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# On Generalized Order Statistics From Kumaraswamy Distribution*

Mridula Garg†
*Department of Mathematics,University of Rajasthan, Jaipur-302055, India*

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**Abstract**

In the present paper, we study the generalized order statistics introduced recently by Kamps [27]. We obtain, the joint distribution, distribution of product and distribution of ratio of two generalized order statistics from Kumaraswamy distribution [24]. The method used is based on Mellin transform and its inverse. Several interesting special cases, which give the study when generalized order statistics are from uniform distribution and the distribution of product and quotient of extreme order statistics and consecutive order statistics have also been obtained.

**Keywords and Phrases:** *Fox H- function ,Generalized Order Statistics, Kumaraswamy Distribution, Mellin Transform,Random Variable.*

# 1. Introduction

The distributions of the product and ratio of random variables find an important place in the literature and much work is done when the random variables are independent and come from a particular probability distribution.

If the random variables $X_1$, $X_2,\cdots$, $X_n$ are arranged in ascending order of magnitudes and then written as $X_{(1)} \leq X_{(2)} \ldots \leq X_{(n)}$, then $X_{(i)}$ is called the $i^{th}$ order

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† E-mail: gargmridula@gmail.com

statistics (i=1,2, ⋯,n) and the ordered random variables are necessarily dependent. The distribution of product and quotient of the extreme order statistics and that of consecutive order statistics are useful in ranking and selection problems. Subrahmaniam [16] has made the study of product and quotient of order statistics from uniform distribution and exponential distribution, whereas Malik and Trudel [14] studied the cases when the order statistics are from Pareto, power and Weibull distributions.Recently the author [21] has studied order statistics from Kumaraswami distribution.

The subject of order statistics has been further generalized and the concept of generalized order statistics is introduced and studied by Kamps in a series of papers and books [26, 27, 28, 29]. The order statistics, record values and sequential order statistics are special cases of generalized order statistics. This concept is widely studied by many research workers namely Ahsanullah [17, 18, 19, 20], El-Baset, Ahmed and Al-Matrofi [2], Cramer and Kamps [7, 8], Cramer, Kamps and Rychlik [9, 10, 11, 12], Kamps and Cramer [29] and Reiess [25].

In the present paper we shall obtain the joint distribution, distribution of product and distribution of ratio of two generalized order statistics from the family of distributions known as Kumaraswamy distribution [24].

# 2. Definitions

### (i) Generalized Order Statistics

Let F(x) denote an absolutely continuous distribution function with density function f(x) and $X_{1;\,n,\,m,k}, X_{2;\,n,\,m,k}, \ldots, X_{n;\,n,\,m,k}$ ($k \geq 1$, m is a real number) be 'n' generalized order statistics. Then the joint probability density function (p.d.f.) $f_{1,\ldots,n}(x_1,\ldots,x_n)$ can be written as [26]

$$f_{1,\ldots,n}(x_1,\ldots,x_n) = \begin{cases} k\prod_{j=1}^{n-1}\gamma_j \prod_{i=1}^{n-1}\left[\left(1-F(x_i)\right)^m f(x_i)\right]\left(1-F(x_n)\right)^{k-1} f(x_n), \\ \qquad\qquad \text{for } F^{-1}(0) < x_1 < \ldots < x_n < F^{-1}(1) \\ 0\ ,\ \text{otherwise} \end{cases} \tag{1}$$

where,

$$\gamma_j = k+(n-j)(m+1) \text{ and } f(x) = \frac{dF(x)}{dx}$$

If m=0 and k=1 it gives the joint p.d.f. of 'n' ordinary order statistics $X_{1,n} \le ... \le X_{n,n}$. If m = -1 and k = 1, it gives the joint p.d.f. of the first 'n' upper records of the independent and identically distributed random variables. Various distributional properties of generalized order statistics are studied by Kamps [27] and that of record values by Ahsanullah [17, 19], Arnold, Balakrishnan and Nagaraja [4] and Raqab [24].

