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Improvements and Generalizations of Some Euler Grüss Type Inequalities and Applications*

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Abstract

A sharp bound for some Euler-Grüss type inequalities are established and some applications are given.

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1. Introduction

Let $f:[a,b]\to R$ be such that $f^{(n)}$ $(n\geq 1)$ is continuous on $[a,b]$ and $m_n\leq f^{(n)}(t)\leq M_n$, $t\in[a,b]$, for some real numbers m_n and M_n. In [1], M. Matić et al. proved the following inequality:

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\frac{(b-a)^2}{4\sqrt{3}}(M_1-m_1),\tag{1.1}$$

$$\left|\int_a^b f(t)dt-(b-a)f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{4\sqrt{3}}(M_1-m_1),\tag{1.2}$$

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(a)+f(b)]+\frac{(b-a)^2}{12}[f'(b)-f'(a)]\right|\leq\frac{(b-a)^2}{6\sqrt{5}}(M_2-m_2),\tag{1.3}$$

and

$$\left|\int_a^b f(t)dt-(b-a)f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\right|\leq\frac{(b-a)^2}{24\sqrt{5}}(M_2-m_2).\tag{1.4}$$

Further, in [2], C. E. M. Pearce et al. proved the following inequality, for $n=1,2,3$:

$$\left|\int_a^b f(t)dt-\frac{(b-a)}{6}\left[f(b)+4f\left(\frac{a+b}{2}\right)+f(a)\right]\right|\leq C_n(b-a)^{n+1}(M_n-m_n),\tag{1.5}$$

where

$$C_1=\frac{1}{12},\quad C_2=\frac{1}{24\sqrt{30}},\quad C_3=\frac{1}{96\sqrt{105}}.$$

Recently, in [3], Lj. Dedić et al. established some inequalities of Euler-Grüss type to generalize all the above inequalities and improve the inequality (1.3) with the factor $\frac{1}{6\sqrt{5}}$ replaced by $\frac{1}{24\sqrt{5}}$. Further, in [4], Xiao-Liang Cheng improve the inequalities (1.1), (1.2) and (1.4) with the factors $\frac{1}{4\sqrt{3}}$, $\frac{1}{4\sqrt{3}}$ and $\frac{1}{24\sqrt{5}}$ replaced by $\frac{1}{8}$, $\frac{1}{8}$ and $\frac{1}{72\sqrt{3}}$, respectively. The other inequalities of Euler type see [5, 6, 7].

In this paper, using some Euler formulas, we shall establish some new generalization of all the above inequalities and improve inequalities (1.3) and (1.5). In section 4 and 5, we apply the obtained results to estimate the error bounds for composite quadrature rule and to apply for expectation.

2. Some Identities

Let $B_k(t),\ k \geq 0$ be the Bernoulli polynomials, and $B_k = B_k(0),\ k \geq 0$, the Bernoulli numbers. The first few Bernoulli polynomials are

$$B_0(t)=1,\ \ B_1(t)=t-\frac{1}{2},\ \ B_2(t)=t^2-t+\frac{1}{6},\ \ B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t,$$

and the first few Bernoulli numbers are

$$B_0=1,\ \ B_1=-\frac{1}{2},\ \ B_2=\frac{1}{6},\ \ B_3=0,\ \ B_4=-\frac{1}{30},\ \ B_5=0,\ \ B_6=\frac{1}{42}$$

For some details on the Bernoulli polynomials and the Bernoulli numbers, see for example [8, 9].

Further, let the function $B_k^*(t),\ k \geq 0$, be periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t)=B_k(t),\ \ 0 \leq t < 1,$$

$$B_k^*(t+1)=B_k^*(t),\ \ t \in R,$$

so that $B_0^*=1$, B_1^* is a discontinuous function with a jump of –1 at each integer, and $B_k^*,\ k \geq 2$, is a continuous function.

As stated in [3], the following Euler type identities hold.

Let $a,\ b \in R,\ a < b,\ x \in [a,b]$, and let $f:[a,b] \to R$ be such that $f^{(n)}$ is continuous on $[a,b]$ for some $n \geq 1$. Then the following formula for expansion in Bernoulli polynomials is valid.

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\left(\frac{x-a}{b-a}-\frac{1}{2}\right)[f(b)-f(a)]+T_n(x)+R_n(x) \tag{2.1}$$

where $T_1(x) = 0$,

$$T_n(x) = \sum_{k=2}^{n} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \tag{2.2}$$

for $n \geq 2$ and

$$R_n(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b B_n^*\left(\frac{x-t}{b-a}\right) f^{(n)}(t)\,dt$$

for $n \geq 1$.

Let $x = b$ in (2.1). Then the following trapezoid type identity holds.

Trapezoid type identity:

$$\int_a^b f(t)\,dt = \frac{b-a}{2}\left[f(a) + f(b)\right] + S_n^T(a,b) + \rho_n^T(a,b), \tag{2.3}$$

where $S_1^T(a,b) = 0$,

$$S_n^T(a,b) = -(b-a)T_n(b) = -\sum_{j=1}^{[n/2]} \frac{(b-a)^{2j}}{(2j)!} B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geq 2$, and

$$\rho_n^T(a,b) = -(b-a)R_n(b) = \frac{(b-a)^n}{n!} \int_a^b B_n^*\left(1 - \frac{t-a}{b-a}\right) f^{(n)}(t)\,dt\,,$$

for $n \geq 1$.

