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A Note on Unique Range Sets of Meromorphic Functions with Deficient Values *

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Abstract

With the help of the notion of weighted sharing of sets we deal with the problem of Unique Range Sets for meromorphic functions and obtain a result which improve some previous results.

Keywords and Phrases: *Meromorphic functions, Uniqueness, Weighted sharing, Shared set.*

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1. Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We shall use the standard notations of value distribution theory :

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\ldots$$

(see [8]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty, r\notin E$.

For any constant a, we define

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

If for some $a\in\mathbb{C}\cup\{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM.

We know that in a given domain D only a single analytic function exists that assumes specified values in the sequence of points $\{z_n\}$ convergent to a point $\alpha\in D$. In 1926, Prof. R. Nevanlinna proved that a non-constant meromorphic function is uniquely determined by the inverse image of 5 distinct values (including the infinity) IM. The above theory, known as Nevanlinna's five value theory can be considered as a threshold of uniqueness theory of entire and meromorphic functions. Gross [7] extended the study by considering pre-images of sets counting multiplicities.

We recall that a set S is called a unique range set for meromorphic functions (URSM) if for any pair of non-constant meromorphic functions f and g, the condition $E_f(S)=E_g(S)$ implies $f\equiv g$. Similarly a set S is called a unique range set for entire functions (URSE) if for any pair of non-constant entire functions f and g, the condition $E_f(S)=E_g(S)$ implies $f\equiv g$.

We will call any set $S \subset \mathbb{C}$ a unique range set for meromorphic functions ignoring multiplicities (URSM-IM) for which $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$ for any pair of non-constant meromorphic functions.

In the recent past the inquisition to characterize different URSE, URSM and URSM-IM under weaker hypothesis by many researchers further add essence to-wards the prosperity of uniqueness theory. A glance in the references [3]-[6], [15]-[17], [19]-[23] also authenticate the statement.

The introduction of the new notion of scaling between CM and IM, known as weighted sharing of values, by I. Lahiri [11]-[12] in 2001 further expedite the investigation process in the above direction. Below we are giving the following definitions:

Definition 1.1. [11, 12] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [11] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. Let $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

We start the discussion with a result of Y. Xu [19], in which he proved the following theorem.

Theorem A. *[19] If f and g are two non-constant meromorphic functions and $\Theta(\infty; f) > \frac{3}{4}$, $\Theta(\infty; g) > \frac{3}{4}$, then there exists a set with seven elements such that $E_f(S, \infty) = E_g(S, \infty)$ implies $f \equiv g$.*

Dealing with the question of Yi raised in [22], Lahiri and Banerjee exhibited a unique range set S with smaller cardinalities than that obtained by Xu [19], imposing some restrictions on the poles of f and g. We now state their result.

Theorem B. *[13] Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where n (≥ 9) be an integer and a, b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If $E_f(S, 2) = E_g(S, 2)$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$ then $f \equiv g$.*

In [3] and [5] Bartels and Fang-Guo both independently proved the existence of a URSM-IM with 17 elements.

Suppose that the polynomial $P(w)$ is defined by

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2, \tag{1.1}$$

where $n \geq 3$ is an integer, a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$.

In fact we consider the following rational function

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.2}$$

where α_1 and α_2 are two distinct roots of

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

We have from (1.2) that

$$R^{'}(w) = \frac{(n-2)aw^{n-1}\ (w-b)^2}{n(n-1)\ (w-\alpha_1)^2\ (w-\alpha_2)^2}. \tag{1.3}$$

From (1.3) we know that $w = 0$ is a root with multiplicity n of the equation $R(w) = 0$ and $w = b$ is a root with multiplicity 3 of the equation $R(w) - c = 0$, where $c = \frac{ab^{n-2}}{2} \neq \frac{1}{2}, 1$. Then

$$R(w) - c = \frac{a(w-b)^3\ Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.4}$$

where $Q_{n-3}(w)$ is a polynomial of degree $n-3$.

Moreover from (1.1) and (1.2) we have

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \tag{1.5}$$

Noting that $c = \frac{ab^{n-2}}{2} \neq 1$, from (1.3) and (1.5) we have

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2$$

has only simple zeros.

In 2007 Thamir C.Alzahary [2] improved *Theorem A* and *Theorem B* and obtained the following result:

Theorem C. *[2] Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1), where $n(\geq 6)$ is an integer. Then $f \equiv g$, if f and g are non-constant meromorphic functions which satisfying one of the following conditions:*

(i) $\frac{16-n}{6} < \Theta_f, \frac{16-n}{6} < \Theta_g$ *and* $E_f(S,0) = E_g(S,0)$.