Further integrating out $x_1,...,x_{r-1},x_{r+1},...,x_n$ from (1), we get p.d.f. $f_{r;n,m,k}$ of $X_{r;n,m,k}$ [26] as

$$f_{r,n,m,k}(x) = \frac{C_r}{(r-1)!}[1-F(x_r)]^{\gamma_r -1} g_m^{r-1}[F(x_r)]\, f(x_r) \tag{2}$$

where

$$C_r = \prod_{j=1}^{r}\gamma_j, \qquad \gamma_r = k+(n-r)(m+1) \tag{3}$$

and

$$g_m(x) = \begin{cases} \dfrac{1}{m+1}[1-(1-x)^{m+1}], & m \ne -1 \\ -\ln(1-x), & m=-1,\ x \in (0,1) \end{cases} \tag{4}$$

Since $\lim\limits_{m\to -1}\dfrac{1}{m+1}[1-(1-x)^{m+1}] = -\ln(1-x)$

We shall write

$$g_m(x) = \frac{1}{m+1}[1-(1-x)^{m+1}] \text{ for all } x \in (0,1) \text{ and for all } m$$

with

$$g_{-1}(x) = \lim_{m\to -1} g_m(x) \tag{5}$$

Also the joint distribution of of $i^{th}$ and $j^{th}$ generalized order statistics is given by

$$f_{i,j;\,n,m,k}(x_i, x_j) = \frac{C_j}{(i-1)!(j-i-1)!}[1-F(x_i)]^m [1-F(x_j)]^{\gamma_j - 1}[g_m(F(x_i))]^{i-1} \tag{6}$$
$$[g_m(F(x_j)) - g_m(F(x_i))]^{j-i-1} f(x_i)\ f(x_j)$$
$$0 < x_i < x_j < \infty,\ 1 \le i < j \le n$$

symbols $C_j$ , $\gamma_j$ , $g_m(x)$ are as defined above.

The result (6), on taking m=0 and k=1 reduces to the joint p.d.f. of $i^{th}$ and $j^{th}$ order statistics as given in David [13].

**(ii) Kumaraswamy Distribution**

In this distribution, the probability density function of a random variable X is given by

$$f(x) = \begin{cases} abx^{a-1}(1-x^a)^{b-1}, & a > 0,\ b > 0,\ 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \tag{7}$$

with the cumulative density function (or distribution function), given as

$$F(x) = 1-(1-x^a)^b \tag{8}$$

In probability theory Kumaraswamy`s double bounded distribution is as versatile as the beta distribution, but much simpler to use especially in simulation studies as it has a simple closed form for both the p.d.f. and c.d.f.

**(iii) The Mellin Transform**

Let ($X_1$, $X_2$) be a two dimensional random variable having the joint probability density function f($x_1$, $x_2$) that is positive in the first quadrant and zero elsewhere. The Mellin transform of f($x_1$, $x_2$) is defined by Fox [5] as

$$M_{s_1,s_2}[f(x_1,x_2)] = \int_0^\infty \int_0^\infty x_1^{s_1-1} x_2^{s_2-1} f(x_1, x_2)\ dx_1 dx_2 \tag{9}$$

with the inverse

$$f(x_1, x_2) = \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} x_1^{-s_1} x_2^{-s_2} \ \mathrm{M}_{s_1, s_2}[f(x_1, x_2)] \, \mathrm{ds_1 ds_2} \tag{10}$$

under the appropriate conditions discussed by Fox.

In this paper, we are interested in the following two particular cases [16].

If Y = $X_1$ $X_2$, then h(y), the p.d.f. of Y, has the Mellin transform

$$\mathrm{M}_{s_1, s_2}[\mathrm{h(y)}] = \mathrm{M}_{s, s}[\mathrm{h(y)}] \tag{11}$$

and if Z = $X_1 / X_2$ , then g(z), the p. d. f. of Z, has the Mellin transform

$$\mathrm{M}_{s_1, s_2}[\mathrm{g}(z)] = \mathrm{M}_{s, -s+2}[\mathrm{g}(z)] \tag{12}$$

