Tamsui Oxford Journal of Mathematical Sciences **25(2)** (2009) 178-206
Aletheia University

q-Analogue of the Dunkl Transform on the Real Line*

Néji Bettaibi† and Rym Bettaieb‡
Institut Préparatoire aux Études d'Ingénieur de Monastir,
5000 Monastir, Tunisia

Received August 13, 2007, Accepted September 21, 2007.

Abstract

In this paper, we consider a q-analogue of the Dunkl operator on $\mathbb{R}$, we define and study its associated Fourier transform which is a q-analogue of the Dunkl transform. In addition to several properties, we establish an inversion formula and prove a Plancherel theorem for this q-Dunkl transform. Next, we study the q-Dunkl intertwining operator and its dual via the q-analogues of the Riemann-Liouville and Weyl transforms. Using this dual intertwining operator, we provide a relation between the q-Dunkl transform and the q^2-analogue Fourier transform introduced and studied in [17, 18].

Keywords and Phrases: *q-Dunkl operator, q-Dunkl transform, q-Dunkl intertwining operator.*

1. Introduction

The Dunkl operator on $\mathbb{R}$ of index $\left(\alpha+\frac{1}{2}\right)$ associated with the reflection group $\mathbb{Z}_2$ is the differential-difference operator Λ_α introduced by C. F. Dunkl

*2000 *Mathematics Subject Classification.* 33D15, 39A12, 42A38, 44A15, 44A20.
†E-mail: neji.bettaibi@ipein.rnu.tn
‡E-mail: rym.bettaieb@yahoo.fr

in [3] by

$$\Lambda_\alpha(f)(x) = \frac{df(x)}{dx} + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}, \quad \alpha \geq -\frac{1}{2}. \tag{1}$$

These operators are very important in pure mathematics and physics. They provide a useful tool in the study of special functions with root systems [4, 2] and they are closely related to certain representations of degenerate affine Heke algebras [1, 16], moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Suterland-Moser models, which deal with systems of identical particles in a one dimensional space [15, 13].

In [5], C. F. Dunkl has introduced and studied a Fourier transform associated with the operator Λ_α, called Dunkl transform, but the basic results such as inversion formula and Placherel theorem were established later by M. F. E. de Jeu in [10, 11].

C. F. Dunkl has proved in [4] that there exists a linear isomorphism V_α, called the Dunkl intertwining operator, from the space of polynomials on $\mathbb{R}$ of degree n onto itself, satisfying the transmutation relation

$$\Lambda_\alpha V_\alpha = V_\alpha \frac{d}{dx}, \qquad V_\alpha(1) = 1. \tag{2}$$

Next, K. Trimèche has proved in [19] that the operator V_α can be extended to a topological isomorphism from $\mathcal{E}(\mathbb{R})$, the space of C^∞-functions on $\mathbb{R}$, onto itself satisfying the relation (2).

The goal of this paper is to provide a similar construction for a q-analogue context. The analogue transform we employ to make our construction is based on some q-Bessel functions and orthogonality results from [14], which have important applications to q-deformed mechanics. The q-analogue of the Bessel operator and the Dunkl operator are defined in terms of the q^2-analogue differential operator, ∂_q, introduced in [18].

This paper is organized as follows: In Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we establish some results associated with the q-Bessel transform and study the q-Riemann-Liouville and the q-Weyl operators. In Section 4, we introduce and study a q-analogue of the Dunkl operator (1) and we deal with its eigenfunctions by giving some of their properties and providing for them a

q-integral representations of Mehler type as well as an orthogonality relation. In section 5, we define and study the q-Dunkl intertwining operator and its dual via the q-Riemann-Liouville and the q-Weyl transforms. Finally, in Section 6, we study the Fourier transform associated with the q-Dunkl operator (q-Dunkl transform), we establish an inversion formula, prove a Plancherel theorem and we provide a relation between the q-Dunkl transform and the q^2-analogue Fourier transform (see [17, 18]).

2. Notations and preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer the reader to the general references [8] and [12], for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions. Throughout this paper, we assume $q \in]0,1[$ and we denote $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$.

2.1 Basic symbols

For $x \in \mathbb{C}$, the q-shifted factorials are defined by

$$(x;q)_0 = 1; \quad (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k), \quad n = 1,2,...; \quad (x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k). \tag{3}$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}. \tag{4}$$

2.2 Operators and elementary special functions

The q-Gamma function is given by (see [9])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0, -1, -2, ...$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1 \quad and \lim_{q\longrightarrow 1^-}\Gamma_q(x) = \Gamma(x), \Re(x) > 0. \tag{5}$$

The q-trigonometric functions q-cosine and q-sine are defined by (see [17, 18])

$$\cos(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n}}{[2n]_q!} \quad , \quad \sin(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n+1}}{[2n+1]_q!}. \tag{6}$$

The q-analogue exponential function is given by (see [17, 18])

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2). \tag{7}$$

These three functions are absolutely convergent for all z in the plane and when q tends to 1 they tend to the corresponding classical ones pointwise and uniformly on compacts.
Note that we have for all $x \in \mathbb{R}_q$ (see [17])

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \quad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty},$$

and

$$|\ e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{8}$$

The q^2-analogue differential operator is (see [17, 18])

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & if \quad z \neq 0 \\ \lim\limits_{x\to 0}\partial_q(f)(x) \qquad (\text{in } \ \mathbb{R}_q) & if \quad z = 0. \end{cases} \tag{9}$$

Remark that if f is differentiable at z, then $\lim\limits_{q\to 1}\partial_q(f)(z) = f'(z)$.
A repeated application of the q^2-analogue differential operator n times is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1}f = \partial_q(\partial_q^n f).$$

The following lemma lists some useful computational properties of ∂_q, and reflects the sensitivity of this operator to parity of its argument. The proof is straightforward.

Lemma 1.
1) $\partial_q \sin(x;q^2) = \cos(x;q^2)$, $\partial_q \cos(x;q^2) = -\sin(x;q^2)$ *and* $\partial_q e(x;q^2) = e(x;q^2)$.
2) For all function f *on* $\mathbb{R}_q$, $\partial_q f(z) = \dfrac{f_e(q^{-1}z) - f_e(z)}{(1-q)z} + \dfrac{f_o(z) - f_o(qz)}{(1-q)z}$.
3) For two functions f *and* g *on* $\mathbb{R}_q$, *we have*

- *if* f *even and* g *odd*

$$\partial_q(fg)(z) = q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) = \partial_q(g)(z))f(z) + qg(qz)\partial_q(f)(qz);$$

- *if* f *and* g *are even*

$$\partial_q(fg)(z) = \partial_q(f)(z)g(q^{-1}z) + f(z)\partial_q(g)(z).$$

Here, for a function f defined on $\mathbb{R}_q$, f_e and f_o are its even and odd parts respectively.
The q-Jackson integrals are defined by (see [9])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \tag{10}$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n), \tag{11}$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_{q,+}$,

$$\int_a^{\infty} f(x)d_qx = (1-q)a\sum_{n=-\infty}^{-1} q^n f(aq^n), \tag{12}$$

The following simple result, giving q-analogues of the integration by parts theorem, can be verified by direct calculation.

