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Approximation Error and Generalized Orders of an Entire Function*

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Abstract

The partial sum of Jacobi series of an entire function

$$f(z) = \sum_{k=0}^{\infty} q_k(z)[\gamma(z)]^{k-1},$$

where $\gamma(z)$ is a polynomial of degree ξ and $q_k(z)$ is a uniquely determined polynomial of degree $\xi - 1$ or less, interpolate $f(z)$ at the zeros of $\gamma(z)$. Let B be a Caratheodory domain. For $1 \leq p \leq \infty$, let $L^p(B)$ be the class of all functions f holomorphic in B such that $||f||_{B,p} = [\frac{1}{A}\int\int_B |f(z)|^p dxdy]^{1/p} < \infty$, where A is the area of B. For $f \in L^p(B)$, set

$$E^p_m(f) = \inf_{t \in \pi_m} ||f - t||_{B,p},$$

π_m consists of all polynomials of degree at most $m = \xi k$. This paper deals with generalized growth parameters in terms of above approximation error in $L_p - norm$ on B.

Keywords and Phrases: *Jacobi series, Approximation error, Carath-eodory domain, Leminiscate, Generalized growth parameters.*

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1. Introduction

For a given polynomial $\gamma(z)$ of degree ξ let

$$f(z)=\sum_{k=0}^{\infty} q_k(z)[\gamma(z)]^{k-1} \tag{1.1}$$

be an entire function, where $q_k(z), k = 1, 2, 3, ...$ is a uniquely determined polynomial of degree atmost $\xi - 1$ or less. The partial sums of the series in (1.1) interpolate the function $f(z)$ at the zeros of the polynomial $\gamma(z)$. For $\gamma(z) = z$ the series (1.1) reduces to the Taylor series expansion of $f(z)$ at the origin.

Let B denote a Caratheodory domain, that is, a bounded simply connected domain such that the boundary of B coincides with the boundary of the domain lying in the complement of the closure of B and containing the point ∞. In particular, a domain bounded by a Jordan Curve is a Caratheodory domain. Let $L^p(B), 1 \leq p \leq \infty$, be the class of all functions f holomorphic on B and satisfying

$$||f||_{B,p} = [\frac{1}{A}\int\int_B |f(z)|^p dxdy]^{1/p} < \infty,$$

where the last inequality is understood to be $\sup_{z\in B}|f(z)| < \infty$ for $p = \infty$. Then $||||_{B,p}$ is called the $L^p - norm$ on $L^p(B)$.

Consider the function

$$H_{\alpha^*}(w)=\sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_{\alpha^*}} w^k, \alpha^* < R,$$

where Γ_R be the leminiscate $\Gamma_R = \{z : |\gamma(z)| = R\}, ||\Gamma_R||$ be the length of Γ_R and $M(\Gamma_R, f) = ||f(z)||_{\Gamma_R} = \max_{z\in\Gamma_R} |f(z)|, ||q_k(z)||_{\Gamma_{\alpha^*}} = \max_{z\in\Gamma_{\alpha^*}}\{|q_k(z)|\}$ as $k \to \infty$.

It is known [5, Lemma 2] that if $f(z)$ is analytic in Γ_R, then there exists a polynomial $Q(z)$ of degree $\xi - 1$ independent of k and R such that for $\alpha^* < R$ and $k = 1, 2, ...$

$$||q_k(z)||_{\Gamma_{\alpha^*}} \leq \frac{||\Gamma_R||M(\Gamma_R, f)}{2\pi R^k}||Q(z)||_{\Gamma_R}. \tag{1.2}$$

Using (1.2) we can easily seen that $H_{\alpha^*}(w)$ is entire if and only if

$$[||q_k(z)||_{\Gamma_{\alpha^*}}]^{1/k} = 0. \tag{1.3}$$

Moreover, $H_{\alpha^*}(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_{\alpha^*}} w^k$ holds in the whole complex plane. For $f \in L^p(B)$, we define $E_m^p(f)$, the error in approximating the function f by polynomial of degree at most $m = \xi n$ in $L^p - norm$ as

$$E_m^p(f) = E_m^p(f, B) = \inf_{t \in \pi_m} ||f - t||_{B,p}, n = 0, 1, 2, ... \tag{1.4}$$

where π_m consists of all polynomials of degree at most $m = \xi n$.

Let L^{**} denote the class of functions $h(x)$ satisfying conditions (H,i) and (H,ii):

(H,i) $h(x)$ is defined on $[a, \infty)$, is positive, strictly increasing and differentiable, and tends ∞ as $x \to \infty$.

