beta \neq 1$,
and $4^{\text {th }}$ order of convergence for $a=\frac{1}{2} \& \beta=1$.
(ii) the family (30) has $3^{\text {rd }}$ order of convergence, for
$a \neq \frac{1}{2} \& \beta=\frac{2}{3}, \quad a=\frac{1}{2} \& \beta \neq \frac{2}{3}, \quad a \neq \frac{1}{2} \& \beta \neq \frac{2}{3}$,
and $4^{\text {th }}$ order of convergence for $a=\frac{1}{2} \& \beta=\frac{2}{3}$.
Proof. Since $f(x)$ is sufficiently differentiable, expanding $f\left(x_{n}\right)$ and $f^{\prime}\left(x_{n}\right)$ about $x=r$ by Taylor's expansion, we have

$$
\begin{equation*}
f\left(x_{n}\right)=f^{\prime}(r)\left[e_{n}+c_{2} e_{n}^{2}+c_{3} e_{n}^{3}+c_{4} e_{n}^{4}+c_{5} e_{n}^{5}+O\left(e_{n}^{5}\right)\right] \tag{33}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{\prime}\left(x_{n}\right)=f^{\prime}(r)\left[1+2 c_{2} e_{n}+3 c_{3} e_{n}^{2}+4 c_{4} e_{n}^{3}+5 c_{5} e_{n}^{4}+O\left(e_{n}^{5}\right)\right] \tag{34}
\end{equation*}
$$

where $c_{k}$ and $e_{n}$ are defined earlier.
$f\left(x_{n}-\beta u\right)=(1-\beta) e_{n}+\left\{\left(1-\beta+\beta^{2}\right) c_{2}\right\} e_{n}^{2}-\left\{2 \beta^{2} c_{2}^{2}-\left((1-\beta)^{3}+2 \beta\right) c_{3}\right\} e_{n}^{3}+O\left(e_{n}^{4}\right)$.
Using symbolic computation in the programming package Mathematica, we get the following error equation for the family (26):

$$
\begin{align*}
e_{n+1}= & \left\{2(1-2 a) c_{2}^{2}-(1-\beta) c_{3}\right\} e_{n}^{3}+\left\{\left(28 a-12 a^{2}-9\right) c_{2}^{3}\right. \\
& \left.+(12-24 a-5 \beta+8 a \beta) c_{2} c_{3}-\left(3-4 \beta+\beta^{2}\right) c_{4}\right\} e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{36}
\end{align*}
$$

For $a=\frac{1}{2}$ and $\beta=1$, in equation (36), we get

$$
\begin{equation*}
e_{n+1}=\left(2 c_{2}^{3}-c_{2} c_{3}\right) e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{37}
\end{equation*}
$$

Similarly for scheme (30), we have the following error equation

$$
\begin{align*}
e_{n+1}= & \left\{2(1-2 a) c_{2}^{2}-\left(1-\frac{3 \beta}{2}\right) c_{3}\right\} e_{n}^{3}+\left\{\left(28 a-12 a^{2}-9\right) c_{2}^{3}\right. \\
& \left.+\left(12-24 a-\frac{15 \beta}{2}+12 a \beta\right) c_{2} c_{3}-\left(3-6 \beta+2 \beta^{2}\right) c_{4}\right\} e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{38}
\end{align*}
$$

For $a=\frac{1}{2}$ and $\beta=\frac{2}{3}$, in equation (36), we get

$$
\begin{equation*}
e_{n+1}=\left(2 c_{2}^{3}-c_{2} c_{3}+\frac{1}{9} c_{4}\right) e_{n}^{4}+O\left(e_{n}^{5}\right) \tag{39}
\end{equation*}
$$

## 5. Numerical Results

In this section, we shall present the numerical results obtained by employing the methods namely Newton's method (NM), Chebyshev's method (CM), cubically convergent variant of Chebyshev's method (10) for $a=1$ (CVCM) and quartically convergent variant of Chebyshev's method (19) (QVCM) respectively to solve the nonlinear equations given in Table 1. The results are summarized in Table 2. We also compare Newton's method (NM), Traub-Ostrowski's method (TOM), Jarratt's method (JM), Maheswari's method (MM) with our optimal multipoint methods (29) (MTOM) and (32) (MJM) introduced in this contribution. The results are summarized in Table 3. Computations have been performed using $C^{++}$in double precision arithmetic. We use $\epsilon=10^{-15}$. The following stopping criteria are used for computer programs:

$$
\text { (i) }\left|x_{n+1}-x_{n}\right|<\epsilon, \quad(i i)\left|f\left(x_{n+1}\right)\right|<\epsilon
$$

Table 1: Test Problems

| No Problems | [a, b] | Initial guess | Root (r) |
| :---: | :---: | :---: | :---: |
| 1. $e^{x}-4 x^{2}=0$ | [0.5, 2] | 0.5 | 0.714805901050568 |
|  |  | 2.0 |  |
| 2. $x^{3}+4 x^{2}-10=0$ | $[1,2]$ | 1.0 | 1.3652300134140969 |
|  |  | 2.0 |  |
| 3. $\cos x-x=0$ | $[0,2]$ | 0.0 | 0.7390851332151600 |
|  |  | 2.0 |  |
| 4. $x^{2}-e^{x}-3 x+2=0$ | $[0,1]$ | 0.0 | 0.00000000000000 |
|  |  | 1.0 |  |
| 5. $x e^{x^{2}}-\sin x^{2}+3 \cos x+5=0$ | $\left[\begin{array}{ll}-1.5, & -0.5]\end{array}\right.$ | $-1.5$ | 1.207647800445557 |
|  |  | -0.5 |  |
| 6. $\sin ^{2} x-x^{2}+1=0$ | $[1,3]$ | 1.0 | 1.404491662979126 |
|  |  | 3.0 |  |
| 7. $e^{x^{2}+7 x-30}-1=0$ | [2.9, 3.5] | 2.9 | 3.000000000000000 |
|  |  | 3.5 |  |

Table 2: Results of problems (Number of Iterations)

| Problem | Initial guess | NM | CM | CVCM | QVCM |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1. | 0.5 | 4 | 3 | 3 | 2 |
|  | 2.0 | 5 | 4 | 4 | 3 |
| 2. | 1.0 | 4 | 3 | 3 | 2 |
|  | 2.0 | 4 | 3 | 3 | 2 |
| 3. | 0.0 | 4 | 3 | 3 | 3 |
|  | 2.0 | 3 | 3 | 3 | 2 |
| 4. | 0.0 | 3 | 2 | 2 | 2 |
|  | 1.0 | 3 | 3 | 2 | 2 |
| 5. | -1.5 | 5 | 3 | 4 | 3 |
|  | -0.5 | 9 | Divergent | 7 | 5 |
| 6. | 1.0 | 5 | 4 | 4 | 3 |
| 7. | 3.0 | 5 | 4 | 4 | 3 |
|  | 2.9 | 6 | Divergent | 5 | 3 |

Table 3: Results of problems (D below-stands for divergent)
Number of iterations

| Problem | Initial guess | NM | TOM | JM | MM | MTOM | MJM |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1. | 0.5 | 4 | 2 | 2 | 3 | 2 | 3 |
|  | 2.0 | 5 | 3 | 3 | 3 | 3 | 3 |
| 2. | 1.0 | 4 | 2 | 2 | 3 | 2 | 3 |
|  | 2.0 | 4 | 2 | 2 | 3 | 2 | 3 |
| 3. | 0.0 | 4 | 2 | 3 | 3 | 3 | 3 |
|  | 2.0 | 3 | 2 | 2 | 2 | 2 | 2 |
| 4. | 0.0 | 3 | 2 | 2 | 3 | 2 | 2 |
|  | 1.0 | 3 | 2 | 2 | 2 | 2 | 2 |
| 5. | -1.5 | 5 | 2 | 3 | 3 | 3 | 3 |
|  | -0.5 | 9 | 4 | 3 | D | 6 | 7 |
| 6. | 1.0 | 5 | 3 | 3 | 4 | 3 | 3 |
|  | 3.0 | 5 | 3 | 3 | 3 | 3 | 3 |
| 7. | 2.9 | 6 | 3 | 3 | 36 | 3 | 4 |
|  | 3.5 | 11 | 5 | 5 | 6 | 6 | 6 |

## 6 . Conclusions

In this paper, we obtained a new simple and elegant root-finding family of Chebyshev's method. Chebyshev-Halley type methods are seen as the special cases of our proposed family. Furthermore, we presented a new fourth-order variant of Chebyshev's method. Then we introduced two new multipoint optimal methods of order four. The additional advantage of the presented multipoint methods is similar to that of Traub-Ostrowski's method, Jarratt's method etc. because they do not require the computation of second-order derivative to reach such a high convergence order. Finally, we provide numerical tests showing that these methods are equally competitive to other methods available in literature for finding simple roots of nonlinear equations.

Acknowledgement. Ramandeep Behl gratefully acknowledges the financial support of CSIR, New Delhi.

## References

[1] E. Schróder, Über unendlich viele Algorithmen zur Auflösung der Gleichungen, Math. Ann., 2 (1870), 317-365.
[2] J. F. Traub, Iterative Methods for the Solution of Equations, New Jersey: Prentice-Hall, Englewood Cliffs, 1964.
[3] A. M. Ostrowski, Solutions of Equations and System of Equations, Academic Press, New York 1960.
[4] M. A. Hernández and M. A. Salanova, A family of Chebyshev-Halley type methods, Int. J. Comput. Math., 47 (1993), 59-63.
[5] P. Jarratt, Some fourth order multipoint methods for solving equations, Math Com., 20 (1966), 434-437.
[6] R. F. King, A family of fourth order methods for nonlinear equations, SIAM J. Numer. Anal. 10 (1973), 876-879.
[7] A. K. Maheswari, A fourth-order iterative method for solving nonlinear equations, Appl. Math. Comput., 211 (2009), 383-391.
[8] V. Kanwar and S.K. Tomar, Exponentially fitted variants of Newton's method with quadratic and cubic convergence, Int. J. Comput. Math., 86 (2009), 1603-1611.
[9] H. T. Kung and J. F. Traub, Optimal order of one-point and multipoint iteration, Journal of the ACM, 21 (1974), 643-651.

# Weakly Prime Ideals in Near-Rings * 

P. Dheena ${ }^{\dagger}$<br>Department of Mathematics, Annamalai University, Annamalainagar - 608002, India<br>and<br>\section*{B. Elavarasan ${ }^{\ddagger}$}<br>Department of Mathematics, School of Science and Humanities, Karunya University, Coimbatore - 641 114, Tamilnadu, India

Received July 6, 2011, Accepted December 18, 2012.


#### Abstract

In this short note, we introduce the notion of prime ideals in nearring and obtain equivalent conditions for an ideal to be a weakly prime ideal.


Keywords and Phrases: Near-ring, Prime ideal, M-system, Weakly prime ideal.

## 1. Introduction

Throughout this paper, $N$ denotes a zero-symmetric near-ring not necessarily with identity unless otherwise stated. For $x \in N,\langle x\rangle$ denote the ideal of $N$ generated by $x$, and $P(N)$ denotes the intersection of all prime ideals of $N$. In

[^4][1], D. D. Anderson and E. Smith defined weakly prime ideals in commutative rings, an ideal $P$ of a ring $R$ is weakly prime if $0 \neq a b \in P$ implies $a \in P$ or $b \in P$. In this paper we define a notion of weakly prime ideal in near-ring (not necessarily commutative).

A proper ideal $P$ (i.e., an ideal different from $N$ ) of $N$ is prime if for ideals $A$ and $B$ of $N, A B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. We define a proper ideal $P$ of $N$ to be weakly prime if $0 \neq A B \subseteq P, A$ and $B$ are ideals of $N$, implies $A \subseteq P$ or $B \subseteq P$. Clearly every prime ideal is weakly prime and $\{0\}$ is always weakly prime ideal of $N$. The following example shows that a weakly prime ideal need not be a prime ideal in general.

Example 1.1. Let $N=\{0, a, b, c, d, 1,2,3\}$. Define addition and multiplication in $N$ as follows:

| + | 0 | 1 | 2 | 3 | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 2 | 3 | $a$ | $b$ | $c$ | $d$ |
| 1 | 1 | 2 | 3 | 0 | $d$ | $c$ | $a$ | $b$ |
| 2 | 2 | 3 | 0 | 1 | $b$ | $a$ | $d$ | $c$ |
| 3 | 3 | 0 | 1 | 2 | $c$ | $d$ | $b$ | $a$ |
| $a$ | $a$ | $d$ | $b$ | $c$ | 2 | 0 | 1 | 3 |
| $b$ | $b$ | $c$ | $a$ | $d$ | 0 | 2 | 3 | 1 |
| $c$ | $c$ | $a$ | $d$ | $b$ | 1 | 3 | 0 | 2 |
| $d$ | $d$ | $b$ | $c$ | $a$ | 3 | 1 | 2 | 0 |


| . | 0 | 1 | 2 | 3 | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | $a$ | $b$ | $c$ | $d$ |
| 2 | 0 | 2 | 0 | 2 | 2 | 2 | 0 | 0 |
| 3 | 0 | 3 | 2 | 1 | $b$ | $a$ | $c$ | $d$ |
| $a$ | 0 | $a$ | 2 | $b$ | $a$ | $b$ | $c$ | $d$ |
| $b$ | 0 | $b$ | 2 | $a$ | $b$ | $a$ | $c$ | $d$ |
| $c$ | 0 | $c$ | 0 | $c$ | 0 | 0 | 0 | 0 |
| $d$ | 0 | $d$ | 0 | $d$ | 2 | 2 | 0 | 0 |

Then $(N,+,$.$) is a near-ring (see [2], Library Nearring (8/2, 857)). Here$ $\{0, c\}$ is a weakly prime ideal, but not a prime, since $\{0,2\}^{2} \subseteq\{0, c\}$.

For a less trivial example, let $M$ be a unique maximal ideal of a near-ring $N$ with $M^{2}=0$, then every proper ideal of $N$ is easily seen to be weakly prime. Also in $\mathbb{Z}_{6},\{0\}$ is a weakly prime ideal, but not prime. For basic terminology in near-ring we refer to Pilz [3].

## 2. Main Results

Theorem 2.1. Let $N$ be a near-ring and $P$ a weakly prime ideal of $N$. If $P$ is not a prime, then $P^{2}=0$.

Proof: Suppose that $P^{2} \neq 0$. We show that $P$ is prime. Let $A$ and $B$ be ideals of $N$ such that $A B \subseteq P$. If $A B \neq 0$, then $A \subseteq P$ or $B \subseteq P$. So assume
that $A B=0$. Since $P^{2} \neq 0$, there exist $p_{0}, q_{0} \in P$ such that $<p_{0}><q_{0}>\neq 0$. Then $\left(A+<p_{0}>\right)\left(B+<q_{0}>\right) \neq 0$. Suppose $\left(A+<p_{0}>\right)\left(B+<q_{0}>\right) \nsubseteq P$. Then there exist $a \in A ; b \in B$ and $p_{0}^{\prime} \in<p_{0}>; q_{0}^{\prime} \in<q_{0}>$ such that $\left(a+p_{0}^{\prime}\right)\left(b+q_{0}^{\prime}\right) \notin P$ which implies $a\left(b+q_{0}^{\prime}\right) \notin P$, but $a\left(b+q_{0}^{\prime}\right)=a\left(b+q_{0}^{\prime}\right)-a b \in P$ since $A B=0$, a contradiction. So $0 \neq\left(A+<p_{0}>\right)\left(B+<q_{0}>\right) \subseteq P$ which implies $A \subseteq P$ or $B \subseteq P$.

Corollary 2.2. Let $N$ be a near-ring and $P$ an ideal of $N$. If $P^{2} \neq 0$, then $P$ is prime if and only if $P$ is weakly prime.

Corollary 2.3. Let $P$ be a weakly prime ideal of $N$. Then either $P \subseteq P(N)$ or $P(N) \subseteq P$. If $P \subset P(N)$, then $P$ is not prime, while if $P(N) \subset P$, then $P$ is prime.

It should be noted that a proper ideal $P$ with the property that $P^{2}=\{0\}$ need not be weakly prime. Take $N=\mathbb{Z}_{8}$ and $P=\{\overline{0}, \overline{4}\}$. Clearly $P^{2}=\{0\}$, yet $P$ is not weakly prime.

Lemma 2.4. Let $N$ be a near-ring and $P$ an ideal of $N$. Then the following are equivalent:
i) For any $a, b, c \in N$ with $0 \neq a(<b>+\langle c\rangle) \subseteq P$, we have $a \in P$ or $b$ and $c$ in $P$
ii) For $x \in N \backslash P$, we have $(P:<x>+<y>)=P \cup(0:<x>+<y>)$ for any $y \in N$.
iii) For $x \in N \backslash P$, we have $(P:<x>+<y>)=P$ or $(P:<x>+<y>)=(0:<x>+<y>)$ for any $y \in N$.
iv) $P$ is weakly prime

Proof: $(i) \Rightarrow($ ii $)$ Let $t \in(P:<x>+<y>)$ for any $x \in N \backslash P$ and $y \in N$. Then $t(<x>+<y>) \subseteq P$. If $t(<x>+<y>)=0$, then $t \in(0:<x>+<y>)$. Otherwise $0 \neq t(<x>+<y>) \subseteq P$. Then $t \in P$ by hypothesis. $($ ii $) \Rightarrow($ iii $)$ follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them. $(i i i) \Rightarrow(i v)$ Let $A$ and $B$ be ideals of $N$ such that $A B \subseteq P$ and suppose $A \nsubseteq P$ and $B \nsubseteq P$. Then there exist $a \in A$ and $b \in B$ with $a, b \notin P$. Now we claim that $A B=0$.