Lemma 2.
1) For $a > 0$, *if* $\int_{-a}^{a} (\partial_q f)(x)g(x)d_qx$ *exists, then*

$$\int_{-a}^{a} (\partial_q f)(x)g(x)d_qx = 2\left[f_e(q^{-1}a)g_o(a) + f_o(a)g_e(q^{-1}a)\right] - \int_{-a}^{a} f(x)(\partial_q g)(x)d_qx. \tag{13}$$

2) If $\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx$ *exists,*

$$\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx = -\int_{-\infty}^{\infty} f(x)(\partial_q g)(x)d_qx. \tag{14}$$

2.3 Sets and spaces

By the use of the q^2-analogue differential operator ∂_q, we note:
• $\mathcal{E}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$, satisfying

$$\forall n \in \mathbb{N}, \quad a \geq 0, \qquad P_{n,a}(f) = \sup\left\{|\partial_q^k f(x)|; 0 \leq k \leq n; x \in [-a,a] \cap \mathbb{R}_q\right\} < \infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

We provide it with the topology defined by the semi norms $P_{n,a}$.
• $\mathcal{E}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{E}_q(\mathbb{R}_q)$ constituted of even functions.
• $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n, m \in \mathbb{N}, \qquad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} \mid x^m \partial_q^n f(x) \mid < +\infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

• $\mathcal{S}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of even functions.
• $\mathcal{D}_q(\mathbb{R}_q)$ the space of functions defined on $\mathbb{R}_q$ with compact supports.
• $\mathcal{D}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{D}_q(\mathbb{R}_q)$ constituted of even functions.
Using the q-Jackson integrals, we note for $p > 0$ and $\alpha \in \mathbb{R}$,

• $L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}$,

• $L_q^p(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{p,q} = \left(\int_0^{\infty} |f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}$,

• $L_{\alpha,q}^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_qx\right)^{\frac{1}{p}} < \infty \right\}$,

• $L_{\alpha,q}^p(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_0^{\infty} |f(x)|^p x^{2\alpha+1} d_qx\right)^{\frac{1}{p}} < \infty \right\}$,

- $L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}$,
- $L_q^\infty(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_{q,+}} |f(x)| < \infty \right\}$.

2.4 q^2-Analogue Fourier transform

R. L. Rubin defined in [18] the q^2-analogue Fourier transform as

$$\widehat{f}(x;q^2) = K \int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt, \tag{15}$$

where $K = \dfrac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}$.
Letting $q \uparrow 1$ subject to the condition

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \tag{16}$$

gives, at least formally, the classical Fourier transform. In the remainder of this paper, we assume that the condition (16) holds.

It was shown in [18] that $\widehat{f}(.;q^2)$ verifies the following properties:
1) If $f(u),\quad uf(u) \in L_q^1(\mathbb{R}_q)$, then $\partial_q\left(\widehat{f}\right)(x;q^2) = (-iuf(u))\widehat{\ }(x;q^2)$.
2) If $f,\quad \partial_q f \in L_q^1(\mathbb{R}_q)$, then $(\partial_q f)\widehat{\ }(x;q^2) = ix\widehat{f}\,(x;q^2)$.
3) $\widehat{f}\,(.;q^2)$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ onto itself. For $f \in L_q^2(\mathbb{R}_q)$, we have $\forall x \in \mathbb{R}_q,\quad \left(\widehat{f}\right)^{-1}(x;q^2) = \widehat{f}(-x;q^2)$ and $\|\widehat{f}\,(.;q^2)\|_{2,q} = \|f\|_{2,q}$.

3. q-Bessel Fourier Transform

The normalized q-Bessel function is defined by

$$j_\alpha(x;q^2) = \sum_{n=0}^{+\infty}(-1)^n \frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}. \tag{17}$$

Note that we have

$$j_\alpha(x;q^2) = (1-q^2)^\alpha\Gamma_{q^2}(\alpha+1)\left((1-q)x\right)^{-\alpha} J_\alpha((1-q)x;q^2), \tag{18}$$

where

$$J_\alpha(x;q^2) = \frac{x^\alpha (q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} . {}_1\varphi_1(0;q^{2\alpha+2};q^2,q^2x^2) \tag{19}$$

is the Jackson's third q-Bessel function.
Using the relations (17) and (6), we obtain

$$j_{-\frac{1}{2}}(x;q^2) = \cos(x;q^2), \tag{20}$$

$$j_{\frac{1}{2}}(x;q^2) = \frac{\sin(x;q^2)}{x} \tag{21}$$

and

$$\partial_q j_\alpha(x;q^2) = -\frac{x}{[2\alpha+2]_q} j_{\alpha+1}(x;q^2). \tag{22}$$

In [6], the authors proved the following estimation.

Lemma 3. *For $\alpha \geq -\dfrac{1}{2}$ and $x \in \mathbb{R}_q$,*

- $$|j_\alpha(x;q^2)| \leq \frac{(-q^2;q^2)_\infty (-q^{2\alpha+1};q^2)_\infty}{(q^{2\alpha+1};q^2)_\infty} \begin{cases} 1, & \text{if } |x| \leq \frac{1}{1-q} \\ q^{\left(\frac{Log(1-q)|x|}{Logq}\right)^2}, & \text{if } |x| \geq \frac{1}{1-q} \end{cases}$$

  - *for all* $v \in \mathbb{R}$, $j_\alpha(x;q^2) = o(x^{-v})$ *as* $|x| \longrightarrow +\infty$ (in $\mathbb{R}_q$).

As a consequence of the previous lemma and the relation (22), we have for $\alpha \geq -\dfrac{1}{2}$,

$$j_\alpha(.;q^2) \in \mathcal{S}_{*,q}(\mathbb{R}_q).$$

With the same technique used in [7], we can prove that for $\alpha > -\dfrac{1}{2}$, $j_\alpha(.;q^2)$ has the following q-integral representation of Mehler type

$$j_\alpha(x;q^2) = C(\alpha;q^2) \int_0^1 W_\alpha(t;q^2) \cos(xt;q^2) d_q t, \tag{23}$$

where

$$C(\alpha;q^2) = (1+q) \frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \tag{24}$$

and

$$W_\alpha(t;q^2) = \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}. \tag{25}$$

Remark. Since the functions $W_\alpha(.;q^2)$ and $\cos(.;q^2)$ are even and $\sin(.;q^2)$ is odd, we can write for $\alpha > -\frac{1}{2}$,

$$j_\alpha(x;q^2) = \frac{1}{2}C(\alpha;q^2)\int_{-1}^{1} W_\alpha(t;q^2)e(-ixt;q^2)d_qt. \tag{26}$$

In particular, using the inequality (8), we obtain

$$|j_\alpha(x;q^2)| \le \frac{2}{(q;q)_\infty}, \forall x \in \mathbb{R}_q. \tag{27}$$

Proposition 1. *For $x, y \in \mathbb{R}_{q,+}$, we have*

$$(xy)^{\alpha+1}\int_0^{+\infty} j_\alpha(xt;q^2)j_\alpha(yt;q^2)t^{2\alpha+1}d_qt = \frac{(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)}{(1-q)}\delta_{x,y}. \tag{28}$$

Proof. The result follows from the relation (18) and the orthogonality relation of the Jackson's third q-Bessel function $J_\alpha(.;q^2)$ proved in [14]. ■

Using the same technique as in [7], one can prove the following result.

Proposition 2. *For $\lambda \in \mathbb{C}$, the function $x \mapsto j_\alpha(\lambda x;q^2)$ is the unique even solution of the problem*

$$\begin{cases} \triangle_{\alpha,q}f(x) = -\lambda^2 f(x), \\ f(0) = 1, \end{cases} \tag{29}$$

where $\triangle_{\alpha,q}f(x) = \frac{1}{|x|^{2\alpha+1}}\partial_q[|x|^{2\alpha+1}\partial_q f(x)]$.

Definition 1. *The q-Bessel Fourier transform is defined for $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, by*

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q}\int_0^\infty f(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx \tag{30}$$

where

$$c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}. \tag{31}$$

Letting $q \uparrow 1$ subject to the condition (16), gives, at least formally, the classical Bessel-Fourier transform.
Some properties of the q-Bessel Fourier transform are given in the following result.