Let $x = \frac{a+b}{2}$ in (2.1). Then the following midpoint type identity holds.

Midpoint type identity:

$$\int_a^b f(t)\,dt = (b-a) f\left(\frac{a+b}{2}\right) + S_n^M(a,b) + \rho_n^M(a,b), \tag{2.4}$$

where $S_1^M(a,b) = 0$,

$$S_n^M(a,b) = -(b-a)T_n\left(\frac{a+b}{2}\right) = \sum_{j=1}^{[n/2]} \frac{(b-a)^{2j}}{(2j)!}\left(1-2^{1-2j}\right)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geq 2$, and

$$\rho_n^M(a,b) = -(b-a)R_n\left(\frac{a+b}{2}\right) = \frac{(b-a)^n}{n!}\int_a^b B_n^*\left(\frac{1}{2} - \frac{t-a}{b-a}\right) f^{(n)}(t)dt\,,$$

for $n \geq 1$.

Further, by (2.3) and (2.4) doing as in [3], the following Simpson type, Two-point type and Three-point type identities hold.

Simpson type identity:

$$\int_a^b f(t)dt = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + S_n^S(a,b) + \rho_n^S(a,b), \tag{2.5}$$

where $S_1^S(a,b) = 0$,

$$S_n^S(a,b) = \frac{1}{3}\sum_{j=1}^{[n/2]} \frac{(b-a)^{2j}}{(2j)!}\left(1-2^{2-2j}\right)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geq 2$, and

$$\rho_n^S(a,b) = \frac{(b-a)^n}{3\left(n!\right)}\int_a^b \left[B_n^*\left(1 - \frac{t-a}{b-a}\right) + 2B_n^*\left(\frac{1}{2} - \frac{t-a}{b-a}\right)\right] f^{(n)}(t)dt\,,$$

for $n \geq 1$. Note that

$$S_1^S(a,b) = S_2^S(a,b) = S_3^S(a,b) = 0\,.$$

Tow-point type identity:

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + S_n^{2P}(a,b) + \rho_n^{2P}(a,b) \tag{2.6}$$

where $S_1^{2P}(a,b) = 0$,

$$S_n^{2P}(a,b) = \frac{1}{2}\sum_{j=1}^{[n/2]} \frac{(b-a)^{2j}}{(2j)!}\left(1-3^{-2j}\right)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geq 2$, and

$$\rho_n^{2P}(a,b) = \frac{(b-a)^n}{2\left(n!\right)}\int_a^b \left[B_n^*\left(\frac{1}{3} - \frac{t-a}{b-a}\right) + B_n^*\left(\frac{2}{3} - \frac{t-a}{b-a}\right)\right] f^{(n)}(t)dt\,,$$

for $n \geq 1$.

Three-point type identity:

$$\int_a^b f(t)dt = \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + S_n^{3P}(a,b) + \rho_n^{3P}(a,b), \quad (2.7)$$

where $S_1^{3P}(a,b) = 0$,

$$S_n^{3P}(a,b) = -\frac{1}{3}\sum_{j=1}^{[n/2]}\frac{(b-a)^{2j}}{(2j)!}\left(1-2^{2-2j}\right)\left(1-2^{1-2j}\right)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

for $n \geq 2$, and

$$\rho_n^{3P}(a,b) = \frac{(b-a)^n}{3\left(n!\right)}\int_a^b\left[2B_n^*\left(\frac{1}{4} - \frac{t-a}{b-a}\right) - B_n^*\left(\frac{1}{2} - \frac{t-a}{b-a}\right) + 2B_n^*\left(\frac{3}{4} - \frac{t-a}{b-a}\right)\right]f^{(n)}(t)dt,$$

for $n \geq 1$. Note that

$$S_1^{3P}(a,b) = S_2^{3P}(a,b) = S_3^{3P}(a,b) = 0.$$

3. Integral Inequalities

Throughout the rest of the paper, let $f:[a,b] \to R$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$, and we assume that

$$m_n \leq f^{(n)}(x) \leq M_n, \quad a \leq x \leq b,$$

for some real constants m_n and M_n. Let u^+ and u^- be the positive and negative parts of the mapping u, respectively.

The following Lemma has been obtained in [3].

Lemma 1. *Let* $k \geq 1$ *and* $\gamma \in R$. *Then*

$$\int_0^1 B_k^*(\gamma - t)dt = 0.$$

The following Lemma (see [10]) plays important role in our main results.

Lemma 2. *Let* $F, G:[a,b] \to R$ *be two integrable functions such that*

$$\gamma \leq G(x) \leq \Gamma, \text{ for all } x \in [a,b],$$

where $\gamma, \Gamma \in R$ *are constants and* $\int_a^b F(x)dx = 0$. *Then*

$$\left|\int_a^b F(x)G(x)dx\right| \le (\Gamma - \gamma)\int_a^b F^+(x)dx.$$

Proof. Since $\int_a^b F(x)dx = 0$, we have

$$\int_a^b F^-(x)dx = -\int_a^b F^+(x)dx.$$

Now, $\int_a^b F(x)G(x)dx = \int_a^b F^+(x)G(x)dx + \int_a^b F^-(x)G(x)dx$

$$\le \Gamma\int_a^b F^+(x)dx + \gamma\int_a^b F^-(x)dx$$

$$= (\Gamma - \gamma)\int_a^b F^+(x)dx$$

and

$$\int_a^b F(x)G(x)dx \ge \gamma\int_a^b F^+(x)dx + \Gamma\int_a^b F^-(x)dx$$

$$= -(\Gamma - \gamma)\int_a^b F^+(x)dx,$$

which imply the result of Lemma 2.