**(iv) Fox H- function**

We shall require the following definition of Fox H-function [6]

$$\mathrm{H}_{p,q}^{m,n}\left[ x \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] = \frac{1}{2\pi\omega} \int_L \theta(\mathrm{s}) \, \mathrm{x^s} \, \mathrm{ds} \tag{13}$$

where $\omega = \sqrt{-1}$ , $\mathrm{x}\,(\neq 0)$ is a complex variable and $\mathrm{x^s} = \exp[\mathrm{s}\{\log|x| + \omega \arg x\}]$,

$$\theta(\mathrm{s}) = \frac{\prod_{\mathrm{j=1}}^{\mathrm{m}} \Gamma(\mathrm{b_j} - \beta_\mathrm{j}\,\mathrm{s}) \prod_{\mathrm{j=1}}^{\mathrm{n}} \Gamma(1 - a_\mathrm{j} + \alpha_\mathrm{j}\,\mathrm{s})}{\prod_{\mathrm{j=m+1}}^{\mathrm{q}} \Gamma(1 - \mathrm{b_j} + \beta_\mathrm{j}\,\mathrm{s}) \prod_{\mathrm{j=n+1}}^{\mathrm{p}} \Gamma(a_\mathrm{j} - \alpha_\mathrm{j}\,\mathrm{s})}, \tag{14}$$

m, n, p and q are non-negative integers satisfying, $0 \le n \le p$, $1 \le m \le q$ ; $\alpha_j$ (j = 1, ⋯ , p) and $\beta_j$ ( j = 1, ⋯, q ) are assumed to be positive quantities for standardization purpose.

The definition of the H-function given by (13) will however have meaning even if some of these quantities are zero, giving us in tern simple transformation formulas. The nature of contour L, a set of sufficient conditions for the convergence of this integral, the asymptotic expansion, some of its properties and special cases can be referred to in the book by Srivastava, Gupta and Goyal[15].

# 3. Joint Distribution and Distributions of Product and Ratio of Two Generalized Order Statistics

**Theorem 1.** *Let $X_{i\,;\,n,\,m,\,k}$ and $X_{j\,;\,n,\,m,\,k}$ be $i^{th}$ and $j^{th}$ generalized order statistics with $(i<j)$, based on a random sample of size n from the Kumaraswamy distribution. The joint p.d.f of these generalized order statistics is given by :*

$$f_{i,j;\,n,m,k}(x_i,x_j)=\begin{cases}\dfrac{C_j\,a^2\,b^2\,x_i^{a-1}\,x_j^{a-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}}\displaystyle\sum_{l_1=0}^{i-1}\sum_{l_2=0}^{j-i-1}(-1)^{l_1+l_2}\binom{i-1}{l_1}\binom{j-i-1}{l_2}\\ .\left(1-x_i^{a}\right)^{b(m+1)(l_1-l_2+j-i)-1}\left(1-x_j^{a}\right)^{b(\gamma_j+(m+1)l_2)-1},\ \text{for } m\neq-1\\ \dfrac{k^j\,a^2\,b^j\,x_i^{a-1}\,x_j^{a-1}\left(1-x_j^{a}\right)^{b(k+1)-2}}{(i-1)!(j-i-1)!\left(1-x_i^{a}\right)}\log\left(\dfrac{1}{1-x_i^{a}}\right)^{i-1}\\ .[\log\left(1-x_i^{a}\right)-\log\left(1-x_j^{a}\right)]^{j-i-1},\qquad \text{for } m=-1\end{cases}\tag{15}$$

*provided that $a,\,b>0$, $1\le i<j\le n$, m is a real number, $k\ge 1$ and $n>1$, $0\le x_i<x_j\le 1$ and $C_j$ and $\gamma_j$ are defined by (3).*

**Proof.** The result can easily be established on substituting the values of f(x), F(x), and $g_m(x)$ from equations (7), (8) and (4) respectively in the equation (6) and expressing the values of $[g_m(F(x_i))]^{i-1}$ and $[g_m(F(x_j)) - g_m(F(x_i))]^{j-i-1}$ in their series forms.