(H,ii) $\lim_{x\to\infty} \frac{h[x(1+\phi(x))]}{h(x)} = 1$
for every function $\phi(x)$ such that $\phi(x) \to 0$ as $x \to \infty$.

Let $\triangle$ denote the class of functions $h(x)$ satisfying conditions (H,i) and (H,iii):

(H,iii)$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$ for every $0 < c < \infty$.

Kapoor and Nautiyal [4] defined generalized growth parameters $\rho(\alpha, \beta, f)$ and $\lambda(\alpha, \beta, f)$ of an entire function $f(z)$ as

$$\begin{matrix} \rho(\alpha, \alpha, f) \\ \lambda(\alpha, \alpha, f) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(\Gamma_R, H))}{\beta(R^{1/\xi})} \tag{1.5}$$

where $\alpha(x) \in \Lambda$ and $\beta(x) \in L^{**}$ generalized various results, cf. [1], [2].

The generalized orders of an entire function $f(z)$ have been characterized in terms of $||q_k(z)||_{\Gamma_{\alpha^*}}$. They have obtained these results under the condition :

$$\frac{d[\beta^{-1}(\alpha(x))]}{d(\log x)} = o(1)$$

as $x \to \infty$. Clearly his results fail to exist for the functions $\alpha(x) = \beta(x)$. To include this class of functions we have defined generalized growth parameters analogous to Kapoor and Nautiyal [3] in a new setting as follows:

Let Ω be the class of functions $h(x)$ satisfying (H,i) and (H,iv):

(H,iv) There exists a $\delta(x) \in \Lambda$ and x_0, K_1 and K_2 such that

$$0 < K_1 \leq \frac{d(h(x))}{d(\delta(\log x))} \leq K_2 < \infty, \forall x > x_0.$$

Let $\bar{\Omega}$ be the class of functions $h(x)$ satisfying (H,i) and (H,v) :

(H,v) $\lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} \leq K, 0 < K < \infty.$

The generalized growth parameters of an entire function $f(z)$ are defined as

$$\begin{matrix} \rho(\alpha,\alpha,f) \\ \lambda(\alpha,\alpha,f) \end{matrix} = \lim_{R\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(\Gamma_R, f))}{\alpha(\log R^{1/\xi})} \tag{1.6}$$

where $\alpha(x)$ either belongs to Ω or $\bar{\Omega}$ and

$$\mu(R,f) = \max_{k\geq 0}[||q_k(z)||_{\Gamma_{\alpha^*}} R^k]$$

Kapoor and Nautiyal [4] have characterized generalized growth parameters for entire functions of fast growth in terms of $||q_k(z)||_{\Gamma_{\alpha^*}}$ in sup norm. It has been noticed that, the interrelation between the growth of an entire function in terms of $||q_k(z)||_{\Gamma_{\alpha^*}}$ and approximation error in $L^p - norm$ has been completely neglected.

In this paper we study the approximations of entire functions in $L^p - norm$ on Caratheodory domains. The compact set is a very particular case of Caratheodory domain. The generalized growth parameters of an entire function have been characterized in terms of the approximation error $E_n^p(f)$ defined by (1.4), our results applies satisfactorily for slow growth.

Let B^* be the component of the complement of the closure of the Caratheodory domain B that contains the point ∞. Set $B_R = \{z : |\bar{\phi(z)}| = R\}, R > 1$ where the function $w^* = \bar{\phi(z)}$ maps B^* conformally on to $|w^*| > 1$ such that $\bar{\phi}(\infty) = \infty$ and $\bar{\phi}'(\infty) > 0$. Here B_R is the largest equipotential curve of the modulus of the mapping function associated with the domain B. B_1 corresponds to the boundary of B.

Given $\varepsilon > 0$ there is a lemniscate $\Gamma_{\alpha^*} = \{z : |\gamma(z)| = \alpha^*\}$ so that Γ_{α^*} is interior to $B_{1+\varepsilon}$ and exterior to B_1.

