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On the Generalized Hyers-Ulam Stability of the Generalized Polynomial Function of Degree 3 *

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Abstract

In this paper the generalized Hyers-Ulam stability is proved for the following functional equation

$$\sum_{i=0}^{4} {}_4C_i(-1)^{4-i}f(ix+y)=0$$

following the spirit of the approach that was introduced in the paper of Th.M. Rassias, On, the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc, **72**(1978), 297-300. Also, I investigate the superstability of the functional equation.

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1. Introduction

Throughout this paper, let V be a vector space and Y a Banach space. Let n be a positive integer. For a given mapping $f : V \to Y$, define mappings $C_n f, D_n f : V \times V \to Y$ by

$$E_n f(x, y) := \sum_{i=0}^{n} {}_nC_i(-1)^{n-i} f(ix + y),$$

$$D_n f(x, y) := \sum_{i=0}^{n} {}_nC_i(-1)^{n-i} f(ix + y) - n! f(x)$$

for all $x, y \in X$, where ${}_nC_i = \frac{n!}{i!(n-i)!}$. A mapping $f : X \to Y$ is called a generalized polynomial(monomial, respectively) function of degree $n \in \mathbb{N}$ if f satisfies the functional equation $E_{n+1} f(x, y) = 0(D_n f(x, y) = 0$, respectively). The functional equation $E_{n+1} f(x, y) = 0(D_n f(x, y) = 0$, respectively) is called a generalized polynomial(monomial, respectively) functional equation of degree $n \in \mathbb{N}$. In particular, a mapping $f : V \to Y$ is called an additive (quadratic, cubic, quartic, respectively) mapping if f satisfies the functional equation $D_1 f = 0$ ($D_2 f = 0$, $D_3 f = 0$, $D_4 f = 0$, respectively). The functional equation $D_1 f = 0$ ($D_2 f = 0$, $D_3 f = 0$, $D_4 f = 0$, respectively) is called a Cauchy equation(quadratic functional equation, cubic functional equation, quartic functional equation, respectively). The functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^n$ and $f(x) = \sum_{i=0}^{n-1} a_i x^i$ satisfy the functional equation $D_n f = 0$ and $E_n f = 0$ respectively, where a, a_i are real constants and $\mathbb{R}$ is the set of real numbers.

If we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality must be close to the solutions of the given equation? If the answer is affirmative, we would say that a given functional equation is stable.

In 1941, D.H.Hyers [8] proved the stability of Cauchy equation $D_1 f = 0$ and in 1978, Th.M.Rassias[19] gave a significant generalization of the Hyers' result. Th.M.Rassias[20] during the 27th International Symposium on Functional Equations, that took place in Bielsko-Biala, Poland, in 1990, asked the question whether such a theorem can also be proved for a more general setting. Z.Gadja[6] following Th.M.Rassias's approach[19] gave an affirmative solution to the question. Recently, P.Găvruta[7] obtained a further generalization of Rassias' theorem, the so-called generalized Hyers-Ulam-Rassias stability(See also [4,5,9-11,16-18]).

A stability problem for the quadratic functional equation $D_2 f = 0$ was proved by F.Skof[21] for a function $f : X \to Y$, where X is a normed space. P.W. Cholewa[2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. S.Czerwik[3] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation.

J. C. Parnami, H. L. Vasudeva[14] and J.M. Rassias [15] investigated the stability of the functional equation $D_3 f = 0$. Also, Jun and Kim [12] proved the stability of the functional equation $E_4 f = 0$ under the approximately cubic condition and Baker[1] proved the stability of the functional equation $E_n f = 0$.

In this paper, I solve the general solution of $E_4 f = 0$ and prove the generalized Hyers-Ulam stability of the functional equation $E_4 f = 0$ on the punctured domain $V \setminus \{0\}$ without the approximately cubic condition.

2. General Solution of $E_4 f = 0$

In this section I establish the general solution of $E_4 f = 0$. First I obtain the general solution for the odd cases. Throughout this section, let V and W be vector spaces.