(ii) $\frac{3(12-n)}{14} < \Theta_f, \frac{3(12-n)}{14} < \Theta_g$ *and* $E_f(S,1) = E_g(S,1)$.

(iii) $\frac{10-n}{4} < \Theta_f,\ \frac{10-n}{4} < \Theta_g$ *and* $E_f(S,2) = E_g(S,2)$.

Here $\Theta_f = \Theta(0; f) + \Theta(\infty; f) + \Theta(b; f)$ *and* Θ_g *can be similarly defined.*

Note that the result of Alzahary also improve the results of Bartels [3] and Fang-Guo [5]. In the paper to explore the possibilities of further improving *Theorem C* we obtain the following theorem which in turn produce significant better results than that obtained in [3], [5], [13], [19].

Theorem 1.1. *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1), where $n(\geq 6)$ is an integer. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,m) = E_g(S,m)$. If*

(i) $m \geq 2$ *and* $\Theta_f + \Theta_g > \frac{10-n}{2}$

(ii) or if $m = 1$ *and* $\Theta_f + \Theta_g + \frac{1}{4}\min\{\Theta(0; f) + \Theta(\infty; f) + \Theta(b; f), \Theta(0; g) + \Theta(\infty; g) + \Theta(b; g)\} > \frac{11-n}{2}$

(iii) or if $m = 0$ *and* $\Theta_f + \Theta_g + \frac{1}{3}\min\{\Theta(0; f) + \Theta(b; f) + \Theta(\infty; f), \Theta(0; g) + \Theta(b; g) + \Theta(\infty; g) > \frac{16-n}{3}$.

then $f \equiv g$, where $\Theta_f = \Theta(0; f) + \Theta(\infty; f) + \Theta(b; f)$ and Θ_g can be similarly defined.

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [8]. Throughout this paper, we also need the following definitions:

Definition 1.3. [10] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f| \leq m)(N(r, a; f| \geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f|\leq m)$ $(\overline{N}(r,a;f|\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f|<m)$, $N(r,a;f|>m)$, $\overline{N}(r,a;f|<m)$ *and* $\overline{N}(r,a;f|>m)$ are defined analogously.

Definition 1.4. Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the reduced counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$, by $\overline{N}_E^{(2}(r,a;f)$ the reduced counting function of those a-points of f and g where $p=q\geq 2$. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a\in\mathbb{C}\cup\{\infty\}$. When f and g share (a,m), $m\geq 1$ then $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.

Definition 1.5. [11, 12] Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 1.6. Let $a,b_1,b_2,\ldots,b_q\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i=1,2,\ldots,q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H the following function.

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big).$$

Let f and g be two non-constant meromorphic function and

$$F=R(f),\qquad G=R(g), \tag{2.1}$$

where $R(w)$ is given as (1.2). From (1.2) and (2.1) it is clear that

$$T(r,f)=\frac{1}{n}T(r,F)+S(r,f),\quad T(r,g)=\frac{1}{n}T(r,G)+S(r,g) \tag{2.2}$$

Lemma 2.1. *Let F, G be given by (2.1) and $H\not\equiv 0$. Suppose that F, G share $(1,m)$, where $m\geq 0$ is an integer. Then*

$$\begin{aligned} N_E^{1)}(r,1;F) \;\leq\; & \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)\\ & +\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r.b;g)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'}),\\ & +S(r,f)+S(r,g)\end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ denotes the reduced counting function corresponding to the zeros of $f^{'}$ which are not the zeros of $f(f-b)$ and $F-1$, $\overline{N}_0(r,0;g^{'})$ is defined similarly.

Proof. We omit the proof since it can be carried out in the line of the proof of *Lemma 2.18* [1]. □

Lemma 2.2. *Let f be a non-constant meromorphic function and a_i, $i=1,2,\ldots,n$ be finite distinct complex numbers, where $n\geq 1$. Then*

$$N(r,0;f^{'}) \;\leq\; T(r,f)+\overline{N}(r,\infty;f)-\sum_{i=1}^{n}m(r,a_i;f)+S(r,f)$$