2. Auxiliary Results

In this section we mention certain lemmas which will be used in the sequel.
Lemma 2.1 *Let* $f(z) = \sum_{k=0}^{\infty} q_k(z)[\gamma(z)]^{k-1}$ *be an entire function having generalized growth parameters* $\rho(\alpha,\alpha,f)$ *and* $\lambda(\alpha,\alpha,f)$*. Then*

$$\begin{matrix}\rho(\alpha,\alpha,f)\\ \lambda(\alpha,\alpha,f)\end{matrix} = \lim_{R\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(\log \bar{M}(\Gamma_R, f))}{\alpha(\log R^{1/\xi})}$$

where

$$\bar{M}(\Gamma_R, f) = \max_{z\in B_R} |f(z)|.$$

Proof. Let z_0 be a fixed point of the set B and $R > 1$. Then in view of Winiarski [7],

$$R - 2|B| - |z_0| \leq |z| \leq R + |B| + |z_0|, z \in B_R.$$

Using $\log Kx \sim \log x$ as $x \to \infty, 0 < K < \infty$, we get

$$\log M(T_{\xi^* R}, f) \leq \log \bar{M}(\Gamma_R, f) \leq \log M(\Gamma_{\eta R}, f)$$

for $\xi^* < 1$ and $\eta > 1$. Also, we have that $z \in \Gamma_R$ implies that $|z| = R^{1/\xi}(1 + o(1)), R \to \infty$. Now Lemma 21 is immediate in view of (1.6).

Lemma 2.2 *Let* $f \in L^p(B), 1 \leq p \leq \infty$, *be the restriction to* B *of an entire function having generalized growth parameters* $\rho(\alpha,\alpha,f)$ *and* $\lambda(\alpha,\alpha,f)$*. Then* $H_{\alpha^*}(w) = \sum_{k=1}^{\infty} ||q_k(z)||_{\alpha^*} w^k$ *is an entire function. Further* $\rho(\alpha,\alpha,f) = \rho(\alpha,\alpha,H_{\alpha^*})$ *and* $\lambda(\alpha,\alpha,f) = \lambda(\alpha,\alpha,H_{\alpha^*})$ *also hold.*

Proof. First, we have seen that H_{α^*} is entire by (1.3). From [6,p.77] for $R > \alpha^*$, we have

$$||q_k(z)||_{\Gamma_R} \leq ||q_k(z)||_{\Gamma_{\alpha^*}} R^{\xi-1},$$

for $z \in \Gamma_R$

$$|f(z)| \leq \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_R} ||\gamma(z)||_{\Gamma_R}^{k-1}$$

or

$$\begin{aligned}
\bar{M}(\Gamma_R, f) &\leq \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_{\alpha^*}} R^{k+\xi-2}, z \in B_R \\
&= R^{\xi-2} \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_{\alpha^*}} R^k \\
&= R^{\xi-2} H_{\alpha^*}(R), R > 1. \qquad (2.1)
\end{aligned}$$

Thus using Lemma 2.1 and the fact that either $\alpha \in \Omega$ or $\bar{\Omega}$, (2.1) gives

$$\rho(\alpha, \alpha, f) \leq \rho(\alpha, \alpha, H_{\alpha^*}); \lambda(\alpha, \alpha, f) \leq \lambda(\alpha, \alpha, H_{\alpha^*}) \qquad (2.2)$$

Using the estimate

$$||\Gamma_R|| = 2\pi R^{1/\xi}(1 + o(1)), R \to \infty,$$

we have for every $\varepsilon > 0$,

$$\begin{aligned}
H_{\alpha^*}(R/e^{\varepsilon}) &= \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_{\alpha^*}} (R/e^{\varepsilon})^k \\
&\leq \sum_{k=1}^{\infty} \frac{\bar{M}(\Gamma_R, f)||\Gamma_R||.||Q||_{\Gamma_{\alpha^*}} (R/e^{\varepsilon})^k}{2\pi R^k} \\
&= \bar{M}(\Gamma_R, f) R^{1/\xi}(1 + o(1))||Q||_{\Gamma_{\alpha^*}} \sum_{k=1}^{\infty} \frac{1}{e^{k\varepsilon}} \\
&= \bar{M}(\Gamma_R, f) R^{1/\xi}(1 + o(1))||Q||_{\Gamma_{\alpha^*}} \sum_{k=1}^{\infty} \frac{1}{(e^{\varepsilon} - 1)}
\end{aligned}$$

Thus, using Theorem 3 of [3], Lemma 2.1 and the fact, that either $\alpha \in \Omega$ or $\bar{\Omega}$, we obtain

$$\rho(\alpha, \alpha, H_{\alpha^*}) \leq \rho(\alpha, \alpha, f); \lambda(\alpha, \alpha, H_{\alpha^*}) \leq \lambda(\alpha, \alpha, f). \qquad (2.3)$$

Combining (2.2) and (2.3), result follows for $0 \leq p \leq \infty$.