Let $b_{1} \in B$. Then $A\left(<b>+<b_{1}>\right) \subseteq P$ which implies $A \subseteq(P:<b>$ $\left.+<b_{1}>\right)$. Then by assumption, $A\left(<b>+<b_{1}>\right)=0$ which gives $A b_{1}=0$. Thus $A B=0$ and hence $P$ is weakly prime ideal of $N .(i v) \Rightarrow(i)$ is clear.

Theorem 2.5. Let $N$ be a near-ring and $P$ an ideal of $N$. Then
i) $P$ is weakly prime
ii) For any ideals $I, J$ of $N$ with $P \subset I$ and $P \subset J$, we have either $I J=0$ or $I J \nsubseteq P$.
iii) For any ideals $I, J$ of $N$ with $I \nsubseteq P$ and $J \nsubseteq P$, we have either $I J=0$ or $I J \nsubseteq P$.
Proof: $(i) \Rightarrow(i i)$ and $(i i i) \Rightarrow(i)$ are clear. $(i i) \Rightarrow(i i i)$. Let $I, J$ be ideals of $N$ with $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $i_{1} \in I$ and $j_{1} \in J$ such that $i_{1}, j_{1} \notin P$.

Suppose that $<i><j>\neq 0$ for some $i \in I$ and some $j \in J$. Then $(P+<$ $\left.i>+<i_{1}>\right)\left(P+<j>+<j_{1}>\right) \neq 0$ and $P \subset P+<i>+<i_{1}>; P \subset$ $P+<j>+<j_{1}>$. By hypothesis, $\left(P+<i>+<i_{1}>\right)(P+<j>+<$ $\left.j_{1}>\right) \nsubseteq P$ which implies $<i>\left(P+<j>+<j_{1}>\right)+<i_{1}>(P+<j>$ $\left.+<j_{1}>\right) \nsubseteq P$. So there exist $i^{\prime} \in<i>; i_{1}^{\prime} \in<i_{1}>; j^{\prime}, j^{\prime \prime} \in<j>; j_{1}^{\prime}, j_{1}^{\prime \prime} \in<$ $j_{1}>$ and $p_{1}, p_{2} \in P$ such that $i^{\prime}\left(p_{1}+j^{\prime}+j_{1}^{\prime}\right)+i_{1}^{\prime}\left(p_{2}+j^{\prime \prime}+j_{1}^{\prime \prime}\right) \notin P$. Therefore $i^{\prime}\left(p_{1}+j^{\prime}+j_{1}^{\prime}\right)-i^{\prime}\left(j^{\prime}+j_{1}^{\prime}\right)+i^{\prime}\left(j^{\prime}+j_{1}^{\prime}\right)+i_{1}^{\prime}\left(p_{2}+j^{\prime \prime}+j_{1}^{\prime \prime}\right)-i_{1}^{\prime}\left(j^{\prime \prime}+j_{1}^{\prime \prime}\right)+i_{1}^{\prime}\left(j^{\prime \prime}+j_{1}^{\prime \prime}\right) \notin P$. But since $i^{\prime}\left(p_{1}+j^{\prime}+j_{1}^{\prime}\right)-i^{\prime}\left(j^{\prime}+j_{1}^{\prime}\right) \in P$ and $i_{1}^{\prime}\left(p_{2}+j^{\prime \prime}+j_{1}^{\prime \prime}\right)-i_{1}^{\prime}\left(j^{\prime \prime}+j_{1}^{\prime \prime}\right) \in P$, we have $P$ does not contain either $i^{\prime}\left(j^{\prime}+j_{1}^{\prime}\right)$ or $i_{1}^{\prime}\left(j^{\prime \prime}+j_{1}^{\prime \prime}\right)$ which shows that $I J \nsubseteq P$.
From [3], a subset $M$ of $N$ is called m-system if $a, b \in M$, then there exist $a_{1} \in<a>$ and $b_{1} \in<b>$ such that $a_{1} b_{1} \in M$. A subset $M$ of $N$ is called weakly m-system if $M \cap A \neq \phi$ and $M \cap B \neq \phi$ for any ideals $A, B$ of $N$, then either $A B \cap M \neq \phi$ or $A B=0$. Clearly every m-system is a weakly msystem, but a weakly m-system need not be a m-system, since in Example 1.1, $M=\{1,2,3, a, b, d\}$ is a weakly m-system, but not a m-system since $x_{1} x_{2} \notin M$ for all $x_{1}, x_{2} \in<2>$. It is clear that, an ideal $P$ of $N$ is weakly prime if and only if $N \backslash P$ is weakly $m$ - system. A well known result that, if $M$ is a non-void $m$-system of $N$ and $I$ is an ideal of $N$ with $I \cap M=\phi$, then there exist a prime ideal $P \neq N$ containing $I$ with $P \cap M=\phi$. A similar result does hold for weakly $m$-system.

Theorem 2.6. Let $M \subseteq N$ be a non-void weakly m-system in $N$ and $I$ an ideal of $N$ with $I \cap M=\phi$. Then $I$ is contained in a weakly prime ideal $P \neq N$ with $P \cap M=\phi$.

Proof: Let $\mathbb{A}=\{J: J$ is an ideal of $N$ with $J \cap M=\phi\}$. Clearly $I \in \mathbb{A}$. Then by Zorn's Lemma, $\mathbb{A}$ contains a maximal element (say) $P$ with $P \cap M=\phi$. We show that $P$ is weakly prime ideal of $N$. Let $A$ and $B$ be ideals of $N$ with
$P \subset A$ and $P \subset B$. Then by maximality of $\mathbb{A}, A \cap M \neq \phi$ and $B \cap M \neq \phi$. Since $M$ is weakly m-system, we have $A B=0$ or $A B \cap M \neq \phi$; that is $A B=0$ or $A B \nsubseteq P$ since $P \cap M=\phi$. So by Theorem $2.5, P$ is weakly prime ideal of $N$ and also containing $I$.

Theorem 2.7. Let $N$ be a decomposable near-ring with identity. If $P$ is a weakly prime ideal of $N$, then either $P=0$ or $P$ is prime.

Proof: Suppose that $N=N_{1} \times N_{2}$ and let $P=P_{1} \times P_{2}$ be a weakly prime ideal of $N$. We may assume that $P \neq 0$. Now, let $A$ be a non-zero ideal of $N_{1}$ and $B$ be a non-zero ideal of $N_{2}$ such that $0 \neq(A, B) \subseteq P$. Then $0 \neq\left(A, N_{2}\right)\left(N_{1}, B\right) \subseteq P$ which implies $\left(A, N_{2}\right) \subseteq P$ or $\left(N_{1}, B\right) \subseteq P$. Suppose that $\left(A, N_{2}\right) \subseteq P$. Then $\left(0, N_{2}\right) \subseteq P$ and so $P=P_{1} \times N_{2}$. We show that $P_{1}$ is a prime ideal of $N_{1}$. Let $A_{1}$ and $B_{1}$ be ideals of $N_{1}$ such that $A_{1} B_{1} \subseteq P_{1}$. Then $(0,0) \neq\left(A_{1}, N_{2}\right)\left(B_{1}, N_{2}\right)=\left(A_{1} B_{1}, N_{2}\right) \subseteq P$, so $\left(A_{1}, N_{2}\right) \subseteq P$ or $\left(B_{1}, N_{2}\right) \subseteq P$ and hence $A_{1} \subseteq P_{1}$ or $B_{1} \subseteq P_{1}$. So $P$ is prime ideal of $N$. The case where $\left(N_{1}, B\right) \subseteq P$ is similar.

## References

[1] D. D. Anderson and E. Smith, Weakly prime ideals, Houston J. Math., 29 no. 4 (2003), 831-840.
[2] F. Binder and C. Nöbauer, Table of All Nearrings with Identity Up to Order 15., http://verdi.algebra.uni-linz.ac.at/Sonata/encyclo/ (14 June 2003).
[3] G. Pilz, Near-Rings, North-Holland, Amsterdam, 1983.

# Inclusion Criteria for Subclasses of Functions and Gronwall's Inequality 

Rosihan M. Ali $\dagger$ Mahnaz M. Nargesi ${ }^{\ddagger}$<br>School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia<br>V. Ravichandran ${ }^{\S}$<br>Department of Mathematics, University of Delhi, Delhi 110 007, India<br>and<br>A. Swaminathan ${ }^{〔}$<br>Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247 667, India

Received December 15, 2011, Accepted June 5, 2012.


#### Abstract

A normalized analytic function $f$ is shown to be univalent in the open unit disk $\mathbb{D}$ if its second coefficient is sufficiently small and relates to its Schwarzian derivative through a certain inequality. New criteria for analytic functions to be in certain subclasses of functions are


[^5]62 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

> established in terms of the Schwarzian derivatives and the second coefficients. These include obtaining a sufficient condition for functions to be strongly $\alpha$-Bazilevič of order $\beta$.

Keywords and Phrases: Univalent functions, Bazilevič functions, Gronwall's inequality, Schwarzian derivative, Second coefficient.

## 1. Introduction

Let $\mathcal{A}$ be the set of all normalized analytic functions $f$ of the form $f(z)=z+$ $\sum_{k=2}^{\infty} a_{k} z^{k}$ defined in the open unit disk $\mathbb{D}:=\{z \in \mathbb{C}:|z|<1\}$ and denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ is starlike if it maps $\mathbb{D}$ onto a starlike domain with respect to the origin, and $f$ is convex if $f(\mathbb{D})$ is a convex domain. Analytically, these are respectively equivalent to the conditions $\operatorname{Re}\left(z f^{\prime}(z) / f(z)\right)>0$ and $1+\operatorname{Re}\left(z f^{\prime \prime}(z) / f^{\prime}(z)\right)>0$ in $\mathbb{D}$. Denote by $\mathcal{S T}$ and $\mathcal{C} \mathcal{V}$ the classes of starlike and convex functions respectively. More generally, for $0 \leq \alpha<1$, a function $f \in \mathcal{A}$ is starlike of order $\alpha$ if $\operatorname{Re}\left(z f^{\prime}(z) / f(z)\right)>\alpha$, and is convex of order $\alpha$ if $1+\operatorname{Re}\left(z f^{\prime \prime}(z) / f^{\prime}(z)\right)>\alpha$. We denote these classes by $\mathcal{S T}(\alpha)$ and $\mathcal{C} \mathcal{V}(\alpha)$ respectively. For $0<\alpha \leq 1$, let $\mathcal{S S T}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f$ satisfying the inequality

$$
\left|\arg \frac{z f^{\prime}(z)}{f(z)}\right| \leq \frac{\alpha \pi}{2}
$$

Functions in $\mathcal{S S T}(\alpha)$ are called strongly starlike functions of order $\alpha$.

The Schwarzian derivative $S(f, z)$ of a locally univalent analytic function $f$ is defined by

$$
S(f, z):=\left(\frac{f^{\prime \prime}(z)}{f^{\prime}(z)}\right)^{\prime}-\frac{1}{2}\left(\frac{f^{\prime \prime}(z)}{f^{\prime}(z)}\right)^{2}
$$

The Schwarzian derivative is invariant under Möbius transformations. Also, the Schwarzian derivative of an analytic function $f$ is identically zero if and only if it is a Möbius transformation.

Nehari showed that the univalence of an analytic function in $\mathbb{D}$ can be guaranteed if its Schwarzian derivative is dominated by a suitable positive function [10, Theorem I, p. 700]. In [9], by considering two particular positive functions, a bound on the Schwarzian derivative was obtained that would
ensure univalence of an analytic function in $\mathcal{A}$. In fact, the following theorem was proved.

Theorem 1.1. [9, Theorem II, p. 549] If $f \in \mathcal{A}$ satisfies

$$
|S(f, z)| \leq \frac{\pi^{2}}{2} \quad(z \in \mathbb{D})
$$

then $f \in \mathcal{S}$. The result is sharp for the function $f$ given by $f(z)=(\exp (i \pi z)-$ 1)/ $i \pi$.

The problems of finding similar bounds on the Schwarzian derivatives that would imply univalence, starlikeness or convexity of functions were investigated by a number of authors including Gabriel [4], Friedland and Nehari [3], and Ozaki and Nunokawa [11]. Corresponding results related to meromorphic functions were dealt with in $[4,6,9,12]$. For instance, Kim and Sugawa [8] found sufficient conditions in terms of the Schwarzian derivative for locally univalent meromorphic functions in the unit disk to possess specific geometric properties such as starlikeness and convexity. The method of proof in [8] was based on comparison theorems in the theory of ordinary differential equations with real coefficients.

Chiang [1] investigated strong-starlikeness of order $\alpha$ and convexity of functions $f$ by requiring the Schwarzian derivative $S(f, z)$ and the second coefficient $a_{2}$ of $f$ to satisfy certain inequalities. The following results were proved:

Theorem 1.2. [1, Theorem 1, pp. 108-109] Let $f \in \mathcal{A}, 0<\alpha \leq 1$ and $\left|a_{2}\right|=\eta<\sin (\alpha \pi / 2)$. Suppose

$$
\begin{equation*}
\sup _{z \in \mathbb{D}}|S(f, z)|=2 \delta(\eta) \tag{1.1}
\end{equation*}
$$

where $\delta(\eta)$ satisfies the inequality

$$
\sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \leq \frac{\alpha \pi}{2}
$$

Then $f \in \mathcal{S S T}(\alpha)$. Further, $|\arg (f(z) / z)| \leq \alpha \pi / 2$.
Theorem 1.3. [1, Theorem 2, p. 109] Let $f \in \mathcal{A}$, and $\left|a_{2}\right|=\eta<1 / 3$. Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
6 \eta+5(1+\eta) \delta e^{\delta / 2}<2
$$

64 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

Then

$$
f \in \mathcal{C} \mathcal{V}\left(\frac{2-6 \eta-5(1+\eta) \delta e^{\delta / 2}}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) .
$$

In particular, if $a_{2}=0$ and $2 \delta \leq 0.6712$, then $f \in \mathcal{C V}$.
Chiang's proofs in [1] rely on Gronwall's inequality (see Lemma 2.1 below). In this paper, Gronwall's inequality is used to obtain sufficient conditions for analytic functions to be univalent. Also, certain inequalities related to the Schwarzian derivative and the second coefficient will be formulated that would ensure analytic functions to possess certain specific geometric properties. The sufficient conditions of convexity obtained in [1] will be seen to be a special case of our result, and similar conditions for starlikeness will also be obtained.

## 2. Consequences of Gronwall's Inequality

Gronwall's inequality and certain relationships between the Schwarzian derivative of $f$ and the solution of the linear second-order differential equation $y^{\prime \prime}+A(z) y=0$ with $A(z):=S(f ; z) / 2$ will be revisited in this section. We first state Gronwall's inequality, which is needed in our investigation.

Lemma 2.1. [7, p. 19] Suppose $A$ and $g$ are non-negative continuous real functions for $t \geq 0$. Let $k>0$ be a constant. Then the inequality

$$
g(t) \leq k+\int_{0}^{t} g(s) A(s) d s
$$

implies

$$
g(t) \leq k \exp \left(\int_{0}^{t} A(s) d s\right) \quad(t>0)
$$

For the linear second-order differential equation $y^{\prime \prime}+A(z) y=0$ where $A(z):=\frac{1}{2} S(f ; z)$ is an analytic function, suppose that $u$ and $v$ are two linearly independent solutions with initial conditions $u(0)=v^{\prime}(0)=0$ and $u^{\prime}(0)=v(0)=1$. Such solutions always exist and thus the function $f$ can be represented by

$$
\begin{equation*}
f(z)=\frac{u(z)}{c u(z)+v(z)}, \quad\left(c:=-a_{2}\right) . \tag{2.1}
\end{equation*}
$$

It is evident that

$$
\begin{equation*}
f^{\prime}(z)=\frac{1}{(c u(z)+v(z))^{2}} \tag{2.2}
\end{equation*}
$$

Estimates on bounds for various expressions related to $u$ and $v$ were found in [1]. Indeed, using the integral representation of the fundamental solutions

$$
\begin{align*}
& u(z)=z+\int_{0}^{z}(\eta-z) A(\eta) u(\eta) d \eta  \tag{2.3}\\
& v(z)=1+\int_{0}^{z}(\eta-z) A(\eta) v(\eta) d \eta
\end{align*}
$$

and applying Gronwall's inequality, Chiang obtained the following inequalities [1] which we list for easy reference:

$$
\begin{align*}
& |u(z)|<e^{\delta / 2}  \tag{2.4}\\
& \left|\frac{u(z)}{z}-1\right|<\frac{1}{2} \delta e^{\delta / 2}  \tag{2.5}\\
& |c u(z)+v(z)|<(1+\eta) e^{\delta / 2}  \tag{2.6}\\
& |c u(z)+v(z)-1|<\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2} \tag{2.7}
\end{align*}
$$

For instance, by taking the path of integration $\eta(t)=t e^{i \theta}, t \in[0, r], z=r e^{i \theta}$, Gronwall's inequality shows that, whenever $|A(z)|<\delta$ and $0<r<1$,

$$
\begin{aligned}
|u(z)| & \leq 1+\int_{0}^{r}(r-t)\left|A\left(t e^{i \theta}\right)\right|\left|u\left(t e^{i \theta}\right)\right| d t \\
& \leq \exp \left(\int_{o}^{r}(r-t)\left|A\left(t e^{i \theta}\right)\right| d t\right) \leq \exp (\delta / 2)
\end{aligned}
$$

This proves inequality (2.4). Note that there was a typographical error in [1, Inequality (8), p. 112], and that inequality (2.5) is the right form.