We are ready to prove the following:

Theorem 1. *For* $n \ge 1$ *and for every* $x \in [a,b]$, *we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(\frac{x-a}{b-a} - \frac{1}{2}\right)[f(b) - f(a)] - T_n(x)\right|$$

$$\le \frac{(b-a)^n}{n!}(M_n - m_n)\int_0^1 B_n^+(s)ds \tag{3.1}$$

where $T_n(\cdot)$ *and* $B_n(\cdot)$ *are as in section 2.*

Proof. By Lemma 1, we have

$$\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)dt = \int_a^b B_n^*\left(\frac{x-a}{b-a}-\frac{t-a}{b-a}\right)dt = (b-a)\int_0^1 B_n^*\left(\frac{x-a}{b-a}-s\right)ds = 0$$

Now, using Lemma 2, we have

$$\left|\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)f^{(n)}(t)dt\right| \le (M_n-m_n)\int_a^b B_n^{*+}\left(\frac{x-t}{b-a}\right)dt = (M_n-m_n)(b-a)\int_0^1 B_n^+(s)ds\,.$$

If we multiply this by $\left|-\frac{(b-a)^{n-1}}{n!}\right|$ and use the representation (2.1), we obtain the desired inequality (3.1)

Corollary 1. *For every* $x\in[a,b]$, *we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(\frac{x-a}{b-a}-\frac{1}{2}\right)[f(b)-f(a)]\right| \le \frac{(b-a)}{8}(M_1-m_1) \tag{3.2}$$

Proof. For $n=1$, by (2.1), we have $T_1(x)=0$ and

$$R_1(x)=f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(\frac{x-a}{b-a}-\frac{1}{2}\right)[f(b)-f(a)].$$

Also

$$\int_0^1 B_1^+(s)ds = \int_{1/2}^1\left(s-\frac{1}{2}\right)ds = \frac{1}{8}\,.$$

Thus (3.2) follows from (3.1).

Remark 1. It has been shown that $\frac{1}{8}$ in (3.2) is sharp (see [4]).

Corollary 2. *For every* $x\in[a,b]$, *we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(\frac{x-a}{b-a}-\frac{1}{2}\right)[f(b)-f(a)]\right.$$

$$\left.-\left[\frac{(b-a)}{2}\left(\frac{x-a}{b-a}-\frac{1}{2}\right)^2-\frac{b-a}{24}\right][f'(b)-f'(a)]\right| \le \frac{(b-a)^2}{36\sqrt{3}}(M_2-m_2) \tag{3.3}$$

Proof. We have

$$\int_0^1 B_2^+(s)ds = \int_0^{\frac{3-\sqrt{3}}{6}}\left(s^2 - s + \frac{1}{6}\right)ds + \int_{\frac{3+\sqrt{3}}{6}}^1\left(s^2 - s + \frac{1}{6}\right)ds = \frac{1}{18\sqrt{3}}.$$

From (2.2), we have

$$T_2(x) = \left[\frac{(b-a)}{2}\left(\frac{x-a}{b-a} - \frac{1}{2}\right)^2 - \frac{b-a}{24}\right][f'(b) - f'(a)].$$

Hence (3.3) follows from (3.1) by taking $n = 2$.

Remark 2. We note that Corollary 2 is an improvement of Corollary 2 in [3]. If we choose $x = b$ in (3.3), we have

$$\left|\int_a^b f(t)dt - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)]\right| \le \frac{(b-a)^3}{36\sqrt{3}}(M_2 - m_2), \quad (3.4)$$

which is obtained in [3], with $24\sqrt{5}$ replaced by $36\sqrt{3}$. Similarly, choose $x = \frac{a+b}{2}$ in (3.3), we have

$$\left|\int_a^b f(t)dt - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}[f'(b) - f'(a)]\right| \le \frac{(b-a)^3}{36\sqrt{3}}(M_2 - m_2). \quad (3.5)$$

which is the inequality (1.4) with $24\sqrt{5}$ replaced by $36\sqrt{3}$.

Corollary 3. *Let $S_n^T(a,b)$ be defined as in section 2. For $n \ge 1$, we have*

$$\left|\int_a^b f(t)dt - \frac{(b-a)}{2}[f(b) + f(a)] - S_n^T(a,b)\right|$$

$$\le \frac{(b-a)^{n+1}}{n!}(M_n - m_n)\int_0^1 B_n^+(s)ds \quad (3.6)$$

Proof. This follows from (3.1) by taking $x = b$.

Remark 3. Choose $n = 1$ in (3.6), we have

$$\left|\int_a^b f(t)dt - \frac{b-a}{2}[f(a) + f(b)]\right| \le \frac{(b-a)^2}{8}(M_1 - m_1).$$

The constant $\frac{1}{8}$ is sharp (see [4]). Also, (3.6) reduces to (3.4) when $n = 2$.