Proposition 3. *1) For $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, we have $\mathcal{F}_{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_{q,+})$ and*

$$\|\mathcal{F}_{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,q}.$$

2)For $f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, we have

$$\int_0^\infty f(x)\mathcal{F}_{\alpha,q}(g)(x)x^{2\alpha+1}d_qx = \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)g(\lambda)\lambda^{2\alpha+1}d_q\lambda. \tag{32}$$

3) If f and $\triangle_{\alpha,q}f$ are in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then

$$\mathcal{F}_{\alpha,q}(\triangle_{\alpha,q}f)(\lambda) = -\lambda^2\mathcal{F}_{\alpha,q}(f)(\lambda).$$

4) If f and x^2f are in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then

$$\triangle_{\alpha,q}(\mathcal{F}_{\alpha,q}(f)) = -\mathcal{F}_{\alpha,q}(x^2f).$$

Proof. 1) follows from the definition of $\mathcal{F}_{\alpha,q}$ and the relation (27).
2) Let $f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$.
Since for all $\lambda, x \in \mathbb{R}_{q,+}$, we have $| j_\alpha(\lambda x; q^2) | \leq \dfrac{2}{(q;q)_\infty}$, then

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} | f(x)g(\lambda)j_\alpha(\lambda x;q^2)|x^{2\alpha+1}\lambda^{2\alpha+1}d_qxd_q\lambda \\
\leq\ & \frac{2}{(q;q)_\infty}\|f\|_{1,\alpha,q}\|g\|_{1,\alpha,q} < \infty.
\end{aligned}$$

So, by the Fubini's theorem, we can exchange the order of the q-integrals and obtain,

$$\begin{aligned}
&\int_0^\infty f(x)\mathcal{F}_{\alpha,q}(g)(x)x^{2\alpha+1}d_qx \\
=\ & \int_0^{+\infty}\int_0^{+\infty} f(x)g(\lambda)j_\alpha(\lambda x;q^2)x^{2\alpha+1}\lambda^{2\alpha+1}d_q\lambda d_qx \\
=\ & \int_0^{+\infty} g(\lambda)\left(\int_0^{+\infty} f(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx\right)\lambda^{2\alpha+1}d_q\lambda \\
=\ & \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)g(\lambda)\lambda^{2\alpha+1}d_q\lambda.
\end{aligned}$$

3) For $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$ such that $\triangle_{\alpha,q} f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, let $\widetilde{f}$ be the even function defined on $\mathbb{R}_q$ whose f is its restriction on $\mathbb{R}_{q,+}$. We have $\widetilde{\triangle_{\alpha,q} f} = \triangle_{\alpha,q}\widetilde{f}$ and

$$\begin{aligned} \mathcal{F}_{\alpha,q}(\triangle_{\alpha,q} f)(\lambda) &= c_{\alpha,q}\int_0^\infty (\triangle_{\alpha,q} f)(x) j_\alpha(x\lambda;q^2) x^{2\alpha+1} d_q x & (33)\\ &= \frac{c_{\alpha,q}}{2}\int_{-\infty}^\infty (\triangle_{\alpha,q}\widetilde{f})(x) j_\alpha(x\lambda;q^2)|x|^{2\alpha+1} d_q x. & (34)\end{aligned}$$

So, Proposition 2 and two q-integrations by parts give the result.
4) The result follows from Proposition 2. ■

Proposition 4. *If $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then*

$$\forall x \in \mathbb{R}_{q,+}, \qquad f(x) = c_{\alpha,q}\int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda) j_\alpha(\lambda x;q^2)\lambda^{2\alpha+1} d_q\lambda.$$

Proof. The result follows from the relation (27), Proposition 1 and the Fubini's theorem. ■

Theorem 1. *1) <u>Plancherel formula</u>*
For all $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, we have

$$\|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{35}$$

2) <u>Plancherel theorem</u>
The q-Bessel transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ with $\mathcal{F}^{-1}_{\alpha,q} = \mathcal{F}_{\alpha,q}$.

Proof. 1) Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, it is easy to show that $\mathcal{F}_{\alpha,q}(f)$ is in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$. From Proposition 4, we have $f = \mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q}(f))$, so using the relation (32), we obtain

$$\begin{aligned} \|f\|^2_{2,\alpha,q} &= \int_0^\infty f(x)\overline{f}(x) x^{2\alpha+1} d_q x = \int_0^\infty \mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q} f)(x)\overline{f}(x) x^{2\alpha+1} d_q x\\ &= \int_0^\infty \mathcal{F}_{\alpha,q}(f)(x)\overline{\mathcal{F}_{\alpha,q}(f)}(x) x^{2\alpha+1} d_q x = \|\mathcal{F}_{\alpha,q}(f)\|^2_{2,\alpha,q}.\end{aligned}$$

2) The result follows from 1), Proposition 4 and the density of $\mathcal{D}_{*,q}(\mathbb{R}_q)$ in $L^2_{\alpha,q}(\mathbb{R}_{q,+})$. ■

Definition 2. *For $\alpha > -\frac{1}{2}$, the q-Riemann-Liouville operator $R_{\alpha,q}$ is defined for $f \in \mathcal{E}_{*,q}(\mathbb{R}_q)$ by*

$$R_{\alpha,q}(f)(x) = \frac{1}{2}C(\alpha;q^2)\int_{-1}^{1} W_\alpha(t;q^2)f(xt)d_qt. \tag{36}$$

The q-Weyl operator is defined for $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ by

$${}^tR_{\alpha,q}(f)(t) = \frac{(1+q)^{-\alpha+\frac{1}{2}}}{\Gamma_{q^2}(\alpha+\frac{1}{2})}\int_{q|t|}^{+\infty} W_\alpha\left(\frac{t}{x};q^2\right) f(x)x^{2\alpha}d_qx. \tag{37}$$

In the end of this section, we shall give some useful properties of these two operators. First, by simple calculus, one can easily prove that for $f \in \mathcal{E}_{*,q}(\mathbb{R}_q)$ and $g \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, we have

$$\frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty} R_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = K\int_{-\infty}^{\infty} f(t)\,{}^tR_{\alpha,q}(g)(t)d_qt. \tag{38}$$

Next, using the relation (26), we obtain

$$j_\alpha(.;q^2) = R_{\alpha,q}\left(e(-i.;q^2)\right). \tag{39}$$

Lemma 4. *The operator $R_{\alpha,q}$ is continuous from $\mathcal{E}_{*,q}(\mathbb{R}_q)$ into itself.*

Proof. Let f be in $\mathcal{E}_{*,q}(\mathbb{R}_q)$. The function $x \longmapsto R_{\alpha,q}(f)(x)$ is an even function on $\mathbb{R}_q$.
By q-derivation under the q-integral sign, we deduce that for all $n \in \mathbb{N}$,

$$\partial_q^n R_{\alpha,q}(f)(x) = \frac{1}{2}C(\alpha;q^2)\int_{-1}^{1} W_\alpha(t;q^2)t^n(\partial_q^n f)(xt)d_qt.$$

Then,

$$\forall a \geq 0, \forall n \in \mathbb{N}, P_{n,a}(R_{\alpha,q}(f)) \leq P_{n,a}(f) < \infty.$$

This relation together with the Lebesgue theorem proves that $R_{\alpha,q}(f)$ belongs to $\mathcal{E}_{*,q}(\mathbb{R}_q)$ and it shows that the operator $R_{\alpha,q}$ is continuous from $\mathcal{E}_{*,q}(\mathbb{R}_q)$ into itself. ∎

Using the previous lemma and making a proof as in Theorems 3 and 4 of [7], we obtain the following result.