## 3. Inclusion Criteria for Subclasses of Analytic Functions

The first result leads to sufficient conditions for univalence.
Theorem 3.1. Let $0<\alpha \leq 1,0 \leq \beta<1, f \in \mathcal{A}$ and $\left|a_{2}\right|=\eta$, where $\alpha, \beta$ and $\eta$ satisfy

$$
\begin{equation*}
\sin ^{-1}\left(\beta(1+\eta)^{2}\right)+2 \sin ^{-1} \eta<\frac{\alpha \pi}{2} \tag{3.1}
\end{equation*}
$$

66 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\begin{equation*}
\sin ^{-1}\left(\beta(1+\eta)^{2} e^{\delta}\right)+2 \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \leq \frac{\alpha \pi}{2} \tag{3.2}
\end{equation*}
$$

Then $\left|\arg \left(f^{\prime}(z)-\beta\right)\right| \leq \alpha \pi / 2$.
Proof. Using a limiting argument as $\delta \rightarrow 0$, the condition (3.1) shows that there is a real number $\delta(\eta) \geq 0$ satisfying inequality (3.2). The representation of $f^{\prime}$ in terms of the linearly independent solutions of the differential equation $y^{\prime \prime}+A(z) y=0$ with $A(z):=S(f ; z) / 2$ as given by equation (2.2) yields

$$
\begin{equation*}
f^{\prime}(z)-\beta=\frac{1-\beta(c u(z)+v(z))^{2}}{(c u(z)+v(z))^{2}} \tag{3.3}
\end{equation*}
$$

In view of the fact that for $w \in \mathbb{C}$,

$$
|w-1| \leq r \Leftrightarrow|\arg w| \leq \sin ^{-1} r
$$

inequality (2.6) implies

$$
\begin{equation*}
\left|\arg \left[1-\beta(c u(z)+v(z))^{2}\right]\right| \leq \sin ^{-1}\left(\beta(1+\eta)^{2} e^{\delta}\right) \tag{3.4}
\end{equation*}
$$

Similarly, inequality (2.7) shows

$$
\begin{equation*}
|\arg [c u(z)+v(z)]| \leq \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \tag{3.5}
\end{equation*}
$$

Hence, it follows from (3.3), (3.4) and (3.5) that

$$
\begin{aligned}
\left|\arg \left(f^{\prime}(z)-\beta\right)\right| & \leq\left|\arg \left[1-\beta(c u(z)+v(z))^{2}\right]\right|+2|\arg [c u(z)+v(z)]| \\
& \leq \sin ^{-1}\left(\beta(1+\eta)^{2} e^{\delta}\right)+2 \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \\
& \leq \frac{\alpha \pi}{2}
\end{aligned}
$$

where the last inequality follows from (3.2). This completes the proof.
By taking $\beta=0$ in Theorem 3.1, the following univalence criterion is obtained.

Corollary 3.1. Let $f \in \mathcal{A}$, and $\left|a_{2}\right|=\eta<\sin (\alpha \pi / 4), 0<\alpha \leq 1$. Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2} \leq \sin \left(\frac{\alpha \pi}{4}\right)
$$

Then $\left|\arg f^{\prime}(z)\right| \leq \alpha \pi / 2$, and in particular $f \in \mathcal{S}$.
Example 3.1. Consider the univalent function $g$ given by

$$
g(z)=\frac{z}{1+c z}, \quad|c| \leq 1, \quad z \in \mathbb{D}
$$

Since the Schwarzian derivative of an analytic function is zero if and only if it is a Möbius transformation, it is evident that $S(g, z)=0$. Therefore the condition (1.1) is satisfied with $\delta=0$. It is enough to take $\eta=|c|$ and to assume that $\eta, \alpha$ and $\beta$ satisfy the inequality (3.1). Now

$$
\begin{aligned}
\left|\arg \left(g^{\prime}(z)-\beta\right)\right| & =\left|\arg \frac{1}{(1+c z)^{2}}-\beta\right| \leq\left|\arg \left(1-\beta(1+c z)^{2}\right)\right|+2|\arg (1+c z)| \\
& \leq \sin ^{-1}\left(\beta(1+|c|)^{2}\right)+2 \sin ^{-1}|c|
\end{aligned}
$$

In view of the latter inequality, it is necessary to assume inequality (3.1) for $g$ to satisfy $\left|\arg \left(g^{\prime}(z)-\beta\right)\right| \leq \alpha \pi / 2$.

Let $0 \leq \rho<1,0 \leq \lambda<1$, and $\alpha$ be a positive integer. A function $f \in \mathcal{A}$ is called an $\alpha$-Bazilevič function of order $\rho$ and type $\lambda$, written $f \in \mathcal{B}(\alpha, \rho, \lambda)$, if

$$
\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)^{1-\alpha} g(z)^{\alpha}}\right)>\rho \quad(z \in \mathbb{D})
$$

for some function $g \in \mathcal{S} \mathcal{T}(\lambda)$. The following subclass of $\alpha$-Bazilevič functions is of interest. A function $f \in \mathcal{A}$ is called strongly $\alpha$-Bazilevič of order $\beta$ if

$$
\left|\arg \left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f^{\prime}(z)\right)\right|<\frac{\beta \pi}{2}, \quad(\alpha>0 ; 0<\beta \leq 1)
$$

(see Gao [5]). For the class of strongly $\alpha$-Bazilevič functions of order $\beta$, the following sufficient condition is obtained.

68 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

Theorem 3.2. Let $\alpha>0,0<\beta \leq 1, f \in \mathcal{A}$ and $\left|a_{2}\right|=\eta$, where $\eta, \alpha$ and $\beta$ satisfy

$$
\eta<\sin \left(\frac{\beta \pi}{2(1+\alpha)}\right)
$$

Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\begin{equation*}
|1-\alpha| \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+(1+\alpha) \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \leq \frac{\beta \pi}{2} \tag{3.6}
\end{equation*}
$$

Then $f$ is strongly $\alpha$-Bazilevič of order $\beta$.
Proof. The condition $\eta<\sin (\beta \pi / 2(1+\alpha))$ ensures that there is a real number $\delta(\eta)$ satisfying (3.6). Using (2.1) and (2.2) lead to

$$
\begin{aligned}
\left|\arg \left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f^{\prime}(z)\right)\right| & =\left|\arg \left(\left(\frac{u(z)}{z}\right)^{\alpha-1}(c u(z)+v(z))^{-(\alpha+1)}\right)\right| \\
& \leq|1-\alpha|\left|\arg \left(\frac{u(z)}{z}\right)\right|+|\alpha+1||\arg (c u(z)+v(z))|
\end{aligned}
$$

It now follows from (2.5), (3.5) and (3.6) that

$$
\begin{aligned}
\left|\arg \left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f^{\prime}(z)\right)\right| & \leq|1-\alpha| \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+(1+\alpha) \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \\
& \leq \frac{\beta \pi}{2}
\end{aligned}
$$

For $\alpha \geq 0$, consider the class $R(\alpha)$ defined by

$$
\mathcal{R}(\alpha)=\left\{f \in \mathcal{A}: \operatorname{Re}\left(f^{\prime}(z)+\alpha z f^{\prime \prime}(z)\right)>0, \alpha \geq 0\right\}
$$

For this class, the following sufficient condition is obtained.
Theorem 3.3. Let $\alpha \geq 0, f \in \mathcal{A}$ and $\left|a_{2}\right|=\eta$, where $\eta$ and $\alpha$ satisfy

$$
\begin{equation*}
2 \sin ^{-1} \eta+\sin ^{-1}\left(\frac{2 \eta \alpha}{1-\eta}\right)<\frac{\pi}{2} \tag{3.7}
\end{equation*}
$$

Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\begin{equation*}
2 \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\frac{4 \alpha\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) \leq \frac{\pi}{2} \tag{3.8}
\end{equation*}
$$

Then $f \in \mathcal{R}(\alpha)$.

Proof. Again it is easily seen from a limiting argument that the condition (3.7) guarantees the existence of a real number $\delta(\eta) \geq 0$ satisfying the inequality (3.8). It is sufficient to show that

$$
\left|\arg \left(f^{\prime}(z)\left(1+\alpha \frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)\right)\right|<\frac{\pi}{2} .
$$

The equation (2.2) yields

$$
\begin{equation*}
\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}=-2 z \frac{c u^{\prime}(z)+v^{\prime}(z)}{c u(z)+v(z)} . \tag{3.9}
\end{equation*}
$$

A simple calculation from (2.3) shows that

$$
c u^{\prime}(z)+v^{\prime}(z)=c-\int_{0}^{z} A(\eta)[c u(\eta)+v(\eta)] d \eta,
$$

and an application of (2.6) leads to

$$
\begin{equation*}
\left|c u^{\prime}(z)+v^{\prime}(z)\right| \leq \eta+(1+\eta) \delta e^{\delta / 2} . \tag{3.10}
\end{equation*}
$$

Use of (2.7) results in

$$
\begin{equation*}
|c u(z)+v(z)| \geq 1-|c u(z)+v(z)-1| \geq 1-\eta-\frac{1}{2}(1+\eta) \delta e^{\delta / 2} \tag{3.11}
\end{equation*}
$$

The lower bound in (3.11) is non-negative from the assumption made in (3.8). From (3.9), (3.10) and (3.11), it is evident that

$$
\begin{aligned}
\left|\left(1+\alpha \frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)-1\right| & =\left|2 z \alpha \frac{c u^{\prime}(z)+v^{\prime}(z)}{c u(z)+v(z)}\right| \\
& \leq \frac{2 \alpha\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{1-\eta-\frac{1}{2}(1+\eta) \delta e^{\delta / 2}} .
\end{aligned}
$$

Hence,

$$
\begin{equation*}
\left|\arg \left(1+\alpha \frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)\right| \leq \sin ^{-1}\left(\frac{4 \alpha\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) . \tag{3.12}
\end{equation*}
$$

From (3.5) it follows that

$$
\begin{equation*}
\left|\arg f^{\prime}(z)\right|=2|\arg (c u(z)+v(z))| \leq 2 \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) . \tag{3.13}
\end{equation*}
$$

70 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

Using (2.2) and (3.5), the inequality (3.13) together with (3.12) and (3.8) imply that

$$
\begin{aligned}
& \left|\arg \left(f^{\prime}(z)\left(1+\alpha \frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)\right)\right| \leq\left|\arg f^{\prime}(z)\right|+\left|\arg \left(1+\alpha \frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)\right| \\
& \leq 2 \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\frac{4 \alpha\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) \\
& \leq \frac{\pi}{2}
\end{aligned}
$$

Theorem 3.4. Let $f \in \mathcal{A},\left|a_{2}\right|=\eta \leq 1 / 3$, and $\beta$, $\alpha$ be real numbers satisfying

$$
\begin{equation*}
|\alpha| \sin ^{-1} \eta+|\beta| \sin ^{-1}\left(\frac{2 \eta}{1-\eta}\right)<\frac{\pi}{2} \tag{3.14}
\end{equation*}
$$

Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\begin{align*}
|\alpha| \sin ^{-1} & \left(\frac{1}{2} \delta e^{\delta / 2}\right)+|\alpha| \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \\
& +|\beta| \sin ^{-1}\left(\frac{4\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) \leq \frac{\pi}{2} \tag{3.15}
\end{align*}
$$

Then

$$
\begin{equation*}
\operatorname{Re}\left(\left(\frac{z f^{\prime}(z)}{f(z)}\right)^{\alpha}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)^{\beta}\right)>0 \tag{3.16}
\end{equation*}
$$

Proof. The inequality (3.14) assures the existence of $\delta$ satisfying (3.15). From (2.1) and (2.2) it follows that

$$
\begin{equation*}
\frac{z f^{\prime}(z)}{f(z)}=\frac{z}{u(z)} \frac{1}{c u(z)+v(z)}, \quad z \in \mathbb{D} \tag{3.17}
\end{equation*}
$$

By (2.5) and (3.5),

$$
\begin{equation*}
\left|\arg \left(\frac{z f^{\prime}(z)}{f(z)}\right)\right| \leq \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \tag{3.18}
\end{equation*}
$$

Using (3.12) with $\alpha=1$, (3.18) and (3.15) lead to

$$
\begin{aligned}
& \left|\arg \left(\left(\frac{z f^{\prime}(z)}{f(z)}\right)^{\alpha}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)^{\beta}\right)\right| \leq|\alpha|\left|\arg \left(\frac{z f^{\prime}(z)}{f(z)}\right)\right|+|\beta|\left|\arg \left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)\right| \\
& \leq|\alpha| \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+|\alpha| \sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \\
& \quad+|\beta| \sin ^{-1} \frac{4\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}} \\
& \leq \frac{\pi}{2}
\end{aligned}
$$

This shows that (3.16) holds.
Remark 3.1. Theorem 3.4 yields the following interesting special cases.
(i) If $\alpha=0, \beta=1$, a sufficient condition for convexity is obtained. This case reduces to a result in [1, Theorem 2, p. 109].
(ii) For $\alpha=1, \beta=0$, a sufficient condition for starlikeness is obtained.
(iii) For $\alpha=-1$ and $\beta=1$, then the class of functions satisfying (3.16) reduces to the class of functions

$$
\mathcal{G}:=\left\{f \in \mathcal{A} \left\lvert\, \operatorname{Re}\left(\frac{1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}}{\frac{z f^{\prime}(z)}{f(z)}}\right)>0\right.\right\} .
$$

This class $\mathcal{G}$ was considered by Silverman [14] and Tuneski [15].
Theorem 3.5. Let $\beta \geq 0, f \in \mathcal{A}$ and $\left|a_{2}\right|=\eta$, where $\eta$ satisfies

$$
\begin{equation*}
\sin ^{-1}(\eta)+\sin ^{-1}\left(\frac{2 \beta \eta}{1-\eta}\right)<\frac{\pi}{2} \tag{3.19}
\end{equation*}
$$

Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\begin{aligned}
& \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\frac{4 \beta\left(\eta+(1+\eta) \delta e^{\delta / 2}\right)}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) \\
& \leq \frac{\pi}{2}
\end{aligned}
$$

72 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

Then

$$
\begin{equation*}
\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}+\beta \frac{z^{2} f^{\prime \prime}(z)}{f(z)}\right)>0 \tag{3.20}
\end{equation*}
$$

The proof is similar to the proof of Theorem 3.4, and is therefore omitted. The inequality (3.19) is equivalent to the condition

$$
\eta\left(1+\sqrt{(1-\eta)^{2}-4 \beta^{2} \eta^{2}}+2 \beta \sqrt{1-\eta^{2}}\right)<1
$$

For $\beta=1$, the above equation simplifies to

$$
\eta^{8}-4 \eta^{7}+12 \eta^{6}-12 \eta^{5}+6 \eta^{4}+20 \eta^{3}-4 \eta^{2}-4 \eta+1=0
$$

the value of the root $\eta$ is approximately 0.321336 . Functions satisfying inequality (3.20) were investigated by Ramesha et al. [13].

Consider the class $P(\gamma), 0 \leq \gamma \leq 1$, given by

$$
P(\gamma):=\left\{f \in \mathcal{A}:\left|\arg \left((1-\gamma) \frac{f(z)}{z}+\gamma f^{\prime}(z)\right)\right|<\frac{\pi}{2}, \quad z \in \mathbb{D}\right\}
$$

The same approach applying Gronwall's inequality leads to the following result about the class $P(\gamma)$.

Theorem 3.6. Let $0 \leq \gamma<1, f \in \mathcal{A}$ and $\left|a_{2}\right|=\eta$, where $\eta$ and $\gamma$ satisfy

$$
\begin{equation*}
\sin ^{-1}\left(\frac{\gamma}{1-\gamma} \frac{1}{\eta-1}\right)+\sin ^{-1} \eta<\frac{\pi}{2} \tag{3.21}
\end{equation*}
$$

Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality

$$
\begin{align*}
& \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right)  \tag{3.22}\\
& +\sin ^{-1}\left(\frac{2 \gamma}{1-\gamma} \frac{1}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}} \frac{1}{1-2 e^{\delta / 2}}\right) \leq \frac{\pi}{2}
\end{align*}
$$

Then $f \in P(\gamma)$.

Proof. Condition (3.21) assures the existence of a real number $\delta(\eta) \geq 0$ satisfying the inequality (3.22). A simple calculation from (2.3) and Lemma 2.1 shows that

$$
\begin{aligned}
|u(z)-1| & \leq|z-1|+\left|\int_{0}^{z}(\zeta-z) A(\zeta) u(\zeta) d \zeta\right| \\
& \leq 2 e^{\delta / 2}
\end{aligned}
$$

The above inequality gives

$$
\begin{equation*}
\left|\frac{z}{u(z)}\right| \leq \frac{1}{|u(z)|} \leq \frac{1}{1-|u(z)-1|} \leq \frac{1}{1-2 e^{\delta / 2}} \tag{3.23}
\end{equation*}
$$

Therefore, for some $0<\beta \leq \gamma /(1-\gamma)$, (3.17), (3.23) and (3.11) lead to

$$
\begin{aligned}
\left|1+\beta \frac{z f^{\prime}(z)}{f(z)}-1\right| & =\beta\left|\frac{z}{u(z)}\right| \frac{1}{|c u(z)+v(z)|} \\
& \leq \frac{\beta}{1-2 e^{\delta / 2}} \frac{1}{1-\eta-\frac{1}{2}(1+\eta) \delta e^{\delta / 2}} \\
& =\frac{2 \beta}{1-2 e^{\delta / 2}} \frac{1}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left|\arg \left(1+\beta \frac{z f^{\prime}(z)}{f(z)}\right)\right| \leq \sin ^{-1}\left(\frac{2 \beta}{1-2 e^{\delta / 2}} \frac{1}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) \tag{3.24}
\end{equation*}
$$

Also, (2.5) and (3.5) yield

$$
\begin{align*}
\left|\arg \frac{f(z)}{z}\right| & =\left|\arg \frac{u(z)}{z(c u(z)+v(z))}\right| \\
& \leq\left|\arg \frac{u(z)}{z}\right|+|\arg (c u(z)+v(z))| \\
& \leq \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \tag{3.25}
\end{align*}
$$

74 Rosihan M. Ali, Mahnaz M. Nargesi, V. Ravichandran, and A. Swaminathan

Replacing $\beta$ by $\gamma /(1-\gamma)$ in inequality (3.24), and using (3.25) and (3.22) yield

$$
\begin{aligned}
\left|\arg \left((1-\gamma) \frac{f(z)}{z}+\gamma f^{\prime}(z)\right)\right| \leq & \left|\arg \frac{f(z)}{z}\right|+\left|\arg \left(1+\frac{\gamma}{1-\gamma} \frac{z f^{\prime}(z)}{f(z)}\right)\right| \\
\leq & \sin ^{-1}\left(\frac{1}{2} \delta e^{\delta / 2}\right)+\sin ^{-1}\left(\eta+\frac{1}{2}(1+\eta) \delta e^{\delta / 2}\right) \\
& +\sin ^{-1}\left(\frac{2 \gamma}{1-\gamma} \frac{1}{1-2 e^{\delta / 2}} \frac{1}{2-2 \eta-(1+\eta) \delta e^{\delta / 2}}\right) \\
\leq & \frac{\pi}{2}
\end{aligned}
$$

and hence $f \in P(\gamma)$.