Proof. Let $F=\sum\limits_{i=1}^{n}\frac{1}{f-a_i}$. Then $\sum\limits_{i=1}^{n}m(r,a_i;f)=m(r,F)+O(1)$. Note that

$$\begin{aligned} m(r,F) &\leq m(r,0;f^{'})+m(r,\sum_{i=1}^{n}\frac{f^{'}}{f-a_i})\\ &= T(r,f^{'})-N(r,0;f^{'})+S(r,f).\end{aligned}$$

Also we observe that

$$\begin{aligned} T(r,f^{'}) &= m(r,f^{'})+N(r,f^{'})\\ &\leq m(r,f)+m(r,\frac{f^{'}}{f})+N(r,f)+\overline{N}(r,f)\\ &= T(r,f)+\overline{N}(r,f)+S(r,f).\end{aligned}$$

Hence the Lemma follows. □

Lemma 2.3. *[18] Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 2.4. *([2], Lemma 1) Let F, G be given by (2.1). Also let S be given as in* Theorem 1.1*, where $n \geq 3$ is an integer. If $E_f(S, 0) = E_g(S, 0)$ then $S(r, f) = S(r, g)$.*

3. Proofs of the theorem

Proof of Theorem 1.1. Let F, G be given by (2.1). Since $E_f(S, m) = E_g(S, m)$, it follows that F, G share $(1, m)$.
Case 1. Suppose that $H \not\equiv 0$.

Subcase 1.1. $m \geq 1$. While $m \geq 2$, using *Lemma 2.2* with $n = 2$, $a_1 = 0$ and $a_2 = b$ we note that

$$\begin{aligned}
& \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \qquad (3.1)\\
\leq\ & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3)\\
\leq\ & \overline{N}_0(r,0;g^{'}) + \sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g \mid= 2) + 2\overline{N}(r,\omega_j;g \mid\geq 3)\}\\
\leq\ & N(r,0;g^{'} \mid g \neq 0,b)\\
\leq\ & N(r,0;g^{'}) - N(r,0;g) + \overline{N}(r,0;g) - N(r,b;g) + \overline{N}(r,b;g)\\
\leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,b;g) + T(r,g) - N(r,0;g)\\
& -N(r,b;g) - m(r,0;g) - m(r,b;g) + S(r,g)\\
=\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,b;g) - T(r,g) + S(r,g),
\end{aligned}$$

where ω_i $i = 1, 2, \ldots, n$ are the distinct roots of the equation $P(w) = 0$. Hence using (3.1), *Lemmas 2.1* and *2.3* we get from second fundamental theorem for

$\varepsilon > 0$ that

$$\begin{aligned} & (n+1)\, T(r,f) \qquad (3.2) \\ \leq\ & \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geq 2) \\ & -N_0(r,0;f^{'}) + S(r,f) \\ \leq\ & 2\left\{\overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f)\right\} + \overline{N}(r,0;g) + \overline{N}(r,b;g) \\ & +\overline{N}(r,\infty;g) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;g^{'}) \\ & +S(r,f) + S(r,g) \\ \leq\ & 2\big\{\overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) \\ & +\overline{N}(r,b;g)\big\} - T(r,g) + S(r,f) + S(r,g) \\ \leq\ & \left(6 - 2\Theta(0,f) - 2\Theta(b,f) - 2\Theta(\infty,f) + \frac{1}{2}\varepsilon\right) T(r,f) \\ & + \left(5 - 2\Theta(0,g) - 2\Theta(b,g) - 2\Theta(\infty,g) + \frac{1}{2}\varepsilon\right) T(r,g) + S(r,f) + S(r,g) \\ \leq\ & (11 - 2\Theta_f - 2\Theta_g + \varepsilon)\, T(r) + S(r). \end{aligned}$$

In a similar way we can obtain

$$\begin{aligned} & (n+1)\, T(r,g) \qquad (3.3) \\ \leq\ & (11 - 2\Theta_f - 2\Theta_g + \varepsilon)\, T(r) + S(r). \end{aligned}$$

Combining (3.2) and (3.3) we see that

$$(n - 10 + 2\Theta_f + 2\Theta_g - \varepsilon)\, T(r) \leq S(r). \qquad (3.4)$$

Since $\varepsilon > 0$, (3.4) leads to a contradiction.
While $m = 1$, using *Lemma 2.2*, (3.1) changes to

$$\begin{aligned} & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \qquad (3.5) \\ \leq\ & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_L(r,1;G) + \overline{N}_L(r,1;F) \\ \leq\ & N(r,0;g^{'} \mid g \neq 0,b) + \frac{1}{2}N(r,0;f^{'} \mid f \neq 0,b) \\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,b;g) + \overline{N}(r,\infty;g) - T(r,g) + \frac{1}{2}\left\{\overline{N}(r,0;f)\right. \\ & \left. +\overline{N}(r,b;f) + \overline{N}(r,\infty;f)\right\} - \frac{1}{2}T(r,f) + S(r,f) + S(r,g). \end{aligned}$$