Theorem 2. *The q-Riemann-Liouville operator $R_{\alpha,q}$ is a topological isomorphism from $\mathcal{E}_{*,q}(\mathbb{R}_q)$ onto itself and it transmutes the operators $\Delta_{\alpha,q}$ and ∂_q^2 in the following sense*

$$\Delta_{\alpha,q}R_{\alpha,q} = R_{\alpha,q}\partial_q^2. \tag{40}$$

Theorem 3. *The q-Weyl operator ${}^tR_{\alpha,q}$ is an isomorphism from $\mathcal{D}_{*,q}(\mathbb{R}_q)$ onto itself, it transmutes the operators $\Delta_{\alpha,q}$ and ∂_q^2 in the following sense*

$${}^tR_{\alpha,q}\Delta_{\alpha,q} = \partial_q^2({}^tR_{\alpha,q}) \tag{41}$$

and for $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, we have

$$\mathcal{F}_{\alpha,q}(f) = \left({}^tR_{\alpha,q}(f)\right)\widehat{\ }(.;q^2). \tag{42}$$

Proof. The first part of the result can be proved as Proposition 3 of [7] page 158.
The relation (42) is a consequence of the relations (38) and (39).
Let us now, prove the relation (41). Let $g \in \mathcal{D}_{*,q}(\mathbb{R}_q)$. For all $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, we have, using the q-integration by parts theorem, the relations (38) and (40),

$$\begin{aligned}
& K\int_{-\infty}^{\infty}\partial_q^2\left({}^tR_{\alpha,q}g\right)(x)f(x)d_qx \\
= \;& K\int_{-\infty}^{\infty}\left({}^tR_{\alpha,q}g\right)(x)\partial_q^2f(x)d_qx \\
= \;& \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty}g(x)R_{\alpha,q}\partial_q^2f(x)|x|^{2\alpha+1}d_qx \\
= \;& \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty}g(x)\Delta_{\alpha,q}R_{\alpha,q}f(x)|x|^{2\alpha+1}d_qx \\
= \;& -\frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty}\partial_qg(x)\partial_q(R_{\alpha,q}f)(x)|x|^{2\alpha+1}d_qx \\
= \;& \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty}\Delta_{\alpha,q}g(x)R_{\alpha,q}f(x)|x|^{2\alpha+1}d_qx \\
= \;& K\int_{-\infty}^{\infty}{}^tR_{\alpha,q}(\Delta_{\alpha,q}g)(x)f(x)d_qx.
\end{aligned}$$

4. The q-Dunkl operator and its eigenfunctions

For $\alpha \geq -\frac{1}{2}$, consider the operators:

$$H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1} f_o \tag{43}$$

and

$$\Lambda_{\alpha,q}(f)(x) = \partial_q \left[H_{\alpha,q}(f)\right](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x}. \tag{44}$$

It is easy to see that for a differentiable function f, the q-Dunkl operator $\Lambda_{\alpha,q}(f)$ tends, as q tends to 1, to the classical Dunkl operator $\Lambda_\alpha(f)$ given by (1).
In the case $\alpha = -\frac{1}{2}$, $\Lambda_{\alpha,q}$ reduces to the q^2-analogue differential operator ∂_q.
Some properties of the q-Dunkl operator $\Lambda_{\alpha,q}$ are given in the following proposition.

Proposition 5. .
i) If f is odd then $\Lambda_{\alpha,q}(f)(x) = q^{2\alpha+1}\partial_q f(x) + [2\alpha+1]_q \frac{f(x)}{x}$ and if f is even then $\Lambda_{\alpha,q}(f)(x) = \partial_q f(x)$.
ii) If f and g are of the same parity, then

$$\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x) g(x) |x|^{2\alpha+1} d_q x = 0.$$

iii) For all f and g such that $\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x) g(x) |x|^{2\alpha+1} d_q x$ exists, we have

$$\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x) g(x) |x|^{2\alpha+1} d_q x = -\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(g)(x) f(x) |x|^{2\alpha+1} d_q x. \tag{45}$$

iv) The operator $\Lambda_{\alpha,q}$ lives $\mathcal{E}_q(\mathbb{R}_q)$, $\mathcal{S}_q(\mathbb{R}_q)$ and $\mathcal{D}_q(\mathbb{R}_q)$ invariant.

Proof. i) is a direct consequence of the definition of $\Lambda_{\alpha,q}$.
ii) follows from the properties of the q-integrals and the fact that $\Lambda_{\alpha,q}$ change the parity of functions.
iii) From ii) we have the result when f and g are of the same parity.

Now, suppose that f is even and g is odd. Using Lemma 2, the property i) of $\Lambda_{\alpha,q}$ and the properties of the q^2-analogue differential operator ∂_q we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx \\
= &\int_{-\infty}^{+\infty} \partial_q(f)(x)g(x)|x|^{2\alpha+1}d_qx \\
= &-\int_{-\infty}^{+\infty} f(x)\partial_q\left[g(x)|x|^{2\alpha+1}\right]d_qx \\
= &-\int_{-\infty}^{+\infty} f(x)\left[q^{2\alpha+1}\partial_q g(x)+[2\alpha+1]_q\frac{g(x)}{x}\right]|x|^{2\alpha+1}d_qx \\
= &-\int_{-\infty}^{+\infty} f(x)\Lambda_{\alpha,q}(g)(x)|x|^{2\alpha+1}d_qx.
\end{aligned}$$

iv) follows from the facts that for $f \in \mathcal{E}_q(\mathbb{R}_q)$,

$$\Lambda_{\alpha,q}(f)(x) = \partial_q\left[H_{\alpha,q}(f)\right](x) + \frac{[2\alpha+1]_q}{2}\int_{-1}^{1}\partial_q(f)(xt)d_qt$$

and for $f \in \mathcal{S}_q(\mathbb{R}_q)$,

$$\begin{aligned}
\Lambda_{\alpha,q}(f)(x) &= \partial_q\left[H_{\alpha,q}(f)\right](x) + [2\alpha+1]_q\int_{0}^{1}\partial_q(f_o)(xt)d_qt \\
&= \partial_q\left[H_{\alpha,q}(f)\right](x) - [2\alpha+1]_q\int_{1}^{\infty}\partial_q(f_o)(xt)d_qt.
\end{aligned}$$

■

Let us now introduce the eigenfunctions of the q-Dunkl operator.

Theorem 4. *For $\lambda \in \mathbb{C}$, the q-differential-difference equation:*

$$\begin{cases} \Lambda_{\alpha,q}(f) &= i\lambda f \\ f(0) &= 1 \end{cases} \tag{46}$$

has as unique solution, the function

$$\psi_\lambda^{\alpha,q} : x \longmapsto j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2). \tag{47}$$

Proof. Let $f = f_e + f_o$. The problem (46) is equivalent to the system

$$\begin{cases} \partial_q f_e(x) + q^{2\alpha+1}\partial_q f_o(x) + [2\alpha+1]_q \dfrac{f_o(x)}{x} & = i\lambda f_e(x) + i\lambda f_o(x) \\ f_e(0) = 1, \end{cases}$$

witch is equivalent to

$$\begin{cases} \partial_q f_e(x) = i\lambda f_o(x) \\ q^{2\alpha+1}\partial_q^2 f_e(x) + [2\alpha+1]_q \dfrac{\partial_q f_e(x)}{x} & = -\lambda^2 f_e(x) \\ f_e(0) = 1. \end{cases}$$

Now, using Proposition 2 and the relation (22), we obtain

$$\begin{cases} f_e(x) = j_\alpha(\lambda x; q^2) \\ f_o(x) = \dfrac{1}{i\lambda}\partial_q(j_\alpha(\lambda x; q^2)) & = \dfrac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2). \end{cases}$$

Finally, for $\lambda \in \mathbb{C}$,

$$\psi_\lambda^{\alpha,q}(x) = f(x) = j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x; q^2). \qquad \blacksquare$$

The function $\psi_\lambda^{\alpha,q}(x)$, called q-Dunkl kernel has an unique extention to $\mathbb{C} \times \mathbb{C}$ and verifies the following properties.

Proposition 6. *1)* $\Lambda_{\alpha,q}\psi_\lambda^{\alpha,q} = i\lambda\psi_\lambda^{\alpha,q}$.