**Theorem 2.** *Let $X_{i\,;\,n,\,m,\,k}$ and $X_{j\,;\,n,\,m,\,k}$ denote the $i^{th}$ and $j^{th}$ generalized order statistics from a random sample of size ‘n’ drawn from Kumaraswamy distribution defined by (7), then the probability density function of the product*

$$Y = X_{i\,;\,n,\,m,\,k}X_{j\,;\,n,\,m,\,k}\tag{16}$$

*and the ratio*

$$Z = \frac{X_{i;\,n,\,m,k}}{X_{j;\,n,\,m,k}}\tag{17}$$

*are given by*

$$g_{i,j,n,m,k}(y) = \frac{C_j\, a\, b^2\, y^{a-1}\left(1-y^a\right)^{b(m+1)(j-i+\gamma_j+1)-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2}$$

$$.\left(1-y^a\right)^{b(m+1)l_1} \frac{\Gamma\left(b(m+1)(l_1-l_2+j\text{-}i)\right)\Gamma\left(b\{\gamma_j+(m+1)l_2\}\right)}{\Gamma\left(b\{(m+1)(l_1+j\text{-}i)+\gamma_j\}\right)}$$

$$\cdot\, {}_2F_1[b(\gamma_j+(m+1)l_2),\, b(m+1)(l_1-l_2+j\text{-}i)\,;\, b\{(m+1)(l_1+j\text{-}i)+\gamma_j\}\,;\, 1-y^a],$$

$$m \neq -1 \qquad (18)$$

*and*

$$h_{i,j;n,m,k}(z) = \frac{C_j\, b^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2} \Gamma\left(b(m+1)(l_1-l_2+j\text{-}i)\right)$$

$$.\Gamma\left(b\left((\gamma_j+(m+1)l_2\right)\right) H^{1,1}_{2,2}\left[\frac{1}{z}\left|\begin{matrix}\left(\frac{1}{a},\frac{1}{a}\right),\left(1+b(\gamma_j+(m+1)l_2)+\frac{1}{a},\frac{1}{a}\right)\\ \left(1+\frac{1}{a},\frac{1}{a}\right),\left(-b(m+1)(l_1-l_2+j\text{-}i)+\frac{1}{a},\frac{1}{a}\right)\end{matrix}\right.\right],\ m \neq -1$$

$$(19)$$

*where H[z] is the Fox H- function defined by (14) and* $j\text{-}i+l_1-l_2>0$ *,* $a>0, b>0$*,* $1 \le i < j \le n,\ k \ge 1$*, m and k are real numbers and the symbols* $\gamma_j$ *and* $C_j$ *are defined by (3).*

**Proof.** To find the p.d.f. of the product Y, we take double Mellin transform of eq.(15) and evaluate the integrals with the help of known result [1, p.311, eq.(31)]. Now using (11) we obtain Mellin transform of g(y) as

$$M_{s,s}\left[g_{i,j,n,m,k}(y)\right] = \frac{C_j\, a\, b^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2}$$

$$\cdot \quad \beta\left(b(m+1)(l_1-l_2+j\text{-}i), \frac{s+a-1}{a}\right) \beta\left(b(\gamma_j+(m+1)l_2), \frac{s+a-1}{a}\right) \qquad (20)$$

where β[ $a$ ,b] is usual beta function. Taking inverse Mellin transform of above equation w.r.t. ‘s’and using a known result [22, p.115] we arrive at (23).

To obtain the p.d.f. of the ratio i.e. $h_{i,\,j;\,n,\,m,\,k}(z)$, we use (12) with (9) and (15) to get Mellin transform of h(z) as

$$M_{s,-s+2}[h_{i,\,j;\,n,\,m,k}(z)] = \frac{C_j\, b^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2}$$

$$.\beta\left( b(m+1)(l_1-l_2+j\text{-}i), \frac{s+a-1}{a} \right) \beta\left( b(\gamma_j+(m+1)l_2), \frac{-s+a+1}{a} \right) \quad (21)$$

Taking Mellin inversion of the above result and interpretating with the help of definition of H-function given by (13), we get the desired result (19).