## References

[1] Y. M. Chiang, Properties of analytic functions with small Schwarzian derivative, Complex Variables Theory Appl., 25 no. 2 (1994), 107-118.
[2] S. S. Dragomir, Some Gronwall type inequalities and applications, Nova Sci. Publ., Hauppauge, NY, 2003.
[3] S. Friedland and Z. Nehari, Univalence conditions and Sturm-Liouville eigenvalues, Proc. Amer. Math. Soc., 24 (1970), 595-603.
[4] R. F. Gabriel, The Schwarzian derivative and convex functions, Proc. Amer. Math. Soc., 6 (1955), 58-66.
[5] C. Gao, Fekete-Szegö problem for strongly Bazilevič functions, Northeast. Math. J., 12 no. 4 (1996), 469-474.
[6] D. T. Haimo, A note on convex mappings, Proc. Amer. Math. Soc., 7 (1956), 423-428.
[7] E. Hille, Lectures on ordinary differential equations, Addison-Wesley Publ. Co., Reading, Mass., 1969.
[8] J.-A Kim and T. Sugawa, Geometric properties of functions with small schwarzian derivatives, POSTECH Korea.
[9] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc., 55 (1949), 545-551.
[10] Z. Nehari, Some criteria of univalence, Proc. Amer. Math. Soc., 5 (1954), 700-704.
[11] S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Amer. Math. Soc., 33 (1972), 392-394.
[12] V. V. Pokornyı̆, On some sufficient conditions for univalence, Doklady Akad. Nauk SSSR (N.S.) 79 (1951), 743-746.
[13] C. Ramesha, S. Kumar and K. S. Padmanabhan, A sufficient condition for starlikeness, Chinese J. Math., 23 no. 2 (1995), 167-171.
[14] H. Silverman, Convex and starlike criteria, Int. J. Math. Math. Sci., 22 no. 1 (1999), 75-79.
[15] N. Tuneski, On the quotient of the representations of convexity and starlikeness, Math. Nachr., 248/249 (2003), 200-203.

# A Study of New Mock Theta Functions * 

Bhaskar Srivastava ${ }^{\dagger}$<br>Department of Mathematics and Astronomy<br>Lucknow University Lucknow, India

Received March 5, 2012, Accepted May 9, 2012.


#### Abstract

The mock theta functions of Andrews and the mock theta functions of Bringmann et al are related by half-shift transformation. The generating functions for the partial mock theta functions are given. We extend these mock theta functions to give continued fraction representations. Some interesting expansions are also given in the end.


Keywords and Phrases: Mock theta functions, Generating functions, Continued fractions.

## 1. Introduction

The work of H.M. Srivastava $[\mathbf{1 0}],[\mathbf{1 1}],[\mathbf{1 2}]$ on generating functions motivated me to work on the generating functions. I had the new mock theta functions generated by G.E. Andrews in his paper [2] on orthogonal polynomials and two more mock theta functions generated by Bringmann, Hikami and Lovejoy [3]. The mock theta functions were there and my simple summation identity in [9] was a tool, to give generating functions for the partial mock theta functions. In partial mock theta functions we sum the defining series from 0 to N instead of from 0 to infinity, that is, for the mock theta function $\bar{\psi}_{0}(q)$,

[^6]$$
\bar{\psi}_{0}(q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}}}{(-q ; q)_{2 n}}
$$
the partial mock theta function will be defined and denoted as
$$
\bar{\psi}_{0, N}(q)=\sum_{n=0}^{N} \frac{q^{2 n^{2}}}{(-q ; q)_{2 n}}
$$

The second motivation was to apply the half-shift transformation on these functions. The half-shift transformation was introduced by Gordon and McIntosh [6] to develop eighth order mock theta functions. The application of this method shows that these functions are related to each by half-shift transformation. This is done in section 3.

In my earlier papers I have considered these functions in detail showing they belong to the class of $F_{q}$-functions, their integral representation etc.

In section 5, we give the generating functions for these partial mock theta functions.

In section 6, we represent the generalized functions as continued fraction.
Some expansions for these mock theta functions are given in section 7.
The mock theta functions of Andrews [2]:

$$
\begin{gather*}
\bar{\psi}_{0}(q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}}}{(-q ; q)_{2 n}}={ }_{1} \varphi_{2}\left[\begin{array}{c}
q^{2} \\
-q,-q^{2} ; q^{2}, q^{2}
\end{array}\right]  \tag{1}\\
\bar{\psi}_{1}(q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}+2 n}}{(-q ; q)_{2 n+1}}=\frac{1}{(1+q)}{ }_{1} \varphi_{2}\left[\begin{array}{c}
q^{2} \\
-q^{2},-q^{3} ; q^{2}, q^{4}
\end{array}\right],  \tag{2}\\
\bar{\psi}_{2}(q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}+2 n}\left(q ; q^{2}\right)_{n}}{\left(q^{2} ; q^{2}\right)_{n}(-q ; q)_{2 n}}={ }_{1} \varphi_{2}\left[\begin{array}{c}
q \\
-q,-q^{2} ; q^{2}, q^{4}
\end{array}\right]  \tag{3}\\
\bar{\psi}_{3}(q)=\sum_{n=0}^{\infty} \frac{q^{n^{2}}(-q ; q)_{n}^{2}}{(q ; q)_{2 n}}={ }_{1} \varphi_{2}\left[\begin{array}{c}
-q \\
q^{\frac{1}{2}},-q^{\frac{1}{2}} ; q, q
\end{array}\right] \tag{4}
\end{gather*}
$$

and the mock theta functions of Bringmann, Hikami and Lovejoy [3]:

$$
\bar{\phi}_{0}(q)=\sum_{n=0}^{\infty} q^{n}(-q ; q)_{2 n+1}=(1+q)_{3} \varphi_{2}\left[\begin{array}{c}
q^{2},-q^{2},-q^{3}  \tag{5}\\
0,0
\end{array} ; q^{2}, q\right]
$$

$$
\bar{\phi}_{1}(q)=\sum_{n=0}^{\infty} q^{n}(-q ; q)_{2 n}={ }_{3} \varphi_{2}\left[\begin{array}{c}
q^{2},-q^{2},-q  \tag{6}\\
0,0
\end{array} q^{2}, q\right] .
$$

## 2. Basic Results

The following $q$ - notations have been used
For $\left|q^{k}\right|<1$,

$$
\begin{gathered}
\left(a ; q^{k}\right)_{n}=\prod_{j=0}^{n-1}\left(1-a q^{k j}\right), n \geq 1 \\
\left(a ; q^{k}\right)_{0}=1 \\
\left(a ; q^{k}\right)_{\infty}=\prod_{j=0}^{\infty}\left(1-a q^{k j}\right) \\
(a)_{n}=(a ; q)_{n} \\
\left(a_{1}, a_{2}, \ldots, a_{m} ; q^{k}\right)_{n}=\left(a_{1} ; q^{k}\right)_{n}\left(a_{2} ; q^{k}\right)_{n} \ldots\left(a_{m} ; q^{k}\right)_{n}
\end{gathered}
$$

A generalized basic hypergeometric series with base $q$ is defined as

$$
\begin{gathered}
{ }_{r} \varphi_{s}\left[a_{1}, a_{2}, \ldots, a_{r} ; b_{1}, b_{2}, \ldots, b_{s} ; q, z\right] \\
=\sum_{n=0}^{\infty} \frac{\left(a_{1} ; q\right)_{n} \ldots\left(a_{r} ; q\right)_{n} z^{n}}{\left(b_{1} ; q\right)_{n} \ldots\left(b_{s} ; q\right)_{n}(q ; q)_{n}}\left[(-1)^{n} q^{\frac{n(n-1)}{2}}\right]^{1+s-r},
\end{gathered}
$$

where $q \neq 0$ when $r>s+1$.

## 3. Half-Shift Transformation

(i) We first obtain $\bar{\psi}_{0}(q)$ by applying left half-shift transformation on $\bar{\psi}_{1}(q)$.

Now

$$
\begin{align*}
\bar{\psi}_{1}(q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}+2 n}}{(-q ; q)_{2 n+1}} & =\frac{1}{(-q ; q)_{\infty}} \sum_{n=0}^{\infty} q^{2 n^{2}+2 n}\left(-q^{2 n+2} ; q\right)_{\infty} \\
& =\sum_{n=0}^{\infty} a_{n} \quad(\text { say }) \tag{7}
\end{align*}
$$

where $a_{n}$ is defined for all real $n$. Making a left half-shift transformation and summing $a_{n}$ over the positive half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, instead of the nonnegative integers. Define $b_{n}=a_{n-\frac{1}{2}}$. Then

$$
\begin{gather*}
\sum_{n=0}^{\infty} b_{n}=\sum_{n=0}^{\infty} a_{n-\frac{1}{2}}=\frac{q^{-\frac{1}{2}}}{(-q ; q)_{\infty}} \sum_{n=0}^{\infty} q^{2 n^{2}}\left(-q^{2 n+1} ; q\right)_{\infty} \\
=q^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{q^{2 n^{2}}}{(-q ; q)_{2 n}}=q^{-\frac{1}{2}} \bar{\psi}_{0}(q) \tag{8}
\end{gather*}
$$

Thus by (7) and (8) we obtain $\underline{q}^{-\frac{1}{2}} \bar{\psi}_{0}(q)$ by applying a left half-shift on $\bar{\psi}_{1}(q)$. This implies that $\bar{\psi}_{0}(q)$ and $\bar{\psi}_{1}(q)$ are related by a half-shift transformation.
(ii) Now we obtain $\bar{\varphi}_{0}(q)$ by applying left half-shift transformation on $\bar{\varphi}_{1}(q)$.

By definition

$$
\begin{gather*}
\bar{\varphi}_{1}(q)=\sum_{n=0}^{\infty} q^{n}(-q ; q)_{2 n}=(-q ; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n}}{\left(-q^{2 n+1} ; q\right)_{\infty}} \\
=\sum_{n=0}^{\infty} a_{n} \quad(\text { say }) \tag{9}
\end{gather*}
$$

where $a_{n}$ is defined for all real $n$. Making a left half-shift transformation and summing $a_{n}$ over the positive half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, instead of the nonnegative integers. Define $b_{n}=a_{n-\frac{1}{2}}$. Then

$$
\sum_{n=0}^{\infty} b_{n}=(-q ; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n-\frac{1}{2}}}{\left(-q^{2 n} ; q\right)_{\infty}}=q^{-\frac{1}{2}} \sum_{n=0}^{\infty} q^{n}(-q ; q)_{2 n-1}
$$

$$
\begin{gather*}
=q^{-\frac{1}{2}} \sum_{n=1}^{\infty} q^{n+1}(-q ; q)_{2 n+1} \\
=q^{-\frac{1}{2}}\left[\frac{1}{2}+\sum_{n=0}^{\infty} q^{n+1}(-q ; q)_{2 n+1}\right] \\
=\frac{1}{2 \sqrt{q}}+q^{\frac{1}{2}} \bar{\varphi}_{0}(q) \tag{10}
\end{gather*}
$$

Thus by applying a left half-shift transformation on $q^{\frac{1}{2}} \bar{\varphi}_{0}(q)$ we obtain $\bar{\varphi}_{1}(q)-\frac{1}{2 \sqrt{q}}$.

## 4. Definition of Generalized Mock Theta Functions

We define the generalized mock theta functions as:

$$
\begin{gather*}
\bar{\psi}_{0}(z, \alpha, q)=\frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_{n} q^{2 n^{2}-n+n \alpha}}{(-q ; q)_{2 n}}  \tag{11}\\
\bar{\psi}_{1}(z, \alpha, q)=\frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_{n} q^{2 n^{2}+n+n \alpha}}{(-q ; q)_{2 n+1}}  \tag{12}\\
\bar{\psi}_{2}(z, \alpha, q)=\frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_{n} q^{2 n^{2}+n+n \alpha}\left(q ; q^{2}\right)_{n}}{\left(q^{2} ; q^{2}\right)_{n}(-q ; q)_{2 n}}  \tag{13}\\
\bar{\psi}_{3}(z, \alpha, q)=\frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_{n} q^{n^{2}-n+n \alpha}(-q ; q)_{n}^{2}}{\left(q^{2} ; q^{2}\right)_{n}\left(q ; q^{2}\right)_{n}}  \tag{14}\\
\bar{\varphi}_{0}(z, \alpha, q)=\frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty}(z)_{n} q^{n+n \alpha}(-q ; q)_{2 n+1} \tag{15}
\end{gather*}
$$

and

$$
\begin{equation*}
\bar{\varphi}_{1}(z, \alpha, q)=\frac{1}{(z)_{\infty}} \sum_{n=0}^{\infty}(z)_{n} q^{n+n \alpha}(-q ; q)_{2 n} \tag{16}
\end{equation*}
$$

For $z=0$ and $\alpha=1$, the generalized functions defined in (11)-(14) reduce to mock theta functions $\bar{\psi}_{0}(q), \bar{\psi}_{1}(q), \bar{\psi}_{2}(q)$ and $\bar{\psi}_{3}(q)$ respectively. For $z=0$, $\alpha=0$ the generalized functions defined in (15)-(16) reduce to the mock theta functions $\bar{\varphi}_{0}(q)$ and $\bar{\varphi}_{1}(q)$, respectively.

## 5. Generating Functions for Partial Generalized Functions and Partial Mock Theta Functions

I shall be using the following summation identity, which I deduced in [10] to give the generating functions for the generalized functions.

$$
\begin{equation*}
\sum_{r=0}^{p} \alpha_{r} \beta_{r}=\beta_{p+1} \sum_{r=0}^{p} \alpha_{r}+\sum_{m=0}^{p}\left(\beta_{m}-\beta_{m+1}\right) \sum_{r=0}^{m} \alpha_{r} . \tag{17}
\end{equation*}
$$

Taking $\beta_{n}=z^{n},|z|<1$ in (17), we have

$$
\begin{equation*}
\sum_{r=0}^{p} \alpha_{r} z^{r}=z^{p+1} \sum_{r=0}^{p} \alpha_{r}+(1-z) \sum_{m=0}^{p} z^{m} \sum_{r=0}^{m} \alpha_{r} \tag{18}
\end{equation*}
$$

Letting $p \rightarrow \infty$ in (18)

$$
\begin{gather*}
\sum_{m=0}^{\infty} z^{m} \sum_{r=0}^{m} \alpha_{r}=\frac{1}{(1-z)} \sum_{r=0}^{\infty} \alpha_{r} z^{r} \\
=\left[\sum_{r=0}^{\infty} \alpha_{r} z^{r}\right] \sum_{n=0}^{\infty} z^{n} \tag{19}
\end{gather*}
$$

Now we define $\alpha_{r}$ such that $\sum_{r=0}^{m} \alpha_{r}$ is a partial generalized function.
(i) Take $\alpha_{r}=\frac{q^{2 r^{2}-r+r \alpha}}{(-q ; q)_{2 r}}$ in (19), to have

$$
\sum_{m=0}^{\infty} z^{m} \bar{\psi}_{0, m}(0, \alpha, q)=\left[\sum_{r=0}^{\infty} \frac{q^{2 r^{2}-r+r \alpha}}{(-q ; q)_{2 r}} z^{r}\right] \sum_{n=0}^{\infty} z^{n}
$$

$$
={ }_{1} \phi_{2}\left[\begin{array}{c}
q^{2}  \tag{20}\\
-q,-q^{2}
\end{array} ; q^{2}, z q^{\alpha+1}\right] \sum_{n=0}^{\infty} z^{n} .
$$

Here $\bar{\psi}_{0, m}(0, \alpha, q)$ is the partial generalized function. Thus we have the generating function for $\bar{\psi}_{0, m}(0, \alpha, q)$.