2) $\psi_\lambda^{\alpha,q}(x) = \psi_x^{\alpha,q}(\lambda)$, $\psi_{a\lambda}^{\alpha,q}(x) = \psi_\lambda^{\alpha,q}(ax)$ *and* $\overline{\psi_\lambda^{\alpha,q}(x)} = \psi_{-\lambda}^{\alpha,q}(x)$, *for* $\lambda, x \in \mathbb{R}$ *and* $a \in \mathbb{C}$.

3) If $\alpha = -\frac{1}{2}$, *then* $\psi_\lambda^{\alpha,q}(x) = e(i\lambda x; q^2)$.
For $\alpha > -\frac{1}{2}$, $\psi_\lambda^{\alpha,q}$ *has the following q-integral representation of Mehler type*

$$\psi_\lambda^{\alpha,q}(x) = \frac{1}{2}C(\alpha; q^2)\int_{-1}^{1} W_\alpha(t; q^2)(1+t)e(i\lambda xt; q^2)d_q t, \tag{48}$$

where $C(\alpha; q^2)$ *and* $W_\alpha(t; q^2)$ *are given respectively by (24) and (25).*
4) For all $n \in \mathbb{N}$ *we have*

$$|\ \partial_q^n \psi_\lambda^{\alpha,q}(x)\ | \leq \frac{4\ |\ \lambda\ |^n}{(q;q)_\infty}, \quad \forall \lambda, x \in \mathbb{R}_q. \tag{49}$$

In particular for all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q}$ *is bounded on* $\mathbb{R}_q$ *and we have*

$$|\ \psi_\lambda^{\alpha,q}(x)\ | \leq \frac{4}{(q;q)_\infty}, \quad \forall x \in \mathbb{R}_q. \tag{50}$$

5) For all $\lambda \in \mathbb{R}_q$, $\psi_\lambda^{\alpha,q} \in \mathcal{S}_q(\mathbb{R}_q)$.

Proof. 1) and 2) are immediate consequences of the definition of $\psi_\lambda^{\alpha,q}$.
3) If $\alpha = -\frac{1}{2}$ then the relations (20), (21) and (7) give the result.
If $\alpha > -\frac{1}{2}$, using the definition of $\psi_\lambda^{\alpha,q}$, the parity of the function $j_\alpha(.;q^2)$ and the relations (26) and (22), we obtain

$$\begin{aligned} & \psi_\lambda^{\alpha,q}(x) \\ = \; & j_\alpha(\lambda x;q^2) + \frac{1}{i\lambda}\partial_q(j_\alpha(\lambda x;q^2)) \\ = \; & \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)e(i\lambda xt;q^2)d_qt \\ & +\frac{1}{i}\frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)ite(i\lambda xt;q^2)d_qt, \end{aligned}$$

which achieves the proof.
4) By induction on n we prove that
$\partial_q^n\psi_\lambda^{\alpha,q}(x) = \frac{C(\alpha;q^2)}{2}(i\lambda)^n\int_{-1}^{1} W_\alpha(t;q^2)(1+t)t^n e(i\lambda xt;q^2)d_qt.$
So, the fact that $|e(ix;q^2)| \le \frac{2}{(q;q)_\infty}$ gives the result.
5) The result follows from Lemma 3, the relation (22) and the properties of ∂_q. ■
The function $\psi_\lambda^{\alpha,q}$ verifies the following orthogonality relation.

Proposition 7. *For all* $x, y \in \mathbb{R}_q$, *we have*

$$\int_{-\infty}^{+\infty} \psi_\lambda^{\alpha,q}(x)\overline{\psi_\lambda^{\alpha,q}(y)}|\lambda|^{2\alpha+1}dq\lambda = \frac{4(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)\delta_{x,y}}{(1-q)|xy|^{\alpha+1}}. \tag{51}$$

Proof. Let $x, y \in \mathbb{R}_q$, the use of the relation (28) and the properties of the

q-Jackson's integral lead to

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \psi_\lambda^{\alpha,q}(x)\overline{\psi_\lambda^{\alpha,q}(y)}|\lambda|^{2\alpha+1}dq\lambda \\
= &\int_{-\infty}^{+\infty} j_\alpha(\lambda x;q^2)j_\alpha(\lambda y;q^2)|\lambda|^{2\alpha+1}dq\lambda \\
&+\frac{xy}{[2\alpha+2]_q^2}\int_{-\infty}^{+\infty} j_{\alpha+1}(\lambda x;q^2)j_{\alpha+1}(\lambda y;q^2)|\lambda|^{2\alpha+3}dq\lambda \\
= &\frac{2(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)\delta_{|x|,|y|}}{(1-q)|xy|^{\alpha+1}} + \frac{2xy(1+q)^{2\alpha+2}\Gamma_{q^2}^2(\alpha+2)\delta_{|x|,|y|}}{[2\alpha+2]_q^2(1-q)|xy|^{\alpha+2}} \\
= &\frac{2(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)\delta_{|x|,|y|}}{(1-q)|xy|^{\alpha+1}}(1+sgn(xy)) = \frac{4(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)\delta_{x,y}}{(1-q)|xy|^{\alpha+1}}.
\end{aligned}$$

∎

5. $q-$Dunkl intertwining operator

Definition 3. *We define the $q-$Dunkl intertwining operator $V_{\alpha,q}$ on $\mathcal{E}_q(\mathbb{R}_q)$ by*

$$\forall x\in\mathbb{R}_q, V_{\alpha,q}(f)(x) = \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)(1+t)f(xt)d_qt, \tag{52}$$

where $C(\alpha;q^2)$ and $W_\alpha(t;q^2)$ are given by (24) and (25) respectively.

Theorem 5. *We have*
i) $V_{\alpha,q}(e(-i\lambda x;q^2)) = \psi_{-\lambda}^{\alpha,q}(x)$, $\lambda, x\in\mathbb{R}_q$.
ii) $V_{\alpha,q}$ verifies the following transmutation relation

$$\Lambda_{\alpha,q}V_{\alpha,q}(f) = V_{\alpha,q}(\partial_q f), \qquad V_{\alpha,q}(f)(0) = f(0). \tag{53}$$

Proof. i) follows from the relation (48).
ii) Let $f = f_o + f_e \in \mathcal{E}_q(\mathbb{R}_q)$, we have on the one hand

$$\begin{aligned}
&V_{\alpha,q}(\partial_q f)(x) \\
= &\frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)\partial_q f_o(xt)d_qt \\
&+\frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)t\partial_q f_e(xt)d_qt.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\Lambda_{\alpha,q}V_{\alpha,q}(f)(x)\\
= \quad &\frac{C(\alpha;q^2)}{2}\int_{-1}^{1}W_\alpha(t;q^2)t\partial_q f_e(xt)d_qt\\
&+\frac{q^{2\alpha+1}C(\alpha;q^2)}{2}\int_{-1}^{1}W_\alpha(t;q^2)t^2\partial_q f_o(xt)d_qt\\
&+\frac{[2\alpha+1]_qC(\alpha;q^2)}{2x}\int_{-1}^{1}W_\alpha(t;q^2)tf_o(xt)d_qt.
\end{aligned}$$