# 4. Special Cases

**Corollary 1.** *If we take a = b = 1 in Theorems 1 and 2, we get the joint p.d.f. and p.d.f. of product and ratio of $i^{th}$ and $j^{th}$ generalized order statistics from uniform distribution. The results are given by*

$$f_{i,\,j;\,n,m,k}(x_i, x_j) = \begin{cases} \dfrac{C_j}{(i-1)!(j-i-1)!(m+1)^{j-2}} \displaystyle\sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2} \\ \quad .(1-x_i)^{(m+1)(l_1-l_2+j-i)-1} (1-x_j)^{\gamma_j+(m+1)l_2-1}, \quad m \neq -1 \\ \dfrac{k^j (1-x_j)^{k-1}}{(i-1)!(j-i-1)!(1-x_i)} \left( \log\dfrac{1}{1-x_i} \right)^{(i-1)} \left( \log\dfrac{1-x_i}{1-x_j} \right)^{j-i-1}, m=-1 \end{cases}$$

(22)

$$g_{i,\,j,n,m,k}(y) = \frac{C_j\,(1-y)^{(m+1)(j-i+\gamma_j+1)-1}}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{l_1=0}^{i-1} \sum_{l_2=0}^{j-i-1} (-1)^{l_1+l_2} \binom{i-1}{l_1} \binom{j-i-1}{l_2}$$

$$.\left(1-\mathrm{y}\right)^{(m+1)l_1}\frac{\Gamma\left((\mathrm{m}+1)(l_1-l_2+\mathrm{j}\text{-}\mathrm{i})\right)\Gamma\left((\gamma_{\mathrm{j}}+(\mathrm{m}+1)l_2\right)}{\Gamma\left((\mathrm{m}+1)(l_1+\mathrm{j}\text{-}\mathrm{i})+\gamma_{\mathrm{j}}\right)}$$

$$\cdot\ {}_2F_1[(\gamma_{\mathrm{j}}+(\mathrm{m}+1)l_2),(\mathrm{m}+1)(l_1-l_2+\mathrm{j}\text{-}\mathrm{i});\ \{(\mathrm{m}+1)(l_1+\mathrm{j}\text{-}\mathrm{i})+\gamma_{\mathrm{j}}\};1-\mathrm{y}] \tag{23}$$

*and*

$$\mathrm{h}_{\mathrm{i,j;\,n,m,k}}(\mathrm{z})=\frac{\mathrm{C}_{\mathrm{j}}}{(\mathrm{i}-1)!(\mathrm{j}\text{-}\mathrm{i}\text{-}1)!(\mathrm{m}+1)^{\mathrm{j}-2}}\sum_{l_1=0}^{\mathrm{i}\text{-}1}\ \sum_{l_2=0}^{\mathrm{j}\text{-}\mathrm{i}\text{-}1}(-1)^{l_1+l_2}\binom{\mathrm{i}\text{-}1}{l_1}\binom{\mathrm{j}\text{-}\mathrm{i}\text{-}1}{l_2}\Gamma\left((\gamma_{\mathrm{j}}+(\mathrm{m}+1)l_2\right)$$

$$.\Gamma\left((\mathrm{m}+1)(l_1-l_2+\mathrm{j}\text{-}\mathrm{i})\right)G^{1,\,1}_{2\,,2}\left[\frac{1}{\mathrm{z}}\left|\begin{matrix}1,\ 2+\gamma_{\mathrm{j}}+(\mathrm{m}+1)l_2\\ 2,\ 1-(\mathrm{m}+1)(l_1-l_2+\mathrm{j}\text{-}\mathrm{i})\end{matrix}\right.\right] \tag{24}$$

where $G^{1,1}_{2,2}[z]$ is Meijer G function [3].

**Corollary 2.** *If we take j = i + 1 in Theorems 1 and 2, we get the joint distribution and distribution of product and ratio of consecutive generalized order statistics based on a random sample of size n from the Kumaraswamy distribution and are given by*