Taking $\alpha=1$ in (20), we have

$$
\sum_{m=0}^{\infty} z^{m} \bar{\psi}_{0, m}(q)={ }_{1} \phi_{2}\left[\begin{array}{c}
q^{2} \\
-q,-q^{2}
\end{array} ; q^{2}, z q^{2}\right] \sum_{n=0}^{\infty} z^{n}
$$

We list the generating functions for other generalized and partial mock theta functions, omitting calculations, only giving the values of $\alpha_{r}$ in the parenthesis.
(ii)

$$
\begin{gather*}
(1+q) \sum_{m=0}^{\infty} z^{m} \bar{\psi}_{1, m}(0, \alpha, q)={ }_{1} \phi_{2}\left[\begin{array}{c}
q^{2} \\
-q^{2},-q^{3}
\end{array} ; q^{2}, z q^{3+\alpha}\right] \sum_{n=0}^{\infty} z^{n}  \tag{21}\\
\left(\alpha_{r}=\frac{q^{2 r^{2}+r+r \alpha}}{(-q ; q)_{2 r+1}} \text { in }(19)\right)
\end{gather*}
$$

(iii)

$$
\begin{gathered}
(1+q) \sum_{m=0}^{\infty} z^{m} \bar{\psi}_{1, m}(q)={ }_{1} \phi_{2}\left[\begin{array}{c}
q^{2} \\
-q^{2},-q^{3}
\end{array} ; q^{2}, z q^{4}\right] \sum_{n=0}^{\infty} z^{n} . \\
(\alpha=1 \text { in }(21))
\end{gathered}
$$

(iv)

$$
\begin{gather*}
\sum_{m=0}^{\infty} z^{m} \bar{\psi}_{2, m}(0, \alpha, q)={ }_{1} \phi_{2}\left[\begin{array}{c}
q \\
-q,-q^{2}
\end{array} ; q^{2}, z q^{3+\alpha}\right] \sum_{n=0}^{\infty} z^{n}  \tag{23}\\
\left(\alpha_{r}=\frac{q^{2 r^{2}+r+r \alpha}\left(q ; q^{2}\right)_{r}}{\left(q^{2} ; q^{2}\right)_{r}(-q ; q)_{2 r}} \text { in }(19)\right)
\end{gather*}
$$

(v)

$$
\sum_{m=0}^{\infty} z^{m} \bar{\psi}_{2, m}(q)={ }_{1} \phi_{2}\left[\begin{array}{c}
q  \tag{24}\\
-q,-q^{2}
\end{array} ; q^{2}, z q^{4}\right] \sum_{n=0}^{\infty} z^{n}
$$

$$
(\alpha=1 \text { in }(23))
$$

(vi)

$$
\begin{aligned}
& \sum_{m=0}^{\infty} z^{m} \bar{\psi}_{3, m}(0, \alpha, q)={ }_{1} \phi_{2}\left[\begin{array}{c}
-q \\
q^{\frac{1}{2}},-q^{\frac{1}{2}}
\end{array} ; q, z q^{\alpha}\right] \sum_{n=0}^{\infty} z^{n} . \\
& \quad\left(\alpha_{r}=\frac{q^{r^{2}-r+r \alpha}(-q ; q)_{r}}{(q ; q)_{r}\left(q^{\frac{1}{2}} ; q\right)_{r}\left(-q^{\frac{1}{2}} ; q\right)_{r}} \text { in }(19)\right)
\end{aligned}
$$

(vii)

$$
\begin{align*}
\sum_{m=0}^{\infty} z^{m} \bar{\psi}_{3, m}(q) & ={ }_{1} \phi_{2}\left[\begin{array}{c}
-q \\
q^{\frac{1}{2}},-q^{\frac{1}{2}}
\end{array} ; q, z q\right] \sum_{n=0}^{\infty} z^{n}  \tag{26}\\
(\alpha & =1 \text { in }(25))
\end{align*}
$$

(viii)

$$
\begin{gather*}
\sum_{m=0}^{\infty} z^{m} \bar{\varphi}_{0, m}(0, \alpha, q)=(1+q){ }_{3} \phi_{2}\left[\begin{array}{c}
-q^{2},-q^{3}, q^{2} \\
0,0
\end{array} ; q^{2}, z q^{\alpha+1}\right] \sum_{n=0}^{\infty} z^{n}  \tag{27}\\
\left(\alpha_{r}=q^{r+r \alpha}(-q ; q)_{2 r+1} \text { in }(19)\right)
\end{gather*}
$$

(ix)

$$
\begin{gather*}
\sum_{m=0}^{\infty} z^{m} \bar{\varphi}_{0, m}(q)=(1+q){ }_{3} \phi_{2}\left[\begin{array}{c}
-q^{2},-q^{3}, q^{2} \\
0,0
\end{array} ; q^{2}, z q\right] \sum_{n=0}^{\infty} z^{n}  \tag{28}\\
(\alpha=0 \text { in }(27))
\end{gather*}
$$

(x)

$$
\begin{gathered}
\sum_{m=0}^{\infty} z^{m} \bar{\varphi}_{1, m}(0, \alpha, q)={ }_{3} \phi_{2}\left[\begin{array}{c}
q^{2},-q^{2},-q \\
0,0
\end{array} ; q^{2}, z q^{\alpha+1}\right] \sum_{n=0}^{\infty} z^{n} \\
\left(\alpha_{r}=q^{r+r \alpha}(-q ; q)_{2 r} \text { in }(19)\right)
\end{gathered}
$$

(xi)

$$
\begin{gather*}
\sum_{m=0}^{\infty} z^{m} \bar{\varphi}_{1, m}(q)={ }_{3} \phi_{2}\left[\begin{array}{c}
q^{2},-q^{2},-q \\
0,0
\end{array} ; q^{2}, z q\right] \sum_{n=0}^{\infty} z^{n} .  \tag{30}\\
(\alpha=0 \text { in }(29))
\end{gather*}
$$

## 6. Continued Fraction Representation

We shall give the continued fraction representation for generalized functions.
(i) Representation of $\bar{\psi}_{0}(z, \alpha, q)$ as continued fraction

By definition

$$
\begin{equation*}
\bar{\psi}_{0}(0, \alpha, q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}-n+n \alpha}}{(-q ; q)_{2 n}} \tag{31}
\end{equation*}
$$

Letting $q \rightarrow q^{2}, \lambda=0, b=q^{2}, c=1 / q, a=1$ in $[\mathbf{1},(5.26)$, p. 97], we have

$$
\begin{equation*}
\frac{\sum_{n=0}^{\infty} \frac{q^{2 n^{2}-n}}{(-q ; q)_{2 n}}}{2 \sum_{n=0}^{\infty} \frac{q^{2 n^{2}+n}}{\left(-q ; q^{2}\right)_{n}\left(-q^{2} ; q^{2}\right)_{n+1}}}=1+\frac{\left(1-1 / q^{2}\right) q^{2}}{(1+1)+} \frac{q}{1+} \frac{\left(1-1 / q^{4}\right) q^{4}}{2+} \frac{q^{3}}{1+\ldots} \tag{32}
\end{equation*}
$$

Taking $\alpha=0$ in (31), we have from (32)

$$
\begin{equation*}
\frac{\bar{\psi}_{0}(0,0, q)}{2 \sum_{n=0}^{\infty} \frac{q^{2 n^{2}+n}}{\left(-q ; q^{2}\right)_{n}\left(-q^{2} ; q^{2}\right)_{n+1}}}=1+\frac{\left(1-1 / q^{2}\right) q^{2}}{(1+1)+} \frac{q}{1+} \frac{\left(1-1 / q^{4}\right) q^{4}}{2+} \frac{q^{3}}{1+\ldots} \tag{33}
\end{equation*}
$$

## (ii) Representation of $\bar{\psi}_{1}(z, \alpha, q)$ as continued fraction

By definition

$$
\begin{equation*}
\bar{\psi}_{1}(0, \alpha, q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}+n+n \alpha}}{(-q ; q)_{2 n+1}}=\frac{1}{(1+q)} \sum_{n=0}^{\infty} \frac{q^{2 n^{2}+n+n \alpha}}{\left(-q^{2} ; q\right)_{2 n}} \tag{34}
\end{equation*}
$$

Letting $q \rightarrow q^{2}, \lambda=0, b=q^{2}, c=q, a=1$ in $[\mathbf{1},(5.26)$, p. 97], we have

$$
\begin{equation*}
\frac{\sum_{n=0}^{\infty} \frac{q^{2 n^{2}+n}}{\left(-q^{2} ; q\right)_{2 n}}}{2 \sum_{n=0}^{\infty} \frac{q^{2 n^{2}+3 n}}{\left(-q^{3} ; q^{2}\right)_{n}\left(-q^{2} ; q^{2}\right)_{n+1}}}=1+\frac{\left(1-1 / q^{2}\right) q^{2}}{(1+1)+} \frac{q^{3}}{1+} \frac{\left(1-1 / q^{4}\right) q^{4}}{2+} \frac{q^{5}}{1+\ldots} \tag{35}
\end{equation*}
$$

Taking $\alpha=0$ in (34), we have from (35)

$$
\begin{equation*}
\frac{\bar{\psi}_{1}(0,0, q)}{2 \sum_{n=0}^{\infty} \frac{q^{2 n^{2}+3 n}}{\left(-q ; q^{2}\right)_{n+1}\left(-q^{2} ; q^{2}\right)_{n+1}}}=1+\frac{\left(1-1 / q^{2}\right) q^{2}}{(1+1)+} \frac{q^{3}}{1+} \frac{\left(1-1 / q^{4}\right) q^{4}}{2+} \frac{q^{5}}{1+\ldots} \tag{36}
\end{equation*}
$$

(iii) Representation of $\bar{\psi}_{2}(z, \alpha, q)$ as continued fraction

By definition

$$
\begin{equation*}
\bar{\psi}_{2}(0, \alpha, q)=\sum_{n=0}^{\infty} \frac{q^{2 n^{2}+n+n \alpha}\left(q ; q^{2}\right)_{n}}{\left(q^{2} ; q^{2}\right)_{n}(-q ; q)_{2 n}} \tag{37}
\end{equation*}
$$

Letting $q \rightarrow q^{2}, \lambda=0, b=q, c=1 / q, a=1$ in $[\mathbf{1},(5.26)$, p. 97$]$, we have

$$
\begin{equation*}
\frac{\sum_{n=0}^{\infty} \frac{q^{2 n^{2}}\left(q ; q^{2}\right)_{n}}{\left(q^{2} ; q^{2}\right)_{n}\left(-q ; q^{2}\right)_{n}\left(-q^{2} ; q^{2}\right)_{n}}}{(1+q) \sum_{n=0}^{\infty} \frac{q^{2 n^{2}+2 n}\left(q ; q^{2}\right)_{n}}{\left(q^{2} ; q^{2}\right)_{n}\left(-q ; q^{2}\right)_{n}\left(-q^{2} ; q^{2}\right)_{n+1}}}=1+\frac{(1-1 / q) q^{2}}{(1+q)+} \frac{q}{1+} \frac{\left(1-1 / q^{3}\right) q^{4}}{1+q+\ldots} \tag{38}
\end{equation*}
$$

Taking $\alpha=-1$ in (37), we have from (38)
(iv) Representation of $\bar{\psi}_{3}(z, \alpha, q)$ as continued fraction

By definition

$$
\begin{equation*}
\bar{\psi}_{3}(0, \alpha, q)=\sum_{n=0}^{\infty} \frac{q^{n^{2}-n+n \alpha}(-q ; q)_{n}^{2}}{(q ; q)_{2 n}}=\sum_{n=0}^{\infty} \frac{q^{n^{2}-n+n \alpha}(-q ; q)_{n}}{(q ; q)_{n}\left(q^{\frac{1}{2}} ; q\right)_{n}\left(-q^{\frac{1}{2}} ; q\right)_{n}} \tag{40}
\end{equation*}
$$

Letting $\lambda=0, b=-q, c=1 / q^{\frac{1}{2}}, a=-1 / q^{\frac{1}{2}}$ in $[\mathbf{1},(5.26)$, p. 97], we have

$$
\begin{equation*}
\frac{\sum_{n=0}^{\infty} \frac{q^{n^{2}-n}(-q ; q)_{n}}{(q ; q)_{n}\left(q^{\frac{1}{2}} ; q\right)_{n}\left(-q^{\frac{1}{2}} ; q\right)_{n}}}{\left(1+1 / q^{\frac{1}{2}}\right) \sum_{n=0}^{\infty} \frac{q^{n^{2}}(-q ; q)_{n}}{(q ; q)_{n}\left(q^{\frac{1}{2}} ; q\right)_{n+1}\left(-q^{\frac{1}{2}} ; q\right)_{n}}}=1+\frac{(1+1 / q)\left(-q^{\frac{1}{2}}\right)}{\left(1+q^{\frac{1}{2}}\right)+} \frac{q^{\frac{1}{2}}}{1+} \frac{\left(1+1 / q^{2}\right)\left(-q^{\frac{3}{2}}\right)}{\left(1+q^{\frac{1}{2}}\right) \ldots} \tag{41}
\end{equation*}
$$

Taking $\alpha=0$ in (40), we have from (41)

$$
\begin{equation*}
\frac{\bar{\psi}_{3}(0,0, q)}{\left(1+1 / q^{\frac{1}{2}}\right) \sum_{n=0}^{\infty} \frac{q^{n^{2}}(-q ; q)_{n}}{(q ; q)_{n}\left(q^{\frac{1}{2}} ; q\right)_{n+1}\left(-q^{\frac{1}{2}} ; q\right)_{n}}}=1+\frac{(1+1 / q)\left(-q^{\frac{1}{2}}\right)}{\left(1+q^{\frac{1}{2}}\right)+} \frac{q^{\frac{1}{2}}}{1+} \frac{\left(1+1 / q^{2}\right)\left(-q^{\frac{3}{2}}\right)}{\left(1+q^{\frac{1}{2}}\right) \ldots} \tag{42}
\end{equation*}
$$

## 7. Expansions of the Mock Theta Functions

We have general expansion formula $[\mathbf{5}, \mathrm{p} 70],[8, \mathrm{p} 56]$

$$
\begin{equation*}
\sum_{k=0}^{\infty} \frac{\left(1-a q^{2 k}\right)(a, b, c, a / b c)_{k} q^{k}}{(1-a)(q, a q / b, a q / c, b c q)_{k}} \sum_{m=0}^{\infty} \alpha_{m+k}=\sum_{m=0}^{\infty} \frac{(a q, b q, c q, a q / b c)_{m}}{(q, a q / b, a q / c, b c q)_{m}} \alpha_{m} \tag{43}
\end{equation*}
$$

Letting $q \rightarrow q^{2}$ and $b, c \rightarrow \infty$ in (43), we have

$$
\begin{equation*}
\sum_{k=0}^{\infty} \frac{(-1)^{k}\left(1-a q^{4 k}\right)\left(a ; q^{2}\right)_{k} q^{k^{2}-k}}{(1-a)\left(q^{2} ; q^{2}\right)_{k}} \sum_{m=0}^{\infty} \alpha_{m+k}=\sum_{m=0}^{\infty} \frac{(-1)^{m}\left(a q^{2} ; q^{2}\right)_{m} q^{m^{2}+m}}{\left(q^{2} ; q^{2}\right)_{m}} \alpha_{m} \tag{44}
\end{equation*}
$$

Putting $a=0$ in (44)

$$
\begin{equation*}
\sum_{k=0}^{\infty} \frac{(-1)^{k} q^{k^{2}-k}}{\left(q^{2} ; q^{2}\right)_{k}} \sum_{m=0}^{\infty} \alpha_{m+k}=\sum_{m=0}^{\infty} \frac{(-1)^{m} q^{m^{2}+m}}{\left(q^{2} ; q^{2}\right)_{m}} \alpha_{m} . \tag{45}
\end{equation*}
$$

Putting $a=1$ in (44)

$$
\begin{equation*}
\sum_{k=0}^{\infty}(-1)^{k}\left(1+q^{2 k}\right) q^{k^{2}-k} \sum_{m=0}^{\infty} \alpha_{m+k}=\sum_{m=0}^{\infty}(-1)^{m} q^{m^{2}+m} \alpha_{m} \tag{46}
\end{equation*}
$$

Writing the inner sum of the left side of (43) as the difference of two series, we have

$$
\begin{gather*}
\frac{(a q, b q, c q, a q / b c)_{\infty}}{(q, a q / b, a q / c, b c q)_{\infty}} \sum_{m=0}^{\infty} \alpha_{m}-\sum_{k=0}^{\infty} \frac{\left(1-a q^{2 k+2}\right)(a, b, c, a / b c)_{k+1} q^{k+1}}{(1-a)(q, a q / b, a q / c, b c q)_{k+1}} \sum_{m=0}^{k} \alpha_{m} \\
=\sum_{m=0}^{\infty} \frac{(a q, b q, c q, a q / b c)_{m}}{(q, a q / b, a q / c, b c q)_{m}} \alpha_{m} \tag{47}
\end{gather*}
$$

Putting $a=0$ in (47), we have

$$
\begin{gather*}
\frac{(b q, c q)_{\infty}}{(q, b c q)_{\infty}} \sum_{m=0}^{\infty} \alpha_{m}-\sum_{k=0}^{\infty} \frac{(b, c)_{k+1} q^{k+1}}{(q, b c q)_{k+1}} \sum_{m=0}^{k} \alpha_{m} \\
=\sum_{m=0}^{\infty} \frac{(b q, c q)_{m}}{(q, b c q)_{m}} \alpha_{m} \tag{48}
\end{gather*}
$$

Taking $b=c=0$ in (48), we have

$$
\begin{equation*}
\frac{1}{(q)_{\infty}} \sum_{m=0}^{\infty} \alpha_{m}-\sum_{k=0}^{\infty} \frac{q^{k+1}}{(q)_{k+1}} \sum_{m=0}^{k} \alpha_{m}=\sum_{m=0}^{\infty} \frac{1}{(q)_{m}} \alpha_{m} \tag{49}
\end{equation*}
$$