Now, using a q-integration by parts and the facts that

$$\partial_q\left[(1-q^2t^2)W_\alpha(qt;q^2)\right]=-[2\alpha+1]_qtW_\alpha(t;q^2)$$

and

$$(1-q^2t^2)W_\alpha(qt;q^2)=(1-t^2q^{2\alpha+1})W_\alpha(t;q^2),$$

we get

$$\begin{aligned}
&[2\alpha+1]_q\frac{C(\alpha;q^2)}{2x}\int_{-1}^{1}W_\alpha(t;q^2)tf_o(xt)d_qt\\
= \quad &\frac{C(\alpha;q^2)}{2}\int_{-1}^{1}(1-q^2t^2)W_\alpha(qt;q^2)\partial_q f_o(xt)d_qt\\
= \quad &\frac{C(\alpha;q^2)}{2}\int_{-1}^{1}(1-t^2q^{2\alpha+1})W_\alpha(t;q^2)\partial_q f_o(xt)d_qt,
\end{aligned}$$

which completes the proof. ∎

Theorem 6. *For all $f\in\mathcal{E}_q(\mathbb{R}_q)$, we have*

$$\forall x\in\mathbb{R}_q, V_{\alpha,q}(f)(x)=R_{\alpha,q}(f_e)(x)+\partial_q R_{\alpha,q}I_q(f_o)(x), \tag{54}$$

where $R_{\alpha,q}$ is given by (36) and I_q is the operator given by

$$\forall x\in\mathbb{R}_q, I_q(f_o)(x)=\int_0^{|qx|}f_o(t)d_qt.$$

Proof. From the definitions of the $q-$Dunkl intertwining and the q-Riemann-Liouville operators, we have

$$\begin{aligned} V_{\alpha,q}(f)(x) &= \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)(1+t)(f_o(xt)+f_e(xt))d_qt \\ &= \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)f_e(xt)d_qt + \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)tf_o(xt)d_qt. \\ &= R_{\alpha,q}(f_e)(x) + \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)tf_o(xt)d_qt. \end{aligned}$$

On the other hand, by q-derivation under the q-integral sign and the fact that $\partial_q(I_qf_o) = f_o$, we obtain

$$\begin{aligned} \partial_q\left[R_{\alpha,q}I_q(f_o)\right](x) &= \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)t\partial_q(I_qf_o)(xt)d_qt \\ &= \frac{C(\alpha;q^2)}{2}\int_{-1}^{1} W_\alpha(t;q^2)tf_o(xt)d_qt. \end{aligned}$$

This gives the result. ■

Theorem 7. *The transform $V_{\alpha,q}$ is an isomorphism from $\mathcal{E}_q(\mathbb{R}_q)$ onto itself, its inverse transform is given by*

$$\forall x \in \mathbb{R}_q, V_{\alpha,q}^{-1}(f)(x) = R_{\alpha,q}^{-1}(f_e)(x) + \partial_q\left(R_{\alpha,q}^{-1}I_q(f_o)\right)(x), \tag{55}$$

where $R_{\alpha,q}^{-1}$ is the inverse transform of $R_{\alpha,q}$.

Proof. Let H be the operator defined on $\mathcal{E}_q(\mathbb{R}_q)$ by

$$H(f) = R_{\alpha,q}^{-1}(f_e) + \partial_q(R_{\alpha,q}^{-1}I_q(f_o)).$$

We have $V_{\alpha,q}(f) = R_{\alpha,q}(f_e)+\partial_q\left(R_{\alpha,q}I_q(f_o)\right)$, $R_{\alpha,q}(f_e)$ is even and $\partial_q(R_{\alpha,q}I_q(f_o))$ is odd, then

$$\begin{aligned} HV_{\alpha,q}(f) &= R_{\alpha,q}^{-1}R_{\alpha,q}f_e + \partial_qR_{\alpha,q}^{-1}I_q(\partial_qR_{\alpha,q}I_q(f_o)) \\ &= f_e + \partial_qR_{\alpha,q}^{-1}I_q(\partial_qR_{\alpha,q}I_q(f_o)). \end{aligned}$$

Using the fact that for $\varphi \in \mathcal{E}_{*,q}(\mathbb{R}_q)$, $I_q(\partial_q\varphi)(x) = \varphi(x) - \lim_{t\to 0}\varphi(t)$, we obtain

$$I_q(\partial_qR_{\alpha,q}I_q(f_o)) = R_{\alpha,q}I_q(f_o).$$

So,

$$R_{\alpha,q}^{-1} I_q(\partial_q R_{\alpha,q} I_q(f_o)) = I_q(f_o)$$

and

$$\partial_q R_{\alpha,q}^{-1} I_q(\partial_q R_{\alpha,q} I_q(f_o)) = \partial_q I_q(f_0) = f_0.$$

Thus,

$$HV_{\alpha,q}(f) = f_e + f_o = f.$$

With the same technique, we prove that $V_{\alpha,q}H(f) = f$. ■

Definition 4. *For $f \in \mathcal{D}_q(\mathbb{R}_q)$ and $\alpha > -\frac{1}{2}$, we define the q-transpose of $V_{\alpha,q}$ by*

$$({}^tV_{\alpha,q})(f)(t) = M_{\alpha,q} \int_{|x| \geq q|t|} W_\alpha\left(\frac{t}{x}; q^2\right)\left(1 + \frac{t}{x}\right) f(x) \frac{|x|^{2\alpha+1}}{x} d_q x, \tag{56}$$

where $W_\alpha(.; q^2)$ is given by (25) and

$$M_{\alpha,q} = \frac{(1+q)^{-\alpha+\frac{1}{2}}}{2\Gamma_{q^2}(\alpha + \frac{1}{2})}. \tag{57}$$

Note that by simple computation, we obtain for $f \in \mathcal{E}_q(\mathbb{R}_q)$ and $g \in \mathcal{D}_q(\mathbb{R}_q)$

$$\frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} V_{\alpha,q}(f)(x) g(x) |x|^{2\alpha+1} d_q x = K \int_{-\infty}^{+\infty} f(t) ({}^tV_{\alpha,q})(g)(t) d_q t. \tag{58}$$

Proposition 8. *For $f \in \mathcal{D}_q(\mathbb{R}_q)$, we have*

$$\partial_q({}^tV_{\alpha,q})(f) = ({}^tV_{\alpha,q})(\Lambda_{\alpha,q})(f). \tag{59}$$

Proof. Using a q-integration by parts and the relations (58), (53) and (45), we get for all $f \in \mathcal{D}_q(\mathbb{R}_q)$ and $g \in \mathcal{E}_q(\mathbb{R}_q)$,

$$\begin{aligned} & K\int_{-\infty}^{+\infty} g(x)\partial_q({}^tV_{\alpha,q})f(x)d_qx \\ = & -K\int_{-\infty}^{+\infty} \partial_q g(x)({}^tV_{\alpha,q})f(x)d_qx \\ = & -\frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} V_{\alpha,q}(\partial_q g)(x)f(x)|x|^{2\alpha+1}d_qx \\ = & -\frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(V_{\alpha,q}g)(x)f(x)|x|^{2\alpha+1}d_qx \\ = & \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} V_{\alpha,q}(g)(x)\Lambda_{\alpha,q}f(x)|x|^{2\alpha+1}d_qx \\ = & K\int_{-\infty}^{+\infty} g(x)({}^tV_{\alpha,q})(\Lambda_{\alpha,q}f)(x)d_qx. \end{aligned}$$

As g is arbitrary in $\mathcal{E}_q(\mathbb{R}_q)$, we obtain the result. ■

Theorem 8. *For $f \in \mathcal{D}_q(\mathbb{R}_q)$, we have*

$$\forall x \in \mathbb{R}_q, ({}^tV_{\alpha,q})(f)(x) = ({}^tR_{\alpha,q})(f_e)(x) + \partial_q\left[{}^tR_{\alpha,q}J_q(f_o)\right](x), \tag{60}$$

where ${}^tR_{\alpha,q}$ is given by (37) and J_q is the operator defined by

$$J_q(f_o)(x) = \int_{-\infty}^{qx} f_o(t)d_qt.$$

Proof. Let $f, g \in \mathcal{D}_q(\mathbb{R}_q)$, using Theorem 6, the relation (38) and a q-integration by parts, we obtain

$$\begin{aligned}
& \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} V_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1}d_qx \\
= & \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} \left[R_{\alpha,q}(g_e)(x)+\partial_q R_{\alpha,q}I_q(g_o)(x)\right] f(x)|x|^{2\alpha+1}d_qx \\
= & \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} R_{\alpha,q}(g_e)(x).f_e(x).|x|^{2\alpha+1}d_qx \\
& +\frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} \partial_q R_{\alpha,q}I_q(g_o)(x).f_o(x).|x|^{2\alpha+1}d_qx \\
= & K\int_{-\infty}^{+\infty} ({}^tR_{\alpha,q})(f_e)(x).g_e(x)d_qx \\
& -\frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} R_{\alpha,q}I_q(g_o)(x).\partial_q\left[f_o(x).|x|^{2\alpha+1}\right]d_qx.
\end{aligned}$$