$$f_{\mathrm{i,i+1;\,n,m,k}}(\mathrm{x_i},\mathrm{x_{i+1}})=\begin{cases}\dfrac{\mathrm{C_{i+1}}\,\mathrm{a}^2\,\mathrm{b}^2\,\mathrm{x}_{\mathrm{i}}^{a\text{-}1}\,\mathrm{x}_{\mathrm{i+1}}^{a\text{-}1}}{(\mathrm{i}-1)!(\mathrm{m}+1)^{\mathrm{i}-1}}\displaystyle\sum_{l_1=0}^{\mathrm{i}\text{-}1}(-1)^{l_1}\binom{\mathrm{i}\text{-}1}{l_1}\left(1-\mathrm{x}_{\mathrm{i}}^{a}\right)^{\mathrm{b(m+1)}(l_1+1)-1}\\ .\left(1-\mathrm{x}_{\mathrm{i+1}}^{a}\right)^{\mathrm{b((k+(n\text{-}i\text{-}1)(m+1))-1}}, & m\neq-1\\ \dfrac{\mathrm{k^{i+1}\,a^2\,b^{i+1}}\,\mathrm{x}_{\mathrm{i}}^{a\text{-}1}\,\mathrm{x}_{\mathrm{i+1}}^{a\text{-}1}(1-\mathrm{x}_{\mathrm{i+1}}^{a\text{-}1})^{\mathrm{b(k+1)\text{-}2}}}{(\mathrm{i}-1)!(1-\mathrm{x}_{\mathrm{i}}^{a\text{-}1})}\left(\log\dfrac{1}{1-\mathrm{x}_{\mathrm{i}}^{a}}\right)^{\mathrm{i}-1}, & \mathrm{m}=-1\end{cases} \tag{25}$$

$$g_{i,i+1,n,m,k}(y) = \frac{C_{i+1}\, a\, b^2\, y^{a-1}\left(1-y^a\right)^{b(m+1)(\gamma_{i+1}+2)-1}}{(i-1)!(m+1)^{i-1}} \sum_{l_1=0}^{i-1} (-1)^{l_1} \binom{i-1}{l_1} \left(1-y^a\right)^{b(m+1)l_1} \frac{\Gamma\left(b(m+1)(l_1+1)\right)\Gamma(b\gamma_{i+1})}{\Gamma\left(b\{(m+1)(l_1+1)+\gamma_{i+1}\}\right)}$$
$$\cdot {}_2F_1[b(m+1)(l_1+1), b(\gamma_{i+1}+1)\,;\, b\{(m+1)(l_1+1)+\gamma_{i+1}\}\,;\, 1-y^a], \tag{26}$$

*and*

$$h_{i,i+1;n,m,k}(z) = \frac{C_{i+1}\, b^2}{(i-1)!(m+1)^{i-1}} \sum_{l_1=0}^{i-1} (-1)^{l_1} \binom{i-1}{l_1} \Gamma\left(b(m+1)(l_1+1)\right)\, \Gamma\left(b\gamma_{i+1}\right)$$
$$.H^{1,1}_{2,2}\left[\frac{1}{z}\left|\begin{matrix}\left(\frac{1}{a},\frac{1}{a}\right),\left(1+b\gamma_{i+1}+\frac{1}{a},\frac{1}{a}\right)\\ \left(1+\frac{1}{a},\frac{1}{a}\right),\left(-b(m+1)(l_1+1)+\frac{1}{a},\frac{1}{a}\right)\end{matrix}\right.\right] \tag{27}$$

**Corollary 3.** *If we take i = 1, j = n in Theorems 1 and 2, we get the joint distribution and distribution of product and ratio of the extreme generalized order statistics based on a random sample of size n, from Kumaraswamy distribution, and are given by*

$$f_{1,n,m,m,k}(x_1,x_n) = \begin{cases} \dfrac{C_n\, a^2\, b^2\, x_1^{a-1}\, x_n^{a-1}}{(n-2)!(m+1)^{n-2}} \displaystyle\sum_{l_2=0}^{n-2} (-1)^{l_2} \binom{n-2}{l_2} \left(1-x_1^a\right)^{b(m+1)(n-l_2-1)-1} \\ .\left(1-x_n^a\right)^{b(k+(m+1)l_2)-1}, \qquad m\neq -1 \\ \dfrac{k^n\, a^2\, b^n\, x_1^{a-1}\, x_n^{a-1}(1-x_1^a)^{b(k+1)-2}}{(n-2)!(1-x_1^a)} \left[\log\left(\dfrac{1-x_1^a}{1-x_n^a}\right)\right]^{n-2}, m = -1 \end{cases} \tag{28}$$