(a) Expansions for $\bar{\psi}_{0}(q)$
(i) Taking $\alpha_{m}=\frac{(-1)^{m} q^{m^{2}-m}\left(q^{2} ; q^{2}\right)_{m}}{(-q ; q)_{2 m}}$ in (45), we have

$$
\begin{align*}
\bar{\psi}_{0}(q) & =\sum_{k=0}^{\infty} \frac{(-1)^{k} q^{2 k^{2}-2 k}}{\left(q^{2} ; q^{2}\right)_{k}} \sum_{m=0}^{\infty} \frac{(-1)^{m+k} q^{m^{2}-m+2 m k}\left(q^{2} ; q^{2}\right)_{m+k}}{(-q ; q)_{2 m+2 k}} \\
& =\sum_{k=0}^{\infty} \frac{q^{2 k^{2}-2 k}}{(-q ; q)_{2 k}} \sum_{m=0}^{\infty} \frac{(-1)^{m} q^{m^{2}-m+2 m k}\left(q^{2 k+2} ; q^{2}\right)_{m}}{\left(-q^{2 k+1} ; q\right)_{2 m}} \\
& =\sum_{k=0}^{\infty} \frac{q^{2 k^{2}-2 k}}{(-q ; q)_{2 k}}{ }_{2} \phi_{2}\left[\begin{array}{c}
q^{2}, q^{2 k+2} \\
-q^{2 k+1},-q^{2 k+2}
\end{array} ; q^{2}, q^{2 k}\right] \tag{50}
\end{align*}
$$

(ii)

$$
\begin{align*}
& \bar{\psi}_{0}(q)= \\
& \sum_{k=0}^{\infty} \frac{\left(1+q^{2 k}\right) q^{2 k^{2}-2 k}}{(-q ; q)_{2 k}} \sum_{m=0}^{\infty} \frac{(-1)^{m} q^{m^{2}-m+2 m k}}{\left(-q^{2 k+1} ; q\right)_{2 m}}  \tag{51}\\
&= \sum_{k=0}^{\infty} \frac{\left(1+q^{2 k}\right) q^{2 k^{2}-2 k}}{(-q ; q)_{2 k}}{ }_{2} \phi_{2}\left[\begin{array}{c}
q^{2}, 0 \\
\left.-q^{2 k+1},-q^{2 k+2} ; q^{2}, q^{2 k}\right] . \\
\\
\quad\left(\alpha_{m}=\frac{(-1)^{m} q^{m^{2}-m}}{(-q ; q)_{2 m}} \text { in }(46)\right)
\end{array}, .\right.
\end{align*}
$$

(iii)

$$
\begin{gather*}
\frac{\left(-q,-q^{2} ; q^{2}\right)_{\infty}}{\left(q, q^{2} ; q^{2}\right)_{\infty}} \bar{\psi}_{0}(q)-2(1+1 / q) \sum_{k=0}^{\infty} \frac{\left(-q,-q^{2} ; q^{2}\right)_{k} q^{k+1}}{\left(q, q^{2} ; q^{2}\right)_{k+1}} \bar{\psi}_{0, k}(q) \\
=\sum_{m=0}^{\infty} \frac{q^{2 m^{2}}}{\left(q ; q^{2}\right)_{m}\left(q^{2} ; q^{2}\right)_{m}}={ }_{0} \phi_{1}\left[\begin{array}{c}
-, \\
q,
\end{array} q^{2}, q^{2}\right]  \tag{52}\\
\left(q \rightarrow q^{2}, b=-1, c=-1 / q, \alpha_{m}=\frac{q^{2 m^{2}}}{(-q ; q)_{2 m}} \text { in }(48)\right)
\end{gather*}
$$

(iv)

$$
\begin{gather*}
\bar{\psi}_{0}(q)-\left(q^{2} ; q^{2}\right)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2 k+2}}{\left(q^{2} ; q^{2}\right)_{k+1}} \bar{\psi}_{0, k}(q)=\left(q^{2} ; q^{2}\right)_{\infty} \sum_{m=0}^{\infty} \frac{q^{2 m^{2}}}{(-q ; q)_{2 m}\left(q^{2} ; q^{2}\right)_{m}} \\
=\left(q^{2} ; q^{2}\right)_{\infty}{ }_{1} \phi_{2}\left[\begin{array}{c}
0, \\
-q,-q^{2}
\end{array} ; q^{2}, q^{2}\right] .  \tag{53}\\
\\
\left(q \rightarrow q^{2}, \alpha_{m}=\frac{q^{2 m^{2}}}{(-q ; q)_{2 m}} \text { in }(49)\right)
\end{gather*}
$$

(b) Expansions for $\bar{\psi}_{1}(q)$
(i)

$$
\begin{gather*}
\bar{\psi}_{1}(q)=\sum_{k=0}^{\infty} \frac{q^{2 k^{2}}}{(-q ; q)_{2 k+1}} \\
\times{ }_{2} \phi_{2}\left[\begin{array}{c}
q^{2}, q^{2 k+2} \\
-q^{2 k+2},-q^{2 k+3} ; q^{2}, q^{2 k+2}
\end{array}\right]  \tag{54}\\
\left(\alpha_{m}=\frac{(-1)^{m} q^{m^{2}+m}\left(q^{2} ; q^{2}\right)_{m}}{\left(-q^{2} ; q\right)_{2 m}} \text { in }(45)\right)
\end{gather*}
$$

(ii)

$$
\begin{gather*}
\bar{\psi}_{1}(q)=\sum_{k=0}^{\infty} \frac{\left(1+q^{2 k}\right) q^{2 k^{2}}}{(-q ; q)_{2 k+1}}{ }_{2} \phi_{2}\left[\begin{array}{c}
q^{2}, 0 \\
\left.-q^{2 k+2},-q^{2 k+3} ; q^{2}, q^{2 k+2}\right] \\
\left(\alpha_{m}=\frac{(-1)^{m} q^{m^{2}+m}}{(-q ; q)_{2 m+1}}\right. \text { in (46))}
\end{array} .\right. \tag{55}
\end{gather*}
$$

(iii)

$$
\begin{gather*}
\frac{\left(-q,-q^{2} ; q^{2}\right)_{\infty}}{\left(q^{2}, q^{3} ; q^{2}\right)_{\infty}} \bar{\psi}_{1}(q)-2(1+q)^{2} \sum_{k=0}^{\infty} \frac{\left(-q^{2},-q^{3} ; q^{2}\right)_{k} q^{2 k+2}}{\left(q^{2}, q^{3} ; q^{2}\right)_{k+1}} \bar{\psi}_{1, k}(q) \\
={ }_{0} \phi_{1}\left[\begin{array}{c}
-, \\
q^{3},
\end{array} ; q^{2}, q^{4}\right]  \tag{56}\\
\left(q \rightarrow q^{2}, b=-1, c=-q, \alpha_{m}=\frac{q^{2 m^{2}+2 m}}{(-q ; q)_{2 m+1}} \text { in }(48)\right)
\end{gather*}
$$

(iv)

$$
\begin{gather*}
\bar{\psi}_{1}(q)-\left(q^{2} ; q^{2}\right)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2 k+2}}{\left(q^{2} ; q^{2}\right)_{k+1}} \bar{\psi}_{1, k}(q)=\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{(1+q)} 1 \phi_{2}\left[\begin{array}{c}
0, \\
\left.-q^{2},-q^{3} ; q^{2}, q^{4}\right] \\
\\
\left(q \rightarrow q^{2}, \alpha_{m}=\frac{q^{2 m^{2}+2 m}}{(-q ; q)_{2 m+1}} \text { in }(49)\right)
\end{array}, .\right. \tag{57}
\end{gather*}
$$

(c) Expansions for $\bar{\psi}_{2}(q)$
(i)

$$
\begin{gather*}
\bar{\psi}_{2}(q)=\sum_{k=0}^{\infty} \frac{q^{2 k^{2}}\left(q ; q^{2}\right)_{k}}{\left(q^{2} ; q^{2}\right)_{k}(-q ; q)_{2 k}}{ }_{2} \phi_{2}\left[\begin{array}{c}
q^{2}, q^{2 k+1} \\
-q^{2 k+1},-q^{2 k+2} ; q^{2}, q^{2 k+2}
\end{array}\right] .  \tag{58}\\
\left(\alpha_{m}=\frac{(-1)^{m} q^{m^{2}+m}\left(q ; q^{2}\right)_{m}}{(-q ; q)_{2 m}}\right. \text { in (45))}
\end{gather*}
$$

(ii)

$$
\begin{gather*}
\bar{\psi}_{2}(q)=\sum_{k=0}^{\infty} \frac{\left(q ; q^{2}\right)_{k}\left(1+q^{2 k}\right) q^{2 k^{2}}}{\left(q^{2} ; q^{2}\right)_{k}(-q ; q)_{2 k}}{ }_{3} \phi_{3}\left[\begin{array}{c}
q^{2}, q^{2 k+1}, 0 \\
\left.q^{2 k+2},-q^{2 k+1},-q^{2 k+2} ; q^{2}, q^{2 k+2}\right]
\end{array}\right.  \tag{59}\\
\left(\alpha_{m}=\frac{(-1)^{m} q^{m^{2}+m}\left(q ; q^{2}\right)_{m}}{\left(q^{2} ; q^{2}\right)_{m}(-q ; q)_{2 m}} \text { in }(46)\right)
\end{gather*}
$$

(iii)

$$
\begin{align*}
& \frac{\left(-q,-q^{2} ; q^{2}\right)_{\infty}}{\left(q, q^{2} ; q^{2}\right)_{\infty}} \bar{\psi}_{2}(q)-\frac{2(1+q)}{q} \sum_{k=0}^{\infty} \frac{\left(-q,-q^{2} ; q^{2}\right)_{k} q^{2 k+2}}{\left(q, q^{2} ; q^{2}\right)_{k+1}} \bar{\psi}_{2, k}(q) \\
& ={ }_{1} \phi_{2}\left[\begin{array}{c}
0, \\
q, q^{2}
\end{array} ; q^{2}, q^{4}\right] .  \tag{60}\\
& \left(q \rightarrow q^{2}, b=-1 / q, c=-1, \alpha_{m}=\frac{q^{2 m^{2}+2 m}\left(q ; q^{2}\right)_{m}}{\left(q^{2} ; q^{2}\right)_{m}(-q ; q)_{2 m}} \text { in }(48)\right)
\end{align*}
$$

(iv)

$$
\begin{align*}
& \bar{\psi}_{2}(q)-\left(q^{2} ; q^{2}\right)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2 k+2}}{\left(q^{2} ; q^{2}\right)_{k+1}} \bar{\psi}_{2, k}(q) \\
& =\left(q^{2} ; q^{2}\right)_{\infty_{2}} \phi_{3}\left[\begin{array}{c}
q, 0, \\
\left.q^{2},-q,-q^{2} ; q^{2}, q^{4}\right]
\end{array}\right.  \tag{61}\\
& \left(q \rightarrow q^{2}, \alpha_{m}=\frac{q^{2 m^{2}+2 m}\left(q ; q^{2}\right)_{m}}{\left(q^{2} ; q^{2}\right)_{m}(-q ; q)_{2 m}} \text { in }(49)\right)
\end{align*}
$$

(d) Expansions for $\bar{\psi}_{3}(q)$
(i)

$$
\begin{gather*}
\bar{\psi}_{3}(q)=\sum_{k=0}^{\infty} \frac{q^{k^{2}}(-q ; q)_{k}}{(q ; q)_{k}\left(q^{\frac{1}{2}} ; q\right)_{k}\left(-q^{\frac{1}{2}} ; q\right)_{k}} \\
\times{ }_{2} \phi_{2}\left[\begin{array}{c}
q,-q^{k+1} \\
q^{k+\frac{1}{2}},-q^{k+\frac{1}{2}} ; q, q^{k}
\end{array}\right] .  \tag{62}\\
\left(q \rightarrow q^{2}, \alpha_{m}=\frac{(-1)^{m} q^{\frac{m^{2}-m}{2}}(-q ; q)_{m}}{\left.\left(q^{\frac{1}{2}} ; q\right)_{m}\left(-q^{\frac{1}{2}} ; q\right)_{m} \text { in }(45)\right)}\right.
\end{gather*}
$$

(ii)

$$
\bar{\psi}_{3}(q)=\sum_{k=0}^{\infty} \frac{(-q ; q)_{k}\left(1+q^{k}\right) q^{k^{2}-k}}{(q ; q)_{k}\left(q^{\frac{1}{2}} ; q\right)_{k}\left(-q^{\frac{1}{2}} ; q\right)_{k}}{ }^{3} \phi_{3}\left[\begin{array}{c}
q,-q^{k+1}, 0  \tag{63}\\
\left.q^{k+1}, q^{k+\frac{1}{2}},-q^{k+\frac{1}{2}} ; q, q^{k}\right] . . . ~ . ~
\end{array}\right.
$$

$$
\left(q \rightarrow q^{2}, \alpha_{m}=\frac{(-1)^{m} q^{\frac{m^{2}-m}{2}}(-q ; q)_{m}}{(q ; q)_{m}\left(q^{\frac{1}{2}} ; q\right)_{m}\left(-q^{\frac{1}{2}} ; q\right)_{m}} \text { in (46)}\right)
$$

(iii)

$$
\begin{gather*}
\frac{\left(q^{\frac{1}{2}},-q^{\frac{1}{2}} ; q\right)_{\infty}}{(q,-1 ; q)_{\infty}} \bar{\psi}_{3}(q)-\sum_{k=0}^{\infty} \frac{\left(q^{-\frac{1}{2}},-q^{-\frac{1}{2}} ; q\right)_{k+1} q^{k+1}}{(q,-1 ; q)_{k+1}} \bar{\psi}_{3, k}(q) \\
={ }_{1} \phi_{2}\left[\begin{array}{c}
-q, \\
-1, q
\end{array} q, q\right]  \tag{64}\\
\left(b=1 / q^{\frac{1}{2}}, c=-1 / q^{\frac{1}{2}}, \alpha_{m}=\frac{q^{m^{2}}(-q ; q)_{m}}{(q ; q)_{m}\left(q^{\frac{1}{2}} ; q\right)_{m}\left(-q^{\frac{1}{2}} ; q\right)_{m}}\right. \text { in (48))}
\end{gather*}
$$

(iv)

$$
\begin{gather*}
\bar{\psi}_{3}(q)-(q ; q)_{\infty} \sum_{k=0}^{\infty} \frac{q^{k+1}}{(q ; q)_{k+1}} \bar{\psi}_{3, k}(q)=(q ; q)_{\infty} \phi_{3}\left[\begin{array}{c}
-q, 0,0 \\
q, q^{\frac{1}{2}},-q^{\frac{1}{2}}
\end{array} q, q\right]  \tag{65}\\
\left(\alpha_{m}=\frac{q^{m^{2}}(-q ; q)_{m}}{(q ; q)_{m}\left(q^{\frac{1}{2}} ; q\right)_{m}\left(-q^{\frac{1}{2}} ; q\right)_{m}} \text { in }(49)\right)
\end{gather*}
$$

(e) Expansions for $\bar{\varphi}_{0}(q)$
(i)

$$
\begin{gather*}
\frac{\left(-q^{2}, q^{3} ; q^{2}\right)_{\infty}}{\left(q^{2},-q^{3} ; q^{2}\right)_{\infty}} \bar{\varphi}_{0}(q)-2(1-q) \sum_{k=0}^{\infty} \frac{\left(-q^{2}, q^{3} ; q^{2}\right)_{k} q^{2 k+2}}{\left(q^{2},-q^{3} ; q^{2}\right)_{k+1}} \bar{\varphi}_{0, k}(q) \\
=(1+q)_{3} \phi_{2}\left[\begin{array}{c}
-q^{2},-q^{2}, q^{3}, \\
0,0,
\end{array} q^{2}, q\right]  \tag{66}\\
\left(q \rightarrow q^{2}, b=-1, c=q, \alpha_{m}=q^{m}(-q ; q)_{2 m+1} \text { in }(48)\right)
\end{gather*}
$$

(ii)

$$
\begin{align*}
& \bar{\varphi}_{0}(q)-\left(q^{2} ; q^{2}\right)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2 k+2}}{\left(q^{2} ; q^{2}\right)_{k+1}} \bar{\varphi}_{0, k}(q) \\
= & (1+q)\left(q^{2} ; q^{2}\right)_{\infty}{ }_{2} \phi_{1}\left[\begin{array}{c}
-q^{2},-q^{3}, \\
0
\end{array} ; q^{2}, q\right] \tag{67}
\end{align*}
$$

$$
\left(q \rightarrow q^{2}, \alpha_{m}=q^{m}(-q ; q)_{2 m+1}\right. \text { in (49)) }
$$

(f) Expansions for $\bar{\varphi}_{1}(q)$
(i)

$$
\begin{gather*}
\frac{\left(-q^{2}, q^{3} ; q^{2}\right)_{\infty}}{\left(q^{2},-q^{3} ; q^{2}\right)_{\infty}} \bar{\varphi}_{1}(q)-\sum_{k=0}^{\infty} \frac{\left(-1, q ; q^{2}\right)_{k+1} q^{2 k+2}}{\left(q^{2},-q^{3} ; q^{2}\right)_{k+1}} \bar{\varphi}_{1, k}(q) \\
={ }_{4} \phi_{3}\left[\begin{array}{c}
-q,-q^{2},-q^{2}, q^{3}, \\
-q^{3}, 0,0,
\end{array} q^{2}, q\right]  \tag{68}\\
\left(q \rightarrow q^{2}, b=-1, c=q, \alpha_{m}=q^{m}(-q ; q)_{2 m} \text { in }(48)\right)
\end{gather*}
$$

(ii)

$$
\bar{\varphi}_{1}(q)-\left(q^{2} ; q^{2}\right)_{\infty} \sum_{k=0}^{\infty} \frac{q^{2 k+2}}{\left(q^{2} ; q^{2}\right)_{k+1}} \bar{\varphi}_{1, k}(q)=\left(q^{2} ; q^{2}\right)_{\infty}{ }_{2} \phi_{1}\left[\begin{array}{c}
-q,-q^{2},  \tag{69}\\
0
\end{array} ; q^{2}, q\right]
$$

$$
\left(q \rightarrow q^{2}, \alpha_{m}=q^{m}(-q ; q)_{2 m} \text { in }(49)\right)
$$

## References

[1] R. P. Agarwal, Resonance of Ramanujan's Mathematics III, New Age International (P) Ltd. New Delhi, 1996.
[2] G. E. Andrews, q-orthogonal polynomials, Rogers-Ramanujan identities, and mock theta functions, preprint.
[3] K. Bringmann, K. Hikami, J. Lovejoy, On the modularity of the unified WRT invariants of certain Seifert manifolds, Adv. Appl. Math., to appear.
[4] A. K. Chongdar, On the unification of a class of bilateral generating functions for certain special functions, Tamkang J. Math., 18 no. 3 (1987), 53-59.
[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.
[6] B. Gordon and R. J. McIntosh, Some eighth order mock theta functions, J. London Math. Soc., 62 no. 2 (2000), 321-335.
[7] E. Mortenson, On three third order mock theta functions and Hecke-type double sums, preprint.
[8] E. D. Rainville, Special Function, Chelsea Publishing Company, Bronx, New York, 1960.
[9] Bhaskar Srivastava, Ramanujan's Mock Theta Functions, Math. J. Okayama Univ., 47 (2005), 163-174.
[10] H. M. Srivastava, Some bilateral generating functions for a certain class of special functions. I and II, Nederl. Akad. Wetensch. Indag. Math., 42 (1980), 221-233 and 234-246.
[11] H. M. Srivastava, A family of $q$-generating functions, Bull. Inst. Math. Acad. Sinica, 12 (1984), 327-336.
[12] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

# Some Sandwich Type Theorems for Analytic Functions Involving the Dziok-Srivastava Operator and Other Related Linear Operators * 

M. K. Aouf ${ }^{\dagger}$<br>Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt<br>and<br>T. M. Seoudy ${ }^{\ddagger}$<br>Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt

Received April 24, 2012, Accepted December 27, 2012.