Lemma 2.3 *Let $f \in L^p(B), 1 \leq p \leq \infty$, be the restriction to B of an entire function having generalized growth parameters $\rho(\alpha, \alpha, f)$ and $\lambda(\alpha, \alpha, f)$. Then $\tilde{H}(t^*) = \sum_{k=1}^{\infty} E_m^p(f) t^{*n}$ is also an entire function Further we have*

$$\rho(\alpha, \alpha, f) = \rho(\alpha, \alpha, \tilde{H}), \lambda(\alpha, \alpha, f) = \lambda(\alpha, \alpha, \tilde{H}).$$

Proof. From the definition of $E_m^p(f)$, since $Q_m \in \pi_m$, we have

$$E_m^p(f) \leq ||f - Q_m||_{B,p} \leq A^{1/p} \max_{z \in B} |f(z) - Q_m(z)|,$$

where A is the area of B. From [6], we have

$$E_m^p(f) \leq \sum_{k=n}^{\infty} ||q_k(z)||_{\Gamma_{\alpha^*}} {\alpha^*}^{k-1}$$

$$\sum_{k=n}^{\infty} \frac{||\Gamma_R|| \bar{M}(\Gamma_R, f)}{2\pi R^k} ||Q(z)||_{\Gamma_R} {\alpha^*}^{k-1}$$

For $\alpha^* > 1$ be fixed constant and $R > \alpha^*$, we get

$$E_m^p(f) \leq \gamma \bar{M}(\Gamma_R, f) (\frac{\alpha^*}{R})^n \left(1/(1 - \frac{\alpha^*}{R}) \right) R^{1/\xi}(1 + o(1)), \tag{2.4}$$

for sufficiently large R, or

$$\mu(R, \alpha^* \tilde{H}) \leq \gamma \bar{M}(\Gamma_R, f) \left(\frac{R^{1+1/\xi}}{R - \alpha^*} \right) (1 + o(1)).$$

Now using the same reasoning as in Lemma 2.2, we obtain

$$\rho(\alpha, \alpha, \tilde{H}) \leq \rho(\alpha, \alpha, f); \lambda(\alpha, \alpha, \tilde{H}) \leq \lambda(\alpha, \alpha, f) \tag{2.5}$$

Further, define the function

$$\tilde{f}(z) = \sum_{k=0}^{\infty} (P_{k+1}(z) - P_k(z)) \tag{2.6}$$

since

$$|P_{k+1}(z) - P_k(z)| \leq ||P_{k+1}(z) - P_k(z|| \leq 2||f - P_k(z)||, z \in B.$$

Using Walsh inequality, [6,p.77], we have

$$|P_{k+1}(z) - P_k(z)| \leq 2||f - P_k(z)||_{B,1}^2 {R'}^k, z \in B_{R'}, R' > 1.$$

On applying Holder's inequality, we get

$$||P_{k+1}(z) - P_k(z)||/{R'}^k \leq 2A^q ||f - P_k(z)||_{B_{R'},q}$$

where A is defined as earlier and $q = 1-1/p, 1 \leq p \leq \infty$. Since above inequality holds for any polynomial $P_k(z)$, so we have

$$||P_{k+1}(z) - P_k(z)||/R^{'k} \leq 2A^q E_{k-1}^p(f), 1 \leq p < \infty.$$

Now using (24), we get

$$|\tilde{f}(z)| \leq \sum_{k=0}^{\infty} |P_{k+1}(z) - P_k(z)|$$

or

$$\tilde{M}(\Gamma_R, f) \leq |a_0| + 2A^q \sum_{k=0}^{\infty} E_{k-1}^p(f)(RR')^k, z \in B_R.$$

Using the fact that $z \in \Gamma_R$ implies that $|z| = R^{1/\xi}\xi(1+o(1)), R \to \infty$. Thus we have

$$\leq |a_0| + 2A^q(RR')^{1/\xi}\mu(R, R', \tilde{H}). \tag{2.7}$$

The right hand side of the series (2.6) converges for every R and therefore, the series on the right of (2.4) converges uniformly on every compact subset of C and so $\tilde{f}(z)$ is entire and $\tilde{f}(z) = f(z)$. Since $\lim_{m\to\infty}[E_m^p(f)]^{1/m} = 0$ by (2.4) it follows that $\tilde{H}(t^*)$ is entire. In view of Lemma 2.1, from (2.7) and $\alpha \in \Omega$ or $\bar{\Omega}$, we have

$$\rho(\alpha, \alpha, f) \leq \rho(\alpha, \alpha, \tilde{H}); \lambda(\alpha, \alpha, f) \leq \lambda(\alpha, \alpha, \tilde{H}) \tag{2.8}$$

On combining (2.5) and (2.8) we get the required result.