Corollary 4. *Let* $S_n^M(a,b)$ *be defined as in section 2. For* $n \geq 1$, *we have*

$$\left|\int_a^b f(t)dt - (b-a) f\left(\frac{a+b}{2}\right) - S_n^M(a,b)\right|$$

$$\leq \frac{(b-a)^{n+1}}{n!}(M_n - m_n)\int_0^1 B_n^+(s)ds \tag{3.7}$$

Proof. This follows from (3.1) by taking $x = \frac{a+b}{2}$.

Remark 4. Choose $n = 1$ in (3.7), we have

$$\left|\int_a^b f(t)dt - (b-a) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8}(M_1 - m_1).$$

The constant $\frac{1}{8}$ is sharp (see [4]). Also, (3.7) reduces to (3.5) when $n = 2$.

Theorem 2. *Let* $S_n^S(a,b)$ *be defined as in section 2. For* $n \geq 1$, *we have*

$$\left|\int_a^b f(t)dt - \frac{(b-a)}{6}\left[f(b) + 4f\left(\frac{a+b}{2}\right) + f(a)\right] - S_n^S(a,b)\right|$$

$$\leq \frac{(b-a)^{n+1}}{3(n!)}(M_n - m_n)\int_0^1 \left[B_n^*(1-s) + 2B_n^*\left(\frac{1}{2} - s\right)\right]^+ ds \tag{3.8}$$

Proof. By Lemma 1, we have

$$\int_a^b \left[B_n^*\left(1 - \frac{t-a}{b-a}\right) + 2B_n^*\left(\frac{1}{2} - \frac{t-a}{b-a}\right)\right]dt$$

$$= \int_a^b B_n^*\left(1 - \frac{t-a}{b-a}\right)dt + 2\int_a^b B_n^*\left(\frac{1}{2} - \frac{t-a}{b-a}\right)dt$$

$$= (b-a)\int_0^1 B_n^*(1-s)ds + 2(b-a)\int_0^1 B_n^*\left(\frac{1}{2} - s\right)ds = 0.$$

Now, using Lemma 2, we have

$$\left|\int_a^b\left[B_n^*\left(1-\frac{t-a}{b-a}\right)+2B_n^*\left(\frac{1}{2}-\frac{t-a}{b-a}\right)\right]f^{(n)}(t)dt\right|$$

$$\leq (M_n-m_n)\int_a^b\left[B_n^*\left(1-\frac{t-a}{b-a}\right)+2B_n^*\left(\frac{1}{2}-\frac{t-a}{b-a}\right)\right]^+dt$$

$$=(b-a)(M_n-m_n)\int_0^1\left[B_n^*(1-s)+2B_n^*\left(\frac{1}{2}-s\right)\right]^+ds.$$

Multiplying this by $\frac{(b-a)^n}{3(n!)}$ and use the representation (2.5), we obtain the desired inequality (3.8).

Corollary 5. $\left|\int_a^b f(t)dt-\frac{(b-a)}{6}\left[f(b)+4f\left(\frac{a+b}{2}\right)+f(a)\right]\right|$

$$\leq\frac{5}{72}(b-a)^2(M_1-m_1). \tag{3.9}$$

Proof. Since

$$B_1^*(1-s)=B_1(1-s)=\frac{1}{2}-s \quad if \quad 0\leq s\leq 1$$

and

$$B_1^*\left(\frac{1}{2}-s\right)=\begin{cases}B_1\left(\frac{1}{2}-s\right)=-s, & if \quad 0\leq s\leq\frac{1}{2}\\ B_1\left(\frac{3}{2}-s\right)=1-s, & if \quad \frac{1}{2}<s\leq 1\end{cases}.$$

We have

$$B_1^*(1-s)+2B_1^*\left(\frac{1}{2}-s\right)=\begin{cases}\frac{1}{2}-3s, & if \quad 0\leq s\leq\frac{1}{2}\\ \frac{5}{2}-3s, & if \quad 0<s\leq 1\end{cases}.$$

Therefore

$$\int_0^1 \left[B_1^*(1-s) + 2B_1^*\left(\frac{1}{2}-s\right) \right]^+ ds = \int_0^{\frac{1}{6}} \left(\frac{1}{2}-3s\right) ds + \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{2}-s\right) ds = \frac{5}{24}.$$

Since $S_1^S(a,b) = 0$, we see that (3.9) follows from (3.8) for $n = 1$.

Corollary 6. $\left| \int_a^b f(t)dt - \frac{(b-a)}{6}\left[f(b) + 4f\left(\frac{a+b}{2}\right) + f(a) \right] \right|$

$$\leq \frac{1}{162}(b-a)^3(M_2 - m_2). \tag{3.10}$$

Proof. Since

$$B_2^*(1-s) = B_2(1-s) = s^2 - s + \frac{1}{6} \quad if \quad 0 \leq s \leq 1$$

and

$$B_2^*\left(\frac{1}{2}-s\right) = \begin{cases} B_2\left(\frac{1}{2}-s\right) = s^2 - \frac{1}{12}, & if \quad 0 \leq s \leq \frac{1}{2} \\ B_2\left(\frac{3}{2}-s\right) = s^2 - 2s + \frac{11}{12}, & if \quad \frac{1}{2} < s \leq 1, \end{cases}$$

we have

$$B_2^*(1-s) + 2B_2^*\left(\frac{1}{2}-s\right) = \begin{cases} 3s^2 - s, & if \quad 0 \leq s \leq \frac{1}{2} \\ 3s^2 - 5s + 2, & if \quad \frac{1}{2} < s \leq 1. \end{cases}$$