#### Abstract

In this paper, we obtain some applications of first order differential subordination and superordination results defined by Dzoik-Srivastava operator and other linear operators for certain normalized analytic functions.


Keywords and Phrases: Analytic function, Hadamard product, Differential subordination, Superordination, Dzoik-Srivastava operator.

[^7]
## 1. Introduction

Let $H(U)$ be the class of analytic functions in the unit disk $U=\{z \in \mathbb{C}:|z|<$ $1\}$ and let $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form:

$$
\begin{equation*}
f(z)=a+a_{n} z^{n}+a_{n+1} z^{n+1} \ldots \quad(a \in \mathbb{C}) \tag{1.1}
\end{equation*}
$$

Also, let $\mathcal{A}$ be the subclass of $H(U)$ consisting of functions of the form:

$$
\begin{equation*}
f(z)=z+a_{2} z^{2}+\ldots \tag{1.2}
\end{equation*}
$$

and Let $S^{*}$ denote the starlike subclass of $\mathcal{A}$.If $f, g \in H(U)$, we say that $f$ is subordinate to $g$ or $f$ is superordinate to $g$, written $f(z) \prec g(z)$ if there exists a Schwarz function $\omega$, which (by definition) is analytic in $U$ with $\omega(0)=0$ and $|\omega(z)|<1$ for all $z \in U$, such that $f(z)=g(\omega(z)), z \in U$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence, (cf., e.g., [3], [13] and [14]):

$$
f(z) \prec g(z) \Leftrightarrow f(0)=g(0) \text { and } f(U) \subset g(U)
$$

Let $\phi: \mathbb{C}^{2} \times U \rightarrow \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the first order differential subordination:

$$
\begin{equation*}
\phi\left(p(z), z p^{\prime}(z) ; z\right) \prec h(z) \tag{1.3}
\end{equation*}
$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), z p^{\prime}(z) ; z\right)$ are univalent in $U$ and if $p(z)$ satisfies first order differential superordination:

$$
\begin{equation*}
h(z) \prec \phi\left(p(z), z p^{\prime}(z) ; z\right) \tag{1.4}
\end{equation*}
$$

then $p(z)$ is a solution of the differential superordination (1.4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant.

For complex parameters $a_{1}, \ldots, a_{q} ; b_{1}, \ldots, b_{s}\left(b j \notin \mathbb{Z}_{0}^{-}=\{0,-1,-2, \ldots\} ; j=\right.$ $1, \ldots, s)$, we define the generalized hypergeometric function ${ }_{q} F_{s}\left(a_{1}, \ldots, a_{i}, \ldots, a_{q} ; b_{1}, \ldots, b_{s} ; z\right)$ by ( see [18] ) the following infinite series:

$$
\begin{gather*}
{ }_{q} F_{s}\left(a_{1}, \ldots, a_{i}, \ldots, a_{q} ; b_{1}, \ldots, b_{s} ; z\right)=\sum_{n=0}^{\infty} \frac{\left(a_{1}\right)_{n} \ldots\left(a_{q}\right)_{n}}{\left(b_{1}\right)_{n} \ldots\left(b_{s}\right)_{n}} \frac{z^{n}}{n!}  \tag{1.5}\\
\left(q \leq s+1 ; q, s \in \mathbb{N}_{0}=\mathbb{N} \cup\{0\} ; z \in U\right)
\end{gather*}
$$

where $(x)_{n}$ is the Pochhammer symbol (or the shift factorial) defined, in terms of the Gamma function $\Gamma$, by

$$
(x)_{n}=\frac{\Gamma(x+n)}{\Gamma(x)}= \begin{cases}1 & (n=0) \\ x(x+1) \ldots(x+n-1) & (n \in \mathbb{N})\end{cases}
$$

Dziok and Srivastava [7] ( see also [8]) considered a linear operator

$$
H\left(a_{1}, \ldots, a_{q} ; b_{1}, \ldots, b_{s}\right): \mathcal{A} \rightarrow \mathcal{A}
$$

defined by the following Hadamard product:

$$
\begin{equation*}
H\left(a_{1}, \ldots, a_{q} ; b_{1}, \ldots, b_{s}\right) f(z)=h\left(a_{1}, \ldots, a_{i}, \ldots, a_{q} ; b_{1}, \ldots, b_{s} ; z\right) * f(z) \tag{1.6}
\end{equation*}
$$

where

$$
\begin{gather*}
h\left(a_{1}, \ldots, a_{i}, \ldots, a_{q} ; b_{1}, \ldots, b_{s} ; z\right)=z_{q} F_{s}\left(a_{1}, \ldots, a_{i}, \ldots, a_{q} ; b_{1}, \ldots, b_{s} ; z\right)  \tag{1.7}\\
\left(q \leq s+1 ; q, s \in \mathbb{N}_{0} ; z \in U\right)
\end{gather*}
$$

if $f(z) \in \mathcal{A}$ is given by $(1.2)$, then we have

$$
\begin{equation*}
H\left(a_{1}, \ldots, a_{q} ; b_{1}, \ldots, b_{s}\right) f(z)=z+\sum_{n=2}^{\infty} \frac{\left(a_{1}\right)_{n-1} \ldots\left(a_{q}\right)_{n-1}}{\left(b_{1}\right)_{n-1} \ldots\left(b_{s}\right)_{n-1}(1)_{n-1}} a_{n} z^{n} \tag{1.8}
\end{equation*}
$$

If, for convenience, we write

$$
H_{q, s}\left[a_{1} ; b_{1}\right]=H\left(a_{1}, \ldots, a_{q} ; b_{1}, \ldots, b_{s}\right)
$$

then one can easily verify from the definition (1.6) or (1.8) that

$$
\begin{equation*}
z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}=a_{1} H_{q, s}\left[a_{1}+1 ; b_{1}\right] f(z)-\left(a_{1}-1\right) H_{q, s}\left[a_{1} ; b_{1}\right] f(z) \tag{1.9}
\end{equation*}
$$

and

$$
\begin{equation*}
z\left(H_{q, s}\left[a_{1} ; b_{1}+1\right] f(z)\right)^{\prime}=b_{1} H_{q, s}\left[a_{1} ; b_{1}\right] f(z)-\left(b_{1}-1\right) H_{q, s}\left[a_{1} ; b_{1}+1\right] f(z) \tag{1.10}
\end{equation*}
$$

It should be remarked that the linear operator $H_{q, s}\left[a_{1} ; b_{1}\right]$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$, we have
(i) $H_{2,1}(a, b ; c) f(z)=\left(I_{c}^{a, b} f\right)(z)\left(a, b \in \mathbb{C} ; c \notin \mathbb{Z}_{0}^{-}\right)$, where the linear operator $I_{c}^{a, b}$ was investigated by Hohlov [9];
(ii) $H_{2,1}(\delta+1,1 ; 1) f(z)=D^{\delta} f(z)(\delta>-1)$, where $D^{\delta}$ is the Ruscheweyh derivative of $f(z)$ (see [16]);
(iii) $H_{2,1}(\mu+1,1 ; \mu+2) f(z)=\mathcal{F}_{\mu}(f)(z)=\frac{\mu+1}{z^{\mu}} \int_{0}^{z} t^{m u-1} f(t) d t(\mu>-1)$, where $\mathcal{F}_{\mu}$ is the Libera integral operator (see [11] and [1]);
(iv) $H_{2,1}(a, 1 ; c) f(z)=L(a, c) f(z)\left(a \in \mathbb{R} ; c \in \mathbb{R} \backslash \mathbb{Z}_{0}^{-}\right)$, where $L(a, c)$ is the Carlson-Shaffer operator ( see [4]);
(vi) $H_{2,1}(\lambda+1, c ; a) f(z)=I^{\lambda}(a, c) f(z)\left(a, c \in \mathbb{R} \backslash \mathbb{Z}_{0}^{-} ; \lambda>-1\right)$, where $I^{\lambda}(a, c) f(z)$ is the Cho-Kwon-Srivastava operator ( see [5]);
(vii) $H_{2,1}(\mu, 1 ; \lambda+1) f(z)=I_{\lambda, \mu} f(z)(\lambda>-1 ; \mu>0)$, where $I_{\lambda, \mu} f(z)$ is the Choi-Saigo-Srivastava operator [6] which is closely related to the Carlson-Shaffer [4] operator $L(\mu, \lambda+1) f(z)$.
(vii) $H_{2,1}(1,1 ; n+1) f(z)=I_{n} f(z)\left(n \in \mathbb{N}_{0}\right)$, where $I_{n} f(z)$ is Noor operator of $n-t h$ order (see [15]).

Definition 1. The function $f \in \mathcal{A}$ belongs to the class $S_{q . s}^{*}\left(a_{1} ; b_{1}\right)$ if and only if $H_{q, s}\left[a_{1} ; b_{1}\right] f(z) \in S^{*}$ for $z \in U$.
Definition 2. The function $f \in \mathcal{A}$ belongs to the class $C_{q . s}\left(a_{1} ; b_{1}\right)$ if and only if there exists $g \in S_{q . s}^{*}\left(a_{1} ; b_{1}\right)$ such that

$$
\Re\left\{\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right\}>0 \quad(z \in U) .
$$

In this paper, we obtain sufficient conditions for normalized analytic functions $f, g$ satisfy

$$
q_{1}(z) \prec \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \prec q_{2}(z)
$$

where $q_{1}(z)$ and $q_{2}(z)$ are given univalent functions in $U$.

## 2. Definitions and Preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 3. [14]. Denote by $Q$, the set of all functions $f$ that are analytic and injective on $U \backslash E(f)$, where

$$
E(f)=\left\{\zeta \in \partial U: \lim _{z \rightarrow \zeta} f(z)=\infty\right\}
$$

and are such that $f^{\prime}(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.
Lemma 1 [14]. Let $q(z)$ be univalent in the unit disk $U$ and $\theta$ and $\varphi$ be analytic in a domain $D$ containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set

$$
\begin{equation*}
\psi(z)=z q^{\prime}(z) \varphi(q(z)) \quad \text { and } \quad h(z)=\theta(q(z))+\psi(z) \tag{2.1}
\end{equation*}
$$

Suppose that
(i) $\psi(z)$ is starlike univalent in $U$,
(ii) $\Re\left\{\frac{z h^{\prime}(z)}{\psi(z)}\right\}>0$ for $z \in U$.

If $p(z)$ is analytic with $p(0)=q(0), p(U) \subset D$ and

$$
\begin{equation*}
\theta(p(z))+z p^{\prime}(z) \varphi(p(z)) \prec \theta(q(z))+z q^{\prime}(z) \varphi(q(z)) \tag{2.2}
\end{equation*}
$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
Taking $\theta(w)=\alpha w$ and $\varphi(w)=\gamma$ in Lemma 1, Shanmugam et al. [17] obtained the following lemma.

Lemma 2 [17]. Let $q(z)$ be univalent in $U$ with $q(0)=1$. Let $\alpha \in \mathbb{C}$; $\gamma \in \mathbb{C}^{*}=\mathbb{C} \backslash\{0\}$, further assume that

$$
\begin{equation*}
\Re\left\{1+\frac{z q^{\prime \prime}(z)}{q^{\prime}(z)}\right\}>\max \left\{0,-\Re\left(\frac{\alpha}{\gamma}\right)\right\} \tag{2.3}
\end{equation*}
$$

If $p(z)$ is analytic in $U$, and

$$
\alpha p(z)+\gamma z p^{\prime}(z) \prec \alpha q(z)+\gamma z q^{\prime}(z)
$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
Lemma 3 [2]. Let $q(z)$ be convex univalent in $U$ and $\vartheta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$. Suppose that
(i) $\Re\left\{\frac{\vartheta^{\prime}(q(z))}{\phi(q(z))}\right\}>0$ for $z \in U$,
(ii) $\Psi(z)=z q^{\prime}(z) \phi(q(z))$ is starlike univalent in $U$.

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z))+z p^{\prime}(z) \phi(p(z))$ is univalent in $U$ and

$$
\begin{equation*}
\vartheta(q(z))+z q^{\prime}(z) \phi(q(z)) \prec \vartheta(p(z))+z p^{\prime}(z) \phi(p(z)), \tag{2.4}
\end{equation*}
$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.
Taking $\vartheta(w)=\alpha w$ and $\phi(w)=\gamma$ in Lemma 3, Shanmugam et al. [17] obtained the following lemma.

Lemma 4 [17]. Let $q(z)$ be convex univalent in $U, q(0)=1$. Let $\alpha \in \mathbb{C}$; $\gamma \in \mathbb{C}^{*}$ and $\Re\left(\frac{\alpha}{\gamma}\right)>0$. If $p(z) \in H[q(0), 1] \cap Q, \alpha p(z)+\gamma z p^{\prime}(z)$ is univalent in $U$ and

$$
\alpha q(z)+\gamma z q^{\prime}(z) \prec \alpha p(z)+\gamma z p^{\prime}(z)
$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

## 3. Sandwich Results

Theorem 1. Let $q(z)$ be univalent in $U$ with $q(0)=1$, and $\gamma \in \mathbb{C}^{*}$. Further assume that

$$
\begin{equation*}
\Re\left\{1+\frac{z q^{\prime \prime}(z)}{q^{\prime}(z)}\right\}>\max \left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\} \tag{3.1}
\end{equation*}
$$

If $f, g \in \mathcal{A}, H_{q, s}\left[a_{1} ; b_{1}\right] g(z) \neq 0$, satisfy the following subordination condition:

$$
\begin{align*}
& \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\{1+\gamma {\left.\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\} }  \tag{3.2}\\
& \prec q(z)+\gamma z q^{\prime}(z),
\end{align*}
$$

then

$$
\begin{equation*}
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \prec q(z) \tag{3.3}
\end{equation*}
$$

and $q(z)$ is the best dominant.
Proof. Define a function $p(z)$ by

$$
\begin{equation*}
p(z)=\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \quad(z \in U) \tag{3.4}
\end{equation*}
$$

Then the function $p(z)$ is analytic in $U$ and $p(0)=1$. Therefore, differentiating (3.4) logarithmically with respect to $z$ and using the the subordination condition (3.2), we get

$$
p(z)+\gamma z p^{\prime}(z) \prec q(z)+\gamma z q^{\prime}(z) .
$$

Therefore, the assertion (3.3) of Theorem 1 now follows by an application of Lemma 2.

Putting $q(z)=\frac{1+A z}{1+B z}(-1 \leq B<A \leq 1)$ in Theorem 1, we have the following corollary.

Corollary 1. Let $\gamma \in \mathbb{C}^{*}$ and

$$
\Re\left\{\frac{1-B z}{1+B z}\right\}>\max \left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}
$$

If $f, g \in \mathcal{A}, H_{q, s}\left[a_{1} ; b_{1}\right] g(z) \neq 0$, satisfy the following subordination condition:

$$
\begin{array}{ll}
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} & \left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\} \\
\quad \prec \quad & \frac{1+A z}{1+B z}+\gamma \frac{(A-B) z}{(1+B z)^{2}}
\end{array}
$$

then

$$
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \prec \frac{1+A z}{1+B z}
$$

and the function $\frac{1+A z}{1+B z}$ is the best dominant.
Taking $A=1, B=-1$ and $g \in S_{\text {q.s }}^{*}\left(a_{1} ; b_{1}\right)$ in Corollary 1, we obtain
Corollary 2. Let $\gamma \in \mathbb{C}^{*}$ with $\Re(\bar{\gamma})>0$. If $g \in \mathcal{A}$ such that $g \in S_{q . s}^{*}\left(a_{1} ; b_{1}\right)$, and $f, g \in \mathcal{A}$ satisfy the following subordination condition:

$$
\begin{array}{ll}
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} & \left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\} \\
\quad \prec \quad & \frac{1+z}{1-z}+\gamma \frac{z}{(1-z)^{2}},
\end{array}
$$

then $f(z) \in C_{q, s}\left(a_{1} ; b_{1}\right)$ and this result best possible.
For $q=2, s=1, a_{1}=a$ and $b_{1}=c \quad\left(a \in \mathbb{R} ; c \in \mathbb{R} \backslash \mathbb{Z}_{0}^{-}\right)$in Theorem 1, we have the following subordination for Carlson-Shaffer operator.