3. Main Results

We shall use the following notations in proving the main theorems:

$P_\phi^* = \max\{1, v\}$ if $\alpha(x) \in \Omega$,
$= \phi + v$ if $\alpha(x) \in \bar{\Omega}$.
We shall write $P^*(v)$ for $P_1^*(v)$.

Theorem 3.1. *Let $f \in L^p(B), 1 \leq p \leq \infty$, be the restriction to B of an entire function having generalized growth parameters $\rho(\alpha, \alpha, f)$ and $\lambda(\alpha, \alpha, f)$. Then* *(i)* $\rho(\alpha, \alpha, f) = P^*(L)$

(ii) $\rho(\alpha, \alpha, f) = P^*(L^*)$,

where

$$L = \lim_{m\to\infty} \sup \frac{\alpha(m)}{\alpha\left(\frac{1}{m}\log E_m^p(f)^{-1}\right)}, m = \xi n,$$

and

$$L^* = \lim_{m\to\infty} \sup \frac{\alpha(m)}{\alpha\left(\log(E_{m-1}^p(f)/E_m^p(f))\right)}.$$

(iii) $\lambda(\alpha, \alpha, f) = P^*(\bar{l})$, *where*

$$\bar{l} = \lim_{m\to\infty} \inf \frac{\alpha(m)}{\alpha\left(\frac{1}{m}\log E_m^p(f)^{-1}\right)},$$

(iv) If we take $\alpha(x) = \alpha(\alpha)$ *on* $(-\infty, a)$, *then* $\lambda(\alpha, \alpha, f) \geq P^*(l^*)$, *where*

$$l^* = \lim_{m\to\infty} \inf \frac{\alpha(m)}{\alpha\left(\log(E_{m-1}^p(f)/E_m^p(f))\right)}.$$

Theorem 3.2. *Let* $f \in L^p(B), 1 \leq p \leq \infty$, *be the restriction to* B *of an entire function having generalized growth parameters* $\rho(\alpha, \alpha, f), \lambda(\alpha, \alpha, f)$ *and if* $(E_m^p(f))/(E_{m+1}^p(f))$ *is nondecreasing, then*

$$\rho(\alpha, \alpha, f) = P^*(L) = P^*(L^*)$$

and

$$\lambda(\alpha, \alpha, f) = P^*(\bar{l}) = P^*(l^*).$$

Theorem 3.3. *Let* $f \in L^p(B), 1 \leq p \leq \infty$, *be the restriction to* B *of an entire function having generalized lower order* $\lambda(\alpha, \alpha, f)$. *Then*

(i) If $\alpha(x) \in \Omega$, *we have*

$$\lambda(\alpha, \alpha, f) = \max_{\{m_k\}}[P_\chi^*(l')], m_k = \xi n_k, \tag{3.1}$$

and if we further take $\alpha(x) = \alpha(a)$ *on* $(-\infty, a)$, *then*

$$\lambda(\alpha, \alpha, f) = \max_{\{m_k\}}[P_\chi(l'^*)], \tag{3.2}$$

where

$$\chi \equiv \chi(\{m_k\}) = \lim_{k\to\infty} \inf \frac{\alpha(m_{k-1})}{\alpha(m_k)}$$

and

$$l' \equiv l'\{(m_k)\} = \lim_{k\to\infty} \inf \frac{\alpha(m_{k-1})}{\alpha\left(\frac{1}{m_k}\log E^p_{m_k}(f)^{-1}\right)},$$

$$l'^* \equiv l'^*(\{m_k\}) = \lim_{k\to\infty} \inf \frac{\alpha(m_{k-1})}{\alpha(\log(E^p_{m_{k-1}}(f)/E^p_{m_k}(f)))}.$$

The maximum in (3.1) and (3.2) is taken over all increasing sequence $\{m_k\}$ of positive integers. Further, $\{m_k\}$ if is the sequence of principal indices of the entire function $\tilde{H}(t^*) = \sum_{m=0}^{\infty} E^p_m(f)t^{*m}$ and $\alpha(m_k) \sim \alpha(m_{k+1})$ as $k \to \infty$, then (3.1) and (3.2) also hold for $\alpha(x) \in \bar{\Omega}$.

Proof of Theorems 3.1, 3.2 and 3.3. The proof of above theorems follows in the same manner as Kapoor and Nautiyal [3, Theorems 4-6, Lemma 1] and Lemma 2.3.

Remark. The characterization of above growth parameters in terms of $||q_k(z)||_{\Gamma_R}$ can be obtain in a similar manner using the Lemma 2.2 by replacing $||q_k(z)||_{\Gamma_R}$ in place of $E^p_m(f)$ in Theorems 3.1, 3.2 and 3.3.

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