It is easily seen that the map J_q is bijective from $\mathcal{D}_q^*(\mathbb{R}_q)$ onto $\mathcal{D}_{*,q}(\mathbb{R}_q)$ and $J_q^{-1}=\partial_q$, where $\mathcal{D}_q^*(\mathbb{R}_q)$ is the subspace of $\mathcal{D}_q(\mathbb{R}_q)$ constituted of odd functions. Hence, by writing $f_o=\partial_qJ_qf_o$ and by making use of (40) and (38) we get

$$\begin{aligned}
& \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} R_{\alpha,q}I_q(g_o)(x).\partial_q\left[f_o(x).|x|^{2\alpha+1}\right]d_qx \\
= & \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} R_{\alpha,q}I_q(g_o)(x).\frac{1}{|x|^{2\alpha+1}}\partial_q\left[|x|^{2\alpha+1}\partial_qJ_qf_o(x)\right]|x|^{2\alpha+1}d_qx \\
= & \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} R_{\alpha,q}I_q(g_o)(x).\Delta_{\alpha,q}J_qf_o(x).|x|^{2\alpha+1}d_qx \\
= & K\int_{-\infty}^{+\infty} I_q(g_o)(x).{}^tR_{\alpha,q}\Delta_{\alpha,q}J_qf_o(x)d_qx \\
= & K\int_{-\infty}^{+\infty} I_q(g_o)(x).\partial_q^2({}^tR_{\alpha,q})J_qf_o(x)d_qx \\
= & -K\int_{-\infty}^{+\infty} \partial_qI_q(g_o)(x).\partial_q({}^tR_{\alpha,q})J_qf_o(x)d_qx.
\end{aligned}$$

Since $\partial_qI_q(g_o)(x)=g_o(x)$, then

$$\begin{aligned}&\frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} V_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1}d_qx\\ =\;& K\int_{-\infty}^{+\infty} g(x)\left[({}^tR_{\alpha,q})f_e(x)+\partial_q({}^tR_{\alpha,q})J_qf_o(x)\right]d_qx.\end{aligned}$$

As g is arbitrary in $\mathcal{D}_q(\mathbb{R}_q)$, this relation when combined with (58) gives the result. ■

Theorem 9. *The transform $({}^tV_{\alpha,q})$ is an isomorphism from $\mathcal{D}_q(\mathbb{R}_q)$ onto itself, its inverse transform is given by*

$$\forall x\in\mathbb{R}_q,\, ({}^tV_{\alpha,q})^{-1}(f)(x)=({}^tR_{\alpha,q})^{-1}(f_e)(x)+\partial_q\left[({}^tR_{\alpha,q})^{-1}J_q(f_o)\right](x), \quad (61)$$

where $({}^tR_{\alpha,q})^{-1}$ is the inverse transform of ${}^tR_{\alpha,q}$.

Proof. Taking account of the relation $J_q\partial_qf(x)=f(x)$ for all $f\in\mathcal{D}_{*,q}(\mathbb{R}_q)$ and proceeding as in Theorem 7 we obtain the result. ■

6. q-Dunkl transform

Definition 5. *Define the q-Dunkl transform for $f\in L^1_{\alpha,q}(\mathbb{R}_q)$ by*

$$F_D^{\alpha,q}(f)(\lambda)=\frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x).|x|^{2\alpha+1}d_qx, \quad (62)$$

where $c_{\alpha,q}$ is given by (31).

Remarks.
1) It is easy to see that in the even case $F_D^{\alpha,q}$ reduces to the q-Bessel Fourier transform given by (30) and in the case $\alpha=-\frac{1}{2}$, it reduces to the q^2-analogue Fourier transform given by (15).
2) Letting $q\uparrow 1$ subject to the condition (16), gives, at least formally, the classical Bessel-Dunkl transform.
Some properties of the q-Dunkl transform are given in the following proposition.

Proposition 9. *i) If $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ then $F_D^{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_q)$,*

$$\|F_D^{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,\alpha,q} \tag{63}$$

and

$$\lim_{\lambda\to\infty} F_D^{\alpha,q}(f)(\lambda) = 0.$$

ii) For $f \in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$F_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F_D^{\alpha,q}(f)(\lambda). \tag{64}$$

iii) For $f, g \in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$\int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)g(\lambda)|\lambda|^{2\alpha+1}d_q\lambda \;=\; \int_{-\infty}^{+\infty} f(x)F_D^{\alpha,q}(g)(x)|x|^{2\alpha+1}d_qx. \tag{65}$$

Proof. i) Follows from the definition of $F_D^{\alpha,q}(f)$, the Lebesgue theorem and the fact that $|\psi^{\alpha,q}_{-\lambda}(x)| \leq \dfrac{4}{(q;q)_\infty}$, for all λ, $x \in \mathbb{R}_q$.
ii) Using the relation (45) and Proposition 6, we obtain the result.
iii) Let $f, g \in L^1_{\alpha,q}(\mathbb{R}_q)$.
Since for all $\lambda, x \in \mathbb{R}_q$,we have $|\ \psi^{\alpha,q}_\lambda(x)\ | \leq \dfrac{4}{(q;q)_\infty}$, then

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\ f(x)g(\lambda)\psi^{\alpha,q}_\lambda(x)||x|^{2\alpha+1}|\lambda|^{2\alpha+1}d_qxd_q\lambda \;\leq\; \frac{4}{(q;q)_\infty}\|f\|_{1,\alpha,q}\|g\|_{1,\alpha,q}.$$

So, by the Fubini's theorem, we can exchange the order of the q-integrals, which gives the result. ∎

Theorem 10. *For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have*

$$\begin{aligned}\forall x \in \mathbb{R}_q, \quad f(x) &= \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)\psi^{\alpha,q}_\lambda(x).|\lambda|^{2\alpha+1}d_q\lambda \\ &= \overline{F_D^{\alpha,q}(\overline{F_D^{\alpha,q}(f)})}(x).\end{aligned} \tag{66}$$

Proof. Let $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ and $x \in \mathbb{R}_q$. Since for all $\lambda, t \in \mathbb{R}_q$, we have $| \psi_\lambda^{\alpha,q}(t) | \leq \dfrac{4}{(q;q)_\infty}$, and $\lambda \mapsto \psi_\lambda^{\alpha,q}(x)$ is in $\mathcal{S}_q(\mathbb{R}_q)$, then

$$\begin{aligned}
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)\psi_{-\lambda}^{\alpha,q}(t)\psi_\lambda^{\alpha,q}(x)||t\lambda|^{2\alpha+1}d_qtd_q\lambda \\
\leq\ & \frac{4}{(q;q)_\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)||\psi_\lambda^{\alpha,q}(x)||t\lambda|^{2\alpha+1}d_qtd_q\lambda \\
=\ & \frac{4}{(q;q)_\infty}\|f\|_{1,\alpha,q}\|\psi_x^{\alpha,q}(\centerdot)\|_{1,\alpha,q}.
\end{aligned}$$

Hence, by the Fubini's theorem, we can exchange the order of the q-integrals and by Proposition 7, we obtain

$$\begin{aligned}
& \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)|\lambda|^{2\alpha+1}d_q\lambda \\
=\ & \left(\frac{c_{\alpha,q}}{2}\right)^2\int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty}\psi_{-\lambda}^{\alpha,q}(t)\psi_\lambda^{\alpha,q}(x)|\lambda|^{2\alpha+1}d_q\lambda\right)|t|^{2\alpha+1}d_qt = f(x).
\end{aligned}$$