Corollary 3. Let $q(z)$ be univalent in $U$ with $q(0)=1$, and $\gamma \in \mathbb{C}^{*}$. Further assume that (3.1) holds. If $f, g \in \mathcal{A}, L(a, c) g(z) \neq 0$, satisfy the following subordination condition:

$$
\begin{aligned}
\frac{z(L(a, c) f(z))^{\prime}}{L(a, c) g(z)}\{1+\gamma & {\left.\left[1+\frac{z(L(a, c) f(z))^{\prime \prime}}{(L(a, c) f(z))^{\prime}}-\frac{z(L(a, c) g(z))^{\prime}}{L(a, c) g(z)}\right]\right\} } \\
& \prec q(z)+\gamma z q^{\prime}(z),
\end{aligned}
$$

then

$$
\frac{z(L(a, c) f(z))^{\prime}}{L(a, c) g(z)} \prec q(z)
$$

and $q(z)$ is the best dominant.
For $q=2, s=1, a_{1}=\lambda+1, a_{2}=c$ and $b_{1}=a\left(a, c \in \mathbb{R} \backslash \mathbb{Z}_{0}^{-} ; \lambda>-1\right)$ in Theorem 1, we obtain the following subordination for Cho-Kwon-Srivastava operator.

Corollary 4. Let $q(z)$ be univalent in $U$ with $q(0)=1$, and $\gamma \in \mathbb{C}^{*}$. Further assume that (3.1) holds. If $f, g \in \mathcal{A}, I^{\lambda}(a, c) g(z) \neq 0$, satisfy the following subordination condition:

$$
\begin{aligned}
\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}\{1+\gamma & {\left.\left[1+\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime \prime}}{\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}-\frac{z\left(I^{\lambda}(a, c) g(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}\right]\right\} } \\
& \prec q(z)+\gamma z q^{\prime}(z)
\end{aligned}
$$

then

$$
\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)} \prec q(z)
$$

and $q(z)$ is the best dominant.
For $q=2, s=1, a_{1}=\mu, a_{2}=1$ and $b_{1}=\lambda+1(\lambda>-1 ; \mu>0)$ in Theorem 1 , we have the following subordination for Choi-Saigo-Srivastava operator.

Corollary 5. Let $q(z)$ be univalent in $U$ with $q(0)=1$, and $\gamma \in \mathbb{C}^{*}$. Further assume that (3.1) holds. If $f, g \in \mathcal{A}, I_{\lambda, \mu} g(z) \neq 0$, satisfy the following subordination condition:

$$
\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}\left\{1+\gamma\left[1+\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime \prime}}{\left(I_{\lambda, \mu} f(z)\right)^{\prime}}-\frac{z\left(I_{\lambda, \mu} g(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}\right]\right\} \prec q(z)+\gamma z q^{\prime}(z),
$$

then

$$
\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)} \prec q(z)
$$

and $q(z)$ is the best dominant.
Remark 1. Taking $q=2, s=1, a_{1}=a_{2}=1$ and $b_{1}=n+1\left(n \in \mathbb{N}_{0}\right)$ in Theorem 1, we obtain the subordination result of Ibrahim and Darus [ 10,Theorem 2] for the Noor operator.

Now, by appealing to Lemma 4 it can be easily prove the following theorem.
Theorem 2. Let $q(z)$ be convex univalent in $U$ with $q(0)=1$. Let $\gamma \in \mathbb{C}$
with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \in H[1,1] \cap Q$,

$$
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\}
$$

is univalent in $U$, and the following superordination condition

$$
\begin{aligned}
& q(z)+\gamma z q^{\prime}(z) \\
\prec & \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\}
\end{aligned}
$$

holds, then

$$
q(z) \prec \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}
$$

and $q(z)$ is the best subordinant.
Taking $q(z)=\frac{1+A z}{1+B z}(-1 \leq B<A \leq 1)$ in Theorem 2, we have the following corollary.

Corollary 6. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that

$$
\begin{gathered}
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \in H[1,1] \cap Q \\
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\}
\end{gathered}
$$

is univalent in $U$, and the following superordination condition

$$
\begin{aligned}
& \frac{1+A z}{1+B z}+\gamma \frac{(A-B) z}{(1+B z)^{2}} \\
\prec & \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\} \text { holds, }
\end{aligned}
$$

then

$$
\frac{1+A z}{1+B z} \prec \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}
$$

and $q(z)$ is the best subordinant.
For $q=2, s=1, a_{1}=a$ and $b_{1}=c \quad\left(a \in \mathbb{R} ; c \in \mathbb{R} \backslash \mathbb{Z}_{0}^{-}\right)$in Theorem 1 , we have the following superordination result for Carlson-Shaffer operator.

Corollary 7. Let $q(z)$ be convex univalent in $U$ with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z(L(a, c) f(z))^{\prime}}{L(a, c) g(z)} \in H[1,1] \cap Q$,

$$
\frac{z(L(a, c) f(z))^{\prime}}{L(a, c) g(z)}\left\{1+\gamma\left[1+\frac{z(L(a, c) f(z))^{\prime \prime}}{(L(a, c) f(z))^{\prime}}-\frac{z(L(a, c) g(z))^{\prime}}{L(a, c) g(z)}\right]\right\}
$$

is univalent in $U$, and the following superordination condition

$$
\begin{gathered}
q(z)+\gamma z q^{\prime}(z) \\
\prec \frac{z(L(a, c) f(z))^{\prime}}{L(a, c) g(z)}\left\{1+\gamma\left[1+\frac{z(L(a, c) f(z))^{\prime \prime}}{(L(a, c) f(z))^{\prime}}-\frac{z(L(a, c) g(z))^{\prime}}{L(a, c) g(z)}\right]\right\}
\end{gathered}
$$

holds,
then

$$
q(z) \prec \frac{z(L(a, c) f(z))^{\prime}}{L(a, c) g(z)}
$$

and $q(z)$ is the best subordinant.
For $q=2, s=1, a_{1}=\lambda+1, a_{2}=c$ and $b_{1}=a\left(a, c \in \mathbb{R} \backslash \mathbb{Z}_{0}^{-} ; \lambda>-1\right)$ in Theorem 2, we obtain the following superordination result for Cho-KwonSrivastava operator.

Corollary 8. Let $q(z)$ be convex univalent in $U$ with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)} \in H[1,1] \cap Q$,

$$
\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}\left\{1+\gamma\left[1+\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime \prime}}{\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}-\frac{z\left(I^{\lambda}(a, c) g(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}\right]\right\}
$$

is univalent in $U$, and the following superordination condition

$$
\begin{gathered}
q(z)+\gamma z q^{\prime}(z) \\
\prec \frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}\left\{1+\gamma\left[1+\frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime \prime}}{\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}-\frac{z\left(I^{\lambda}(a, c) g(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}\right]\right\}
\end{gathered}
$$

holds, then

$$
q(z) \prec \frac{z\left(I^{\lambda}(a, c) f(z)\right)^{\prime}}{I^{\lambda}(a, c) g(z)}
$$

and $q(z)$ is the best subordinant.
For $q=2, s=1, a_{1}=\mu, a_{2}=1$ and $b_{1}=\lambda+1(\lambda>-1 ; \mu>0)$ in Theorem 2, we have the following superordination result for Choi-Saigo-Srivastava operator.

Corollary 9. Let $q(z)$ be convex univalent in $U$ with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)} \in H[1,1] \cap Q$,

$$
\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}\left\{1+\gamma\left[1+\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime \prime}}{\left(I_{\lambda, \mu} f(z)\right)^{\prime}}-\frac{z\left(I_{\lambda, \mu} g(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}\right]\right\}
$$

is univalent in $U$, and the following superordination condition

$$
q(z)+\gamma z q^{\prime}(z) \prec \frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}\left\{1+\gamma\left[1+\frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime \prime}}{\left(I_{\lambda, \mu} f(z)\right)^{\prime}}-\frac{z\left(I_{\lambda, \mu} g(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}\right]\right\}
$$

holds, then

$$
q(z) \prec \frac{z\left(I_{\lambda, \mu} f(z)\right)^{\prime}}{I_{\lambda, \mu} g(z)}
$$

and $q(z)$ is the best subordinant.
Remark 2. Taking $q=2, s=1, a_{1}=a_{2}=1$ and $b_{1}=n+1 \quad\left(n \in \mathbb{N}_{0}\right)$ in Theorem 2, we obtain the superordination result of Ibrahim and Darus 10,Theorem 4] for the Noor operator.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

Theorem 3. Let $q_{i}(z)$ be convex univalent in $U$ with $q_{i}(0)=1(i=1,2)$, $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in A$ such that $\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \in H[1,1] \cap Q$,

$$
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\}
$$

is univalent in $U$, and

$$
\begin{aligned}
& q_{1}(z)+\gamma z q_{1}^{\prime}(z) \\
\prec & \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\} \\
\prec & q_{2}(z)+\gamma z q_{2}^{\prime}(z)
\end{aligned}
$$

holds, then

$$
q_{1}(z) \prec \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \prec q_{2}(z)
$$

and $q_{1}(z)$ and $q_{2}(z)$ are, respectively, the best subordinant and the best dominant.

Taking $q_{i}(z)=\frac{1+A_{i} z}{1+B_{i} z}\left(i=1,2 ;-1 \leq B_{2} \leq B_{1}<A_{1} \leq A_{2} \leq 1\right)$ in Theorem 3 , we have the following corollary.
Corollary 10. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f, g \in \mathcal{A}$ such that $\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \in H[1,1] \cap Q$,

$$
\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\}
$$

is univalent in $U$, and

$$
\begin{aligned}
& \frac{1+A_{1} z}{1+B_{1} z}+\gamma \frac{\left(A_{1}-B_{1}\right) z}{\left(1+B_{1} z\right)^{2}} \\
\prec & \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\left\{1+\gamma\left[1+\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime \prime}}{\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}-\frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] g(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)}\right]\right\} \\
\prec & \frac{1+A_{2} z}{1+B_{2} z}+\gamma \frac{\left(A_{2}-B_{2}\right) z}{\left(1+B_{2} z\right)^{2}}
\end{aligned}
$$

holds, then

$$
\frac{1+A_{1} z}{1+B_{1} z} \prec \frac{z\left(H_{q, s}\left[a_{1} ; b_{1}\right] f(z)\right)^{\prime}}{H_{q, s}\left[a_{1} ; b_{1}\right] g(z)} \prec \frac{1+A_{2} z}{1+B_{2} z}
$$

and $\frac{1+A_{1} z}{1+B_{1} z}$ and $\frac{1+A_{2} z}{1+B_{2} z}$ are, respectively, the best subordinant and the best dominant.

Remark 3. Combining (i) Corollary 3 and Corollary 7; (ii) Corollary 4 and Corollary 8; (iii) Corollary 5 and Corollary 9, we obtain similar sandwich theorems for the corresponding linear operators.

Remark 4. Taking $q=2, s=1, a_{1}=a_{2}=1$ and $b_{1}=n+1 \quad\left(n \in \mathbb{N}_{0}\right)$ in Theorem 3, we obtain the sandwich result of Ibrahim and Darus [10,Theorem 6] for the Noor operator.

Acknowledgement: The authors are grateful to the referees for their valuable suggestions.

## References

[1] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135 (1969), 429-446.
[2] T. Bulboacă, Classes of first order differential superordinations, Demonstratio Math., 35 no. 2 (2002), 287-292.
[3] T. Bulboacă, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.
[4] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., 15 (1984), 737-745.
[5] N. E. Cho,O. H Kwon, and H. M. Srivastava, Inclusion and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292 (2004), 470-483.
[6] J. H. Choi, M. Saigo, and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276 (2002), 432-445.
[7] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput., 103 (1999), 1-13.
[8] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transform. Spec. Funct., 14 (2003), 7-18.
[9] Yu. E. Hohlov, Operators and operations in the class of univalent functions, Izv. Vyss̆h. Uc̆ebn. Zaved. Mat., 10 (1978), 83-89 (in Russian).
[10] R. W. Ibrahim and M. Darus, On sandwich theorems of analytic functions involving Noor operator, J. Math. E3 Stat., 4 no. 1 (2008), 32-36.
[11] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16 (1965), 755-658.
[12] S. S. Miller and P. T Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28 no. 2 (1981), 157-171.
[13] S. S. Miller and P. T Mocanu, Differential Subordination. Theory and Applications, Marcel Dekker Inc., New York, 2000.
[14] S. S. Miller and P. T Mocanu, Subordinates of differential superordinations, Complex Variables 48 no. 10 (2003), 815-826.
[15] K. I. Noor, On new classes of integral operator, J. Nate. Geom., 16 no 1-2 (1999), 71-80.
[16] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49 (1975), 109-115.
[17] T. N. Shanmugam, V. Ravichandran, and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions, Aust. J. Math. Anal. Appl., 3 no. 1 (2006), Art. 8, 11 pp. (electronic).
[18] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Ellis Horwood Ltd., Chichester, Halsted Press (John Wiley \& Sons, Inc.), New York, 1985.

## Information for Authors

## Aim and Scope

Tamsui Oxford Journal of Information and Mathematical Sciences, published by Aletheia University of the Republic of China (Taiwan), is a continuation of Tamsui Oxford Journal of Mathematical Sciences (1997-2010). It aims to publish original research papers and survey articles in all areas of mathematics, statistics, and information sciences. The journal appears quarterly in February, May, August and November, per year.

## Submission of Manuscripts

Submissions should be sent to the Managing Editor or any member of the editorial board. We only take electronic submissions (pdf, Tex, or word file). The submission of a paper implies the author's assurance that it is not being considered for publication in other journals. Once a paper is accepted for publication in Tamsui Oxford Journal of Information and Mathematical Sciences, the authors are assumed to have transferred the copyright to Aletheia University (Taiwan), and have granted Aletheia University the right to have the content of his/her paper included in the academic research databases of internationally renowned organizations which have previously obtained a license agreement with the University.

## Form of Manuscripts

HEADING: The title of the paper should be concise and informative. Successive lines should give the author's name, academic or professional affiliation, and address.

ABSTRACT: Every manuscript must contain a concise abstract, with no more than 150 words.

MSC CODES: The 2000 Mathematical Subject Classification codes that best describe the subject area of the paper should be provided by the author.

KEYWORDS: Three to six Keywords should be provided before the main text.

TABLES AND FIGURES: Tables and figures should be numbered and have titles and captions.

REFERENCES: References should be listed in alphabetical order at the end of the manuscript. A list of references containing items published in the literature; i.e. in journals, books, conference proceedings or technical reports of official institutions, should be provided. Journal names should be written out in full. The following reference styles should be followed:
[1] S. S. Dragomir, Grüss inequality in inner product spaces, The Australian Math. Soc. Gazette, 26 no. 2 (1999), 66-70.
[2] R. L. Eubank, Spline Smoothing and Nonparametric Regression, Marcel Dekker, New York, 1988.

ACKNOWLEDGEMENTS: A brief section near the end of the paper containing the author's acknowledgements can be accommodated.


[^0]:    *2010 Mathematics Subject Classification. Primary 30C45, 30C50, 30C80, 26A33, 33C20.
    $\dagger$ E-mail: sevtaps@dicle.edu.tr
    ${ }^{\ddagger}$ E-mail: bilalseker1980@gmail.com
    ${ }^{\text {§ }}$ Corresponding author. E-mail: owa@math.kindai.ac.jp

[^1]:    *2000 Mathematics Subject Classification. Primary 94A15, 94A17, 94A24, 26D15.
    ${ }^{\dagger}$ Corresponding author. E-mail: arunchoudhary07@gmail.com

[^2]:    *2000 Mathematics Subject Classification. Primary 26D15, 26D99.
    ${ }^{\dagger}$ Corresponding author. E-mail: mwomath@gmail.com
    ${ }^{\ddagger}$ E-mail: sabirhus@gmail.com

[^3]:    *2000 Mathematics Subject Classification. Primary 65H05.
    ${ }^{\dagger}$ Corresponding author: V. Kanwar, E-mail: vmithil@yahoo.co.in

[^4]:    *2000 Mathematics Subject Classification. Primary 16Y30.
    ${ }^{\dagger}$ E-mail: dheenap@yahoo.com
    ${ }^{\ddagger}$ Corresponding author. E-mail: belavarasan@gmail.com

[^5]:    *2010 Mathematics Subject Classification. Primary 30C45.
    The work presented here was supported in part by a grant from Universiti Sains Malaysia. This work was completed during the last two authors' visit to Universiti Sains Malaysia.
    ${ }^{\dagger}$ E-mail: rosihan@cs.usm.my
    ${ }^{\ddagger}$ E-mail: mmn.md08@student.usm.my
    ${ }^{\text {§ }}$ E-mail: vravi@maths.du.ac.in
    【 Corresponding author. E-mail: swamifma@iitr.ernet.in, mathswami@gmail.com

[^6]:    *2000 Mathematics Subject Classification. Primary 33D15.
    ${ }^{\dagger}$ E-mail: bhaskarsrivastav@yahoo.com

[^7]:    *2010 Mathematics Subject Classification. Primary 30C45.
    ${ }^{\dagger}$ E-mail: mkaouf127@yahoo.com
    ${ }^{\ddagger}$ Corresponding author. E-mail: tms00@fayoum.edu.eg