Therefore

$$\int_0^1 \left[B_2^*(1-s) + 2B_2^*\left(\frac{1}{2}-s\right) \right]^+ ds = \int_{\frac{1}{3}}^{\frac{1}{2}} (3s^2 - s) ds + \int_{\frac{1}{2}}^{\frac{2}{3}} (3s^2 - 5s + 2) ds = \frac{1}{27}.$$

Since $S_2^S(a,b) = 0$, we see that (3.10) follows from (3.8) for $n = 2$.

Corollary 7. $\left| \int_a^b f(t)dt - \frac{(b-a)}{6}\left[f(b) + 4f\left(\frac{a+b}{2}\right) + f(a) \right] \right|$

$$\leq \frac{1}{1152}(b-a)^4 (M_3 - m_3). \tag{3.11}$$

Proof. Since

$$B_3^*(1-s) = B_3(1-s) = -s^3 + \frac{3}{2}s^2 - \frac{1}{2}s \quad if \quad 0 \leq s \leq 1$$

and

$$B_3^*\left(\frac{1}{2}-s\right) = \begin{cases} B_3\left(\frac{1}{2}-s\right) = -s^3 + \frac{1}{4}s, & if \quad 0 \leq s \leq \frac{1}{2} \\ B_3\left(\frac{3}{2}-s\right) = -s^3 + 3s^2 - \frac{11}{4}s + \frac{3}{4}, & if \quad \frac{1}{2} < s \leq 1, \end{cases}$$

we have

$$B_3^*(1-s) + 2B_3^*\left(\frac{1}{2}-s\right) = \begin{cases} -3s^3 + \frac{3}{2}s^2, & if \quad 0 \leq s \leq \frac{1}{2} \\ -3s^3 + \frac{15}{2}s^2 - 6s + \frac{3}{2}, & if \quad 0 < s \leq 1. \end{cases}$$

Therefore

$$\int_0^1 \left[B_3^*(1-s) + 2B_3^*\left(\frac{1}{2}-s\right)\right]^+ ds = \int_0^{\frac{1}{2}} \left(-3s^3 + \frac{3}{2}s^2\right) ds = \frac{1}{64}.$$

Since $S_3^S(a,b) = 0$, we see that (3.11) follows from (3.8) for $n = 3$.

Remark 5. Corollary 5, Corollary 6 and Corollary 7 are improvements of (1.5) for $n = 1$, $n = 2$ and $n = 3$, respectively.

Theorem 3. *Let* $S_n^{2P}(a,b)$ *be defined as in section 2. For* $n \geq 1$*, we have*

$$\left| \int_a^b f(t)dt - \frac{(b-a)}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] - S_n^{2P}(a,b) \right|$$

$$\leq \frac{(b-a)^{n+1}}{2(n!)}(M_n - m_n)\int_0^1 \left[B_n^*\left(\frac{1}{3}-s\right) + B_n^*\left(\frac{2}{3}-s\right)\right]^+ ds \tag{3.12}$$

Proof. By Lemma 1, we have

$$\int_a^b \left[B_n^*\left(\frac{1}{3}-\frac{t-a}{b-a}\right)+B_n^*\left(\frac{2}{3}-\frac{t-a}{b-a}\right)\right]dt$$

$$=\int_a^b B_n^*\left(\frac{1}{3}-\frac{t-a}{b-a}\right)dt+\int_a^b B_n^*\left(\frac{2}{3}-\frac{t-a}{b-a}\right)dt$$

$$=(b-a)\int_0^1 B_n^*\left(\frac{1}{3}-s\right)ds+(b-a)\int_0^1 B_n^*\left(\frac{2}{3}-s\right)ds=0.$$

Now, using Lemma 2, we have

$$\left|\int_a^b \left[B_n^*\left(\frac{1}{3}-\frac{t-a}{b-a}\right)+B_n^*\left(\frac{2}{3}-\frac{t-a}{b-a}\right)\right] f^{(n)}(t)dt\right|$$

$$\leq (M_n-m_n)\int_a^b \left[B_n^*\left(\frac{1}{3}-\frac{t-a}{b-a}\right)+B_n^*\left(\frac{2}{3}-\frac{t-a}{b-a}\right)\right]^+ dt$$

$$=(M_n-m_n)(b-a)\int_0^1 \left[B_n^*\left(\frac{1}{3}-s\right)+B_n^*\left(\frac{2}{3}-s\right)\right]^+ ds.$$

Multiplying this by $\dfrac{(b-a)^n}{2(n!)}$ and use the representation (2.6), we obtain the desired inequality (3.12).