The second equality is a direct consequence of the definition of the q-Dunkl transform, Proposition 6 and the definition of the q-Jackson integral. ■

Theorem 11. *i) Plancherel formula*
For $\alpha \geq -1/2$, the q-Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$\|F_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{67}$$

ii) Plancherel theorem
The q-Dunkl transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_q)$. Its inverse transform $(F_D^{\alpha,q})^{-1}$ is given by :

$$(F_D^{\alpha,q})^{-1}(f)(x) = \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} f(\lambda)\psi_\lambda^{\alpha,q}(x).|\lambda|^{2\alpha+1}d_q\lambda = F_D^{\alpha,q}(f)(-x). \tag{68}$$

Proof. i) From Theorem 10, to prove the first part of i) it suffices to prove that $F_D^{\alpha,q}$ lives $\mathcal{S}_q(\mathbb{R}_q)$ invariant. Moreover, from the definition of $\mathcal{S}_q(\mathbb{R}_q)$ and the properties of the operator ∂_q (Lemma 1), one can easily see that $\mathcal{S}_q(\mathbb{R}_q)$ is also the set of all function defined on $\mathbb{R}_q$, such that for all $k,l\in\mathbb{N}$, we have

$$\sup_{x\in\mathbb{R}_q}\left|\partial_q^k\left(x^l f(x)\right)\right|<\infty \quad \text{and} \quad \lim_{x\to 0}\partial_q^k f(x) \quad \text{exists.}$$

Now, let $f\in\mathcal{S}_q(\mathbb{R}_q)$ and $k,l\in\mathbb{N}$. On the one hand, using the notation $\Lambda^0_{\alpha,q}f=f$ and
$\Lambda^{n+1}_{\alpha,q}f=\Lambda_{\alpha,q}(\Lambda^n_{\alpha,q}f)$, $n\in\mathbb{N}$, we obtain from the properties of the operator $\Lambda_{\alpha,q}$ that for all $n\in\mathbb{N}$, $\Lambda^n_{\alpha,q}f\in\mathcal{S}_q(\mathbb{R}_q)\subset L^1_{\beta,q}(\mathbb{R}_q)$ for all $\beta\geq -1/2$.
On the other hand, from the relation (64), we have

$$\begin{aligned}\lambda^l F_D^{\alpha,q}(f)(\lambda) &= (-i)^l F_D^{\alpha,q}(\Lambda^l_{\alpha,q}f)(\lambda)\\ &= (-i)^l\frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty}\Lambda^l_{\alpha,q}f(x)\psi^{\alpha,q}_{-\lambda}(x)|x|^{2\alpha+1}d_qx.\end{aligned}$$

So, using the relation (49), we obtain

$$\begin{aligned}|\partial_q^k(\lambda^l F_D^{\alpha,q}(f)(\lambda))| &= \left|(-i)^l\frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty}\Lambda^l_{\alpha,q}f(x)\partial_q^k\psi^{\alpha,q}_{-x}(\lambda)|x|^{2\alpha+1}d_qx\right|\\ &\leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\int_{-\infty}^{\infty}|\Lambda^l_{\alpha,q}f(x)||x|^{2\alpha+k+1}d_qx<\infty.\end{aligned}$$

This together with the Lebesgue theorem prove that $F_D^{\alpha,q}(f)$ belongs to $\mathcal{S}_q(\mathbb{R}_q)$. By Theorem 10, we deduce that $F_D^{\alpha,q}$ is an isomorphism of $\mathcal{S}_q(\mathbb{R}_q)$ onto itself and for $f\in\mathcal{S}_q(\mathbb{R}_q)$, we have $(F_D^{\alpha,q})^{-1}(f)(x)=F_D^{\alpha,q}(f)(-x)$, $\quad x\in\mathbb{R}_q$.
Finally, the Plancherel formula (67) is a direct consequence of the second equality in Theorem 10 and the relation (65).

ii) The result follows from i), Theorem 10 and the density of $\mathcal{S}_q(\mathbb{R}_q)$ in $L^2_{\alpha,q}(\mathbb{R}_q)$. ∎

Theorem 12. *The q-Dunkl transform and the q^2-analogue Fourier transform are linked by*

$$\forall f\in\mathcal{D}_q(\mathbb{R}_q),\quad F_D^{\alpha,q}(f)=\left[{}^tV_{\alpha,q}(f)\right]\widehat{\ }(.;q^2). \tag{69}$$

Proof. Using the relation (58) and Theorem 5, we obtain for $f \in \mathcal{D}_q(\mathbb{R}_q)$,

$$\begin{aligned}
\left[{}^tV_{\alpha,q}(f)\right]\widehat{\ }(\lambda) &= K\int_{-\infty}^{+\infty} ({}^tV_{\alpha,q})(f)(t)e(-i\lambda t;q^2)d_qt \\
&= \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} V_{\alpha,q}(e(-i\lambda x;q^2))f(x)|x|^{2\alpha+1}d_qx \\
&= \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x).|x|^{2\alpha+1}d_qx \\
&= F_D^{\alpha,q}(f)(\lambda).
\end{aligned}$$

■

References

[1] L. Cherednik, A unification of the Knizhnik-Zamolodchicov equations and Dunkl operators via affine Heke algebras, *Invent. Math.*, **106**(1991), 411-432.

[2] J. F. van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Cologero system with harmonic confinement, *Comm. Math. Phys.* **188**(1997), 467-497.

[3] C. F. Dunkl, Differential-difference operators associated to reflexion groups, *Trans. Amer. Math. Soc.*, **311**(1989), 167-183.

[4] C. F. Dunkl, Integral kernels with reflexion group invariance, *Can. J. Maths.* **43**(1991), 1213-1227.

[5] C. F. Dunkl, Hankel transform associated to finite reflexion groups, *Contemp. Math.* **138**(1992), 123-138.

[6] A. Fitouhi and N. Bettaibi and W. Binous, Inversion formulas for the q-Riemann-Liouville and q-Weyl transforms using wavelets, *J. fract. Anal.* **Vilume 4**, 2007.

[7] A. Fitouhi, M. M. Hamza and F. Bouzeffour, The q-j_α Bessel function, *J. Approx. Theory.*, **115**(2002), 144-166.

[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its application, Vol 35 Cambridge Univ. Press, Cambridge, UK, 1990.

[9] F. H. Jackson, On a q-Definite Integrals, *Quarterly Journal of Pure and Applied Mathematics*, **41**(1910), 193-203.

[10] M. F. E. de Jeu, The Dunkl transform, *Invent. Math.*, **113**(1993), 147-162.

[11] M. F. E. de Jeu, *The Dunkl operators*, Thesis, University of Amesterdam, 1994.

[12] V. G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, (2002).

[13] S. Kakei, Common algebraic structure for the Calogero-Sutherland models, *J. Phys.* **178**(1996), 425-452.

[14] T. H. Koornwinder and R. F. Swarttouw, On q-analogues of the Fourier and Hankel transforms, *Trans. Amer. Math. Soc.* **333**(1992), 445-461.

[15] L. Lapointe and L. Vinet, Exact operator solution of the Calogero-Sutherland model, *Comm. Math. Phys. A*, **29**(1996), 619-624.

[16] E. M. Opdam, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, *Compositio. Math.* **85**(1993), 333-373.

[17] Richard L. Rubin, A q^2-Analogue Operator for q^2-analogue Fourier Analysis,*J. Math. Analys. App.* **212**(1997), 571-582.

[18] Richard L. Rubin, Duhamel Solutions of non-Homogenous q^2-Analogue Wave Equations, *Proc. of Amer. Math. Soc.* **V 135, Nr 3**(2007), 777-785.

[19] K. Trimèche, The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual, *Integ. Transf. Spec. Funct.* **V 12, Nr 4**(2001), 349-374.