So using (3.5), *Lemmas 2.1* and *2.3* and proceeding as in (3.2) we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned}
& (n+1)\, T(r,f) \qquad (3.6)\\
\leq\; & 2\big\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)\\
& +\overline{N}(r,\infty;g)\big\}-T(r,g)-\frac{1}{2}T(r,f)\\
& +\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g)\\
\leq\; & \left(12-2\Theta_f-2\Theta_g-\frac{1}{2}\Theta(0;f)-\frac{1}{2}\Theta(\infty;f)-\frac{1}{2}\Theta(b;f)+\varepsilon\right)T(r)+S(r).
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
& (n+1)\, T(r,g) \qquad (3.7)\\
\leq\; & \left(12-2\Theta_f-2\Theta_g-\frac{1}{2}\Theta(0;g)-\frac{1}{2}\Theta(\infty;g)-\frac{1}{2}\Theta(b;g)+\varepsilon\right)T(r)+S(r).
\end{aligned}$$

Combining (3.6) and (3.7) we see that

$$\begin{aligned}
& (n-11+2\Theta_f+2\Theta_g+\frac{1}{2}\min\{\Theta(0;f)+\Theta(\infty;f)+\Theta(b;f), \qquad (3.8)\\
& \Theta(0;g)+\Theta(\infty;g)+\Theta(b;g)\}-\varepsilon)\; T(r)\leq S(r).
\end{aligned}$$

Since $\varepsilon > 0$, (3.8) leads to a contradiction.

Subcase 1.2. $m = 0$. Using *Lemma 2.2* we note that

$$\begin{aligned}
& \overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F) \qquad (3.9)\\
\leq\; & \overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\leq\; & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\leq\; & N(r,0;g^{'}\mid g\neq 0,b)+\overline{N}(r,1;G\mid\geq 2)+2\overline{N}(r,1;F\mid\geq 2)\\
\leq\; & 2\{N(r,0;g^{'}\mid g\neq 0,b)+N(r,0;f^{'}\mid f\neq 0,b)\}\\
\leq\; & 2\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,b;g)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\\
& +\overline{N}(r,b;f)\}-2T(r,f)-2T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Hence using (3.9), *Lemmas 2.1* and *2.3* we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} & (n+1)\, T(r,f) \qquad (3.10)\\ \leq\ & \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) \\ & + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) - N_0(r,0;f^{'}) + S(r,f) \\ \leq\ & 2\left\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,b;f)\right\} + \overline{N}(r,0;g) + \overline{N}(r,b;g) \\ & + \overline{N}(r,\infty;g) + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \\ & + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g) \\ \leq\ & 4\left\{\overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f)\right\} + 3\{\overline{N}(r,0;g) \\ & + \overline{N}(r,b;g) + \overline{N}(r,\infty;g)\} - 2T(r,f) - 2T(r,g) + S(r,f) + S(r,g) \\ \leq\ & (17 - 3\Theta_f - 3\Theta_g - \Theta(0;f) - \Theta(\infty;f) - \Theta(b;f) + \varepsilon)\, T(r) + S(r). \end{aligned}$$

In a similar manner we can obtain

$$\begin{aligned} & (n+1)\, T(r,g) \qquad (3.11)\\ \leq\ & (17 - 3\Theta_f - 3\Theta_g - \Theta(0;g) - \Theta(\infty;g) - \Theta(b;g) + \varepsilon)\, T(r) + S(r). \end{aligned}$$

Combining (3.10) and (3.11) we see that

$$\begin{aligned} & (n - 16 + 3\Theta_f + 3\Theta_g + \min\{\Theta(0;f) + \Theta(b;f) + \Theta(\infty;f), \qquad (3.12)\\ & \Theta(0;g) + \Theta(b;g) + \Theta(\infty;g)\} - \varepsilon)\, T(r) \leq S(r). \end{aligned}$$

Since $\varepsilon > 0$, (3.12) leads to a contradiction.
Case 2. Suppose that $H \equiv 0$. Now proceeding in the same way as done in the [2] we can prove $f \equiv g$. □

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