$$g_{1,n,n,m,k}(y) = \frac{C_n\, a\, b^2\, y^{a-1}\left(1-y^a\right)^{b(m+1)(n+k)-1}}{(n-2)!\,(m+1)^{n-2}} \sum_{l_2=0}^{n-2} (-1)^{l_2} \binom{n-2}{l_2} . \frac{\Gamma\left(b(m+1)(n-l_2-1)\right)\Gamma\left(b(k+l_2)\right)}{\Gamma\left(b\{(m+1)(n-1)+k\}\right)}$$
$$\cdot {}_2F_1[b(m+1)(n-l_2-1),\, b(k+l_2)\,;\, b\{(m+1)(n-1)+k\};1-y^a] \tag{29}$$

*and*

$$h_{1,n;n,m,k}(z) = \frac{C_n b^2}{(n-2)!\,(m+1)^{n-2}} \sum_{l_2=0}^{n-2} (-1)^{l_2} \binom{n-2}{l_2} \Gamma\left(b(m+1)(n-1-l_2)\right)$$

$$.\Gamma\left(b\left(k+(m+1)l_2\right)\right) H_{2,2}^{1,1}\left[\frac{1}{z}\left|\begin{matrix}\left(\frac{1}{a},\frac{1}{a}\right),\left(1+b\left(k+(m+1)l_2\right)+\frac{1}{a},\frac{1}{a}\right)\\ \left(1+\frac{1}{a},\frac{1}{a}\right),\left(-b(m+1)(n-1-l_2)+\frac{1}{a},\frac{1}{a}\right)\end{matrix}\right.\right] \tag{30}$$

**Corollary 4.** *If we take n to be odd say 2p+1 then putting i = p+1 and j = 2p+1 in Theorem 2. we get the p.d.f. of the product and ratio of peak to median of a random sample of size 2p+1 of generalized order statistics as*

$$g_{p+1,2p+1;2p+1,m,k}(z) = \frac{C_{2p+1}\, a\, b^2\, y^{a-1}\left(1-y^a\right)^{b(m+1)(p+\gamma_{2p+1}+1)-1}}{(p)!(p-1)!(m+1)^{2p-1}} \sum_{l_1=0}^{p}\sum_{l_2=0}^{p-1} (-1)^{l_1+l_2}\binom{p}{l_1}\binom{p-1}{l_2}$$

$$.\left(1-y^a\right)^{b(m+1)l_1} \frac{\Gamma[b(m+1)(l_1-l_2+p)]\Gamma[b(\gamma_{2p+1}+(m+1)l_2]}{\Gamma[b\{(m+1)(l_1+p)+\gamma_{2p+1}\}]}$$

$$._2F_1[b(m+1)(l_1-l_2+p), b(\gamma_{2p+1}+(m+1)l_2); b\{(m+1)(l_1+p)+\gamma_{2p+1}\}; 1-y^a],$$

*and* (31)

$$h_{p+1,2p+1;2p+1,m,k}(z) = \frac{C_{2p+1} b^2}{(p)!(p-1)!(m+1)^{2p-1}} \sum_{l_1=0}^{p}\sum_{l_2=0}^{p-1} (-1)^{l_1+l_2}\binom{p}{l_1}\binom{p-1}{l_2}$$

$$\Gamma\left(b(m+1)(l_1-l_2+p)\right)\Gamma\left(b(\gamma_{2p+1}+(m+1)l_2\right)$$

$$H_{2,2}^{1,1}\left[\frac{1}{z}\left|\begin{matrix}\left(\frac{1}{a},\frac{1}{a}\right),\left(1+b(\gamma_{2p+1}+(m+1)l_2+\frac{1}{a},\frac{1}{a}\right)\\ \left(1+\frac{1}{a},\frac{1}{a}\right),\left(-b(m+1)(l_1-l_2+p)+\frac{1}{a},\frac{1}{a}\right)\end{matrix}\right.\right] \tag{32}$$

**Remark.** If we take m = 0 and k = 1 in Theorems 1 and 2, then generalized order statistics reduces into order statistics and we get the joint distribution and distribution of product and ratio of order statistics $X_{i,n}$ and $X_{n,n}$ from a sample of size n from Kumaraswamy distribution as obtained recently by the author [21].

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