Corollary 8. $\left|\int_a^b f(t)dt-\dfrac{(b-a)}{2}\left[f\left(\dfrac{2a+b}{3}\right)+f\left(\dfrac{a+2b}{3}\right)\right]\right|$

$$\leq \frac{5}{72}(\mathrm{b-a})^2(M_1-m_1). \tag{3.13}$$

Proof. Since

$$B_1^*\left(\frac{1}{3}-s\right)=\begin{cases} B_1\left(\frac{1}{3}-s\right)=-s-\frac{1}{6}, & if \quad 0\leq s\leq \frac{1}{3} \\ B_1\left(\frac{4}{3}-s\right)=-s+\frac{5}{6}, & if \quad \frac{1}{3}<s\leq 1 \end{cases}$$

and

$$B_1^*\left(\frac{2}{3}-s\right)=\begin{cases} B_1\left(\frac{2}{3}-s\right)=-s+\frac{1}{6}, & if \quad 0\le s\le \frac{2}{3} \\ B_1\left(\frac{5}{3}-s\right)=-s+\frac{7}{6}, & if \quad \frac{2}{3}<s\le 1, \end{cases}$$

we have

$$B_1^*\left(\frac{1}{3}-s\right)+B_1^*\left(\frac{2}{3}-s\right)=\begin{cases} -2s, & if \quad 0\le s\le \frac{1}{3} \\ -2s+1, & if \quad \frac{1}{3}<s\le \frac{2}{3} \\ -2s+2 & if \quad \frac{2}{3}<s\le 1. \end{cases}$$

Therefore

$$\int_0^1\left[B_1^*\left(\frac{1}{3}-s\right)+B_1^*\left(\frac{2}{3}-s\right)\right]^+ds=\int_{\frac{1}{3}}^{\frac{1}{2}}(-2s+1)ds+\int_{\frac{2}{3}}^{1}(-2s+2)ds=\frac{5}{36}.$$

Since $S_1^{2p}(a,b)=0$, we see that (3.13) follows from (3.12) for $n=1$.

Corollary 9. $\left|\int_a^b f(t)dt-\frac{(b-a)}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-S_2^{2P}(a,b)\right|$

$$\le\frac{\sqrt{2}}{162}(\mathrm{b}-\mathrm{a})^3(M_2-m_2). \tag{3.14}$$

Proof. Since

$$B_2^*\left(\frac{1}{3}-s\right)=\begin{cases} B_2\left(\frac{1}{3}-s\right)=s^2+\frac{1}{3}s-\frac{1}{18}, & if \quad 0\le s\le \frac{1}{3} \\ B_2\left(\frac{4}{3}-s\right)=s^2-\frac{5}{3}s+\frac{11}{18}, & if \quad \frac{1}{3}<s\le 1 \end{cases}$$

and

$$B_2^*\left(\frac{2}{3}-s\right)=\begin{cases} B_2\left(\frac{2}{3}-s\right)=s^2-\frac{1}{3}s-\frac{1}{18}, & if \quad 0\le s\le \frac{2}{3} \\ B_2\left(\frac{5}{3}-s\right)=s^2-\frac{7}{3}s+\frac{23}{18}, & if \quad \frac{2}{3}<s\le 1, \end{cases}$$

we have

$$B_2^*\left(\frac{1}{3}-s\right)+B_2^*\left(\frac{2}{3}-s\right)=\begin{cases} 2s^2-\frac{1}{9}, & if \quad 0\le s\le\frac{1}{3} \\ 2s^2-2s+\frac{5}{9}, & if \quad \frac{1}{3}<s\le\frac{2}{3} \\ 2s^2-4s+\frac{17}{9} & if \quad \frac{2}{3}<s\le 1. \end{cases}$$

Therefore

$$\int_0^1\left[B_2^*\left(\frac{1}{3}-s\right)+2B_2^*\left(\frac{2}{3}-s\right)\right]^+ ds$$

$$=\int_{\frac{1}{3\sqrt{2}}}^{\frac{1}{3}}\left(2s^2-\frac{1}{9}\right)ds+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(2s^2-2s+\frac{5}{9}\right)ds+\int_{\frac{2}{3}}^{\frac{6-\sqrt{2}}{6}}\left(2s^2-4s+\frac{17}{9}\right)ds=\frac{2\sqrt{2}}{81}.$$

Thus (3.14) follows from (3.12) for $n=2$.

Remark 6. Corollary 8 and Corollary 9 are improvements of Theorem 5 in [3] for $n=1$ and $n=2$ with the factors $\frac{1}{12}$ and $\frac{\sqrt{70}}{360}$ replaced by $\frac{5}{72}$ and $\frac{\sqrt{2}}{162}$, respectively.

Theorem 4. *Let $S_n^{3P}(a,b)$ be defined as in section 2. For $n\ge 1$, we have*

$$\left|\int_a^b f(t)dt-\frac{(b-a)}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+f\left(\frac{a+3b}{4}\right)\right]-S_n^{3P}(a,b)\right|$$

$$\le\frac{(b-a)^{n+1}}{3(n!)}(M_n-m_n)\int_0^1\left[2B_n^*\left(\frac{1}{4}-s\right)-B_n^*\left(\frac{1}{2}-s\right)+2B_n^*\left(\frac{3}{4}-s\right)\right]^+ ds. \qquad (3.15)$$

Proof. By Lemma 1, we have

$$\int_a^b\left[2B_n^*\left(\frac{1}{4}-\frac{t-a}{b-a}\right)-B_n^*\left(\frac{1}{2}-\frac{t-a}{b-a}\right)+2B_n^*\left(\frac{3}{4}-\frac{t-a}{b-a}\right)\right]dt$$

$$=2\int_a^b B_n^*\left(\frac{1}{4}-\frac{t-a}{b-a}\right)dt-\int_a^b B_n^*\left(\frac{1}{2}-\frac{t-a}{b-a}\right)dt+2\int_a^b B_n^*\left(\frac{3}{4}-\frac{t-a}{b-a}\right)dt$$

$$= 2(b-a)\int_0^1 B_n^*\left(\frac{1}{4}-s\right)ds - (b-a)\int_0^1 B_n^*\left(\frac{1}{2}-s\right)ds + 2(b-a)\int_0^1 B_n^*\left(\frac{3}{4}-s\right)ds = 0 .$$

Now, using Lemma 2, we have

$$\left|\int_a^b \left[2B_n^*\left(\frac{1}{4}-\frac{t-a}{b-a}\right) - B_n^*\left(\frac{1}{2}-\frac{t-a}{b-a}\right) + 2B_n^*\left(\frac{3}{4}-\frac{t-a}{b-a}\right)\right] f^{(n)}(t)dt\right|$$

$$\leq (M_n - m_n)\int_a^b \left[2B_n^*\left(\frac{1}{4}-\frac{t-a}{b-a}\right) - B_n^*\left(\frac{1}{2}-\frac{t-a}{b-a}\right) + 2B_n^*\left(\frac{3}{4}-\frac{t-a}{b-a}\right)\right]^+ dt$$

$$= (M_n - m_n)(b-a)\int_0^1 \left[2B_n^*\left(\frac{1}{4}-s\right) - B_n^*\left(\frac{1}{2}-s\right) + 2B_n^*\left(\frac{3}{4}-s\right)\right]^+ ds .$$

Multiplying this by $\dfrac{(b-a)^n}{3\left(n!\right)}$ and use the representation (2.7), we obtain the desired inequality (3.13).

Corollary 10. $\left|\int_a^b f(t)dt - \frac{(b-a)}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + f\left(\frac{a+3b}{4}\right)\right]\right|$

$$\leq \frac{5}{48}(\mathrm{b-a})^2(M_1 - m_1). \tag{3.16}$$

Proof. Since

$$B_1^*\left(\frac{1}{4}-s\right) = \begin{cases} B_1\left(\frac{1}{4}-s\right) = -s-\frac{1}{4}, & if \quad 0 \leq s \leq \frac{1}{4} \\ B_1\left(\frac{5}{4}-s\right) = -s+\frac{3}{4}, & if \quad \frac{1}{4} < s \leq 1, \end{cases}$$

$$B_1^*\left(\frac{1}{2}-s\right) = \begin{cases} B_1\left(\frac{1}{2}-s\right) = -s, & if \quad 0 \leq s \leq \frac{1}{2} \\ B_1\left(\frac{3}{2}-s\right) = -s+1, & if \quad \frac{1}{2} < s \leq 1 \end{cases}$$

and

$$B_1^*\left(\frac{3}{4}-s\right)=\begin{cases} B_1\left(\frac{3}{4}-s\right)=-s+\frac{1}{4}, & if \quad 0\le s\le \frac{3}{4} \\ B_1\left(\frac{7}{4}-s\right)=-s+\frac{5}{4}, & if \quad \frac{3}{4}< s\le 1, \end{cases}$$

we have

$$2B_1^*\left(\frac{1}{4}-s\right)-B_1^*\left(\frac{1}{2}-s\right)+2B_1^*\left(\frac{3}{4}-s\right)=\begin{cases} -3s & if \quad 0\le s\le \frac{1}{4} \\ -3s+2 & if \quad \frac{1}{4}< s\le \frac{1}{2} \\ -3s+1 & if \quad \frac{1}{2}< s\le \frac{3}{4} \\ -3s+3 & if \quad \frac{3}{4}< s\le 1. \end{cases}$$

Therefore

$$\int_0^1\left[2B_1^*\left(\frac{1}{4}-s\right)-B_1^*\left(\frac{1}{2}-s\right)+2B_1^*\left(\frac{3}{4}-s\right)\right]^+ ds=\int_{\frac{1}{4}}^{\frac{1}{2}}(-3s+1)ds+\int_{\frac{3}{4}}^{1}(-3s+3)ds=\frac{5}{16}.$$

Since $S_1^{3P}(a,b)=0$, we see that (3.16) follows from (3.15) for $n=1$.

Corollary 11. $\left|\int_a^b f(t)dt-\frac{(b-a)}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+f\left(\frac{a+3b}{4}\right)\right]\right|$

$$\le \frac{5}{648}(\mathrm{b-a})^3(M_2-m_2). \tag{3.17}$$

Proof. Since

$$B_2^*\left(\frac{1}{4}-s\right)=\begin{cases} B_2\left(\frac{1}{4}-s\right)=s^2+\frac{1}{2}s-\frac{1}{48}, & if \quad 0\le s\le \frac{1}{4} \\ B_2\left(\frac{5}{4}-s\right)=s^2-\frac{3}{2}s+\frac{23}{48}, & if \quad \frac{1}{4}< s\le 1, \end{cases}$$

$$B_2^*\left(\frac{1}{2}-s\right)=\begin{cases} B_2\left(\frac{1}{2}-s\right)=s^2-\frac{1}{12}, & if \quad 0\le s\le\frac{1}{2} \\ B_2\left(\frac{3}{2}-s\right)=s^2-2s+\frac{11}{12}, & if \quad \frac{1}{2}<s\le 1 \end{cases}$$

and

$$B_1^*\left(\frac{3}{4}-s\right)=\begin{cases} B_1\left(\frac{3}{4}-s\right)=s^2-\frac{1}{2}s+\frac{1}{48}, & if \quad 0\le s\le\frac{3}{4} \\ B_1\left(\frac{7}{4}-s\right)=s^2-\frac{5}{2}s+\frac{71}{48}, & if \quad \frac{3}{4}<s\le 1, \end{cases}$$

we have

$$2B_2^*\left(\frac{1}{4}-s\right)-B_2^*\left(\frac{1}{2}-s\right)+2B_2^*\left(\frac{3}{4}-s\right)=\begin{cases} 3s^2, & if \quad 0\le s\le\frac{1}{4} \\ 3s^2-4s+1, & if \quad \frac{1}{4}<s\le\frac{1}{2} \\ 3s^2-2s, & if \quad \frac{1}{2}<s\le\frac{3}{4} \\ 3s^2-6s+3, & if \quad \frac{3}{4}<s\le 1. \end{cases}$$

Therefore

$$\int_0^1\left[2B_1^*\left(\frac{1}{4}-s\right)-B_1^*\left(\frac{1}{2}-s\right)+2B_1^*\left(\frac{3}{4}-s\right)\right]^+ ds$$

$$=\int_0^{\frac{1}{4}}3s^2ds+\int_{\frac{1}{4}}^{\frac{1}{3}}\left(3s^2-4s+1\right)ds+\int_{\frac{2}{3}}^{\frac{3}{4}}\left(3s^2-2s\right)ds+\int_{\frac{3}{4}}^{1}\left(3s^2-6s+3\right)ds$$

$$=\frac{5}{108}.$$

Since $S_2^{3P}(a,b)=0$, we see that (3.17) follows from (3.15) for $n=2$

Remark 7. Corollary 10 and Corollary 11 are improvements of Theorem 6 in [3] for $n=1$ and $n=2$ with the factors $\frac{\sqrt{2}}{12}$ and $\frac{\sqrt{130}}{480}$ replaced by $\frac{5}{48}$ and $\frac{5}{648}$, respectively.

4. Application for The Error Bound for Composite Quadrature Rule

Theorem 5. *Let* I_h *be a partition* $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ *of the interval* $[a,b]$. *Then*

$$\left|\int_a^b f(t) - A_T(f, f', I_h)\right| \le \frac{M_2 - m_2}{36\sqrt{3}} \sum_{i=0}^{m-1} h_i^3 \tag{4.1}$$

where $h_i = x_{i+1} - x_i$ *and* $A_T(f, f', I_h)$ *is perturbed trapezoid quadrature rule defined by*

$$A_T(f, f', I_h) := \frac{1}{2}\sum_{i=1}^{n-1}\left[f(x_i) + f(x_{i+1})\right]\cdot h_i - \frac{1}{12}\sum_{i=0}^{n-1} h_i^2 \left[f'(x_{i+1}) - f'(x_i)\right]$$

Proof. From (3.4), with $[x_i, x_{i+1}]$ in place of $[a,b]$, we get

$$\left|\int_{x_i}^{x_{i+1}} f(t)dt - \frac{x_{i+1} - x_{i+1}}{2}\left[f(x_{i+1}) + f(x_i)\right] + \frac{(x_{i+1} - x_i)^2}{12}\left[f'(x_{i+1}) - f'(x_i)\right]\right|$$

$$\le \frac{(x_{i+1} - x_i)^3}{36\sqrt{3}}(M_2 - m_2)$$

Summing this over $i = 0, 1, \cdots, n-1$, we get the desired result.

Remark 8. The inequality in (4.1) is an improvement of the inequality (4.5) in [1].

5. Applications for Expectation

Theorem 6. *Let* X *be a random variable having the p.d.f.,* $f : [a,b] \to R$ *and the cumulative distribution function* $F : [a,b] \to [0,1]$, *i.e.,*

$$F(x) = \int_a^x f(t)dt, \quad x \in [a,b].$$

If F is absolutely continuous on $[a,b]$ and $m_2 \le f''(x) \le M_2$ for $x \in [a,b]$, then we have the inequality:

$$\left|E(X)-\frac{a+b}{2}-\frac{(b-a)^2}{12}[f(b)-f(a)]\right|$$

$$\leq\frac{(b-a)^3}{36\sqrt{3}}(M_2-m_2) \tag{5.1}$$

Proof. Replaced f by F in (3.4), we have

$$\left|\int_a^b F(t)dt-\frac{F(a)+F(b)}{2}(b-a)+\frac{(b-a)^2}{12}[f(b)-f(a)]\right|$$

$$\leq\frac{(b-a)^3}{36\sqrt{3}}(M_2-m_2). \tag{5.2}$$

However, $F(a)=0$, $F(b)=1$ and

$$\int_a^b F(t)dt=b-E(X),$$

the desired inequality (5.1) follows from (5.2).

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