Theorem 2.1 *Suppose that the odd function $f : V \to W$ satisfies*

$$E_4 f(x,y) = f(4x+y) - 4f(3x+y) + 6f(2x+y) - 4f(x+y) + f(y) = 0$$

(2.1) for all $x, y \in V \setminus \{0\}$ and

$$f(2x) = 2f(x)$$

for all $x \in V$. Then f is an additive function.

Proof. Note that $f(0) = 0$ and $f(x) + f(-x) = \frac{-E_4 f(x,-2x)}{2} = 0$ for all $x \in V$. From (2.1) and $f(2x) = 2f(x)$, we have

$$\begin{aligned} D_1 f(x,y) \;=\; & \frac{1}{132}\big(-28E_4 f(x, y-2x) - 7E_4 f(y, 2x-2y) - 16E_4 f(x, y-3x) \\ & +4E_4 f(y-x, 4x-2y) - 28E_4 f(y, x-2y) - 7E_4 f(x, 2y-2x) \\ & -16E_4 f(y, x-3y) + 4E_4 f(x-y, 4y-2x)\big) = 0 \end{aligned}$$

for all $x, y \in V$ with $x \neq 0, y, 2y, 3y$ and $y \neq 0, x, 2x, 3x$. Since $f(2x) = 2f(x)$ and $f(3x) = E_4 f(x, -x) + 3f(x) = 3f(x)$, f is an additive function.

Theorem 2.2 *Suppose that the odd function $f : V \to W$ satisfies (2.1) for all $x, y \in V \setminus \{0\}$ and*

$$f(2x) = 8f(x)$$

for all $x \in V$. Then f is a cubic function.

Proof. Note that $f(0) = 0$ and $f(x) + f(-x) = \frac{-E_4 f(x,-2x)}{2} = 0$ for all $x \in V$. From (2.1) and $f(2x) = 8f(x)$, we easily get the equality

$$D_3 f(x, y) = E_4 f(x, y - x) - \frac{E_4 f(y + x, -2y)}{4} = 0$$

for all $x, y \in V \setminus \{0\}$ with $x \neq y, -y$. Since $f(2x) = 8f(x)$ and $f(3x) = E_4 f(x, -x) + 27f(x) = 27f(x)$, f is a cubic function.

Theorem 2.3 *Suppose that the odd function $f : V \to W$ satisfies (2.1) for all $x, y \in V \setminus \{0\}$. Then there exist a cubic function $C : V \to W$ and an additive function $A : V \to W$ such that*

$$f(x) = C(x) + A(x)$$

for all $x \in V$, where

$$\begin{aligned} C(x) = &\ \tfrac{-1}{3}[f(x) - \tfrac{1}{2}f(2x)] = \tfrac{4}{3}[f(x) - 2f(\tfrac{1}{2}x)] \\ A(x) = &\ \tfrac{4}{3}[f(x) - \tfrac{1}{8}f(2x)] = \tfrac{-1}{3}[f(x) - 8f(\tfrac{1}{2}x)]. \end{aligned}$$

Proof. Since

$$f(4x) - 10f(2x) + 16f(x) = \frac{1}{4}\big(11E_4 f(x, -x) + E_4 f(2x, -3x) - E_4(x, x)\big) = 0$$

for all $x \in V \setminus \{0\}$, we have

$$f(x) = C(x) + A(x), \quad C(2x) = 8C(x), \quad A(2x) = 2A(x)$$

for all $x \in V$, where

$$C(x) := \frac{-1}{3}[f(x) - \frac{1}{2}f(2x)] \ \text{ and } \ A(x) := \frac{4}{3}[f(x) - \frac{1}{8}f(2x)].$$

By Theorem 2.2 and the equalities

$$E_4C(x,y) = \frac{-1}{6}\big(2E_4f(x,y) - E_4f(2x,2y)\big) = 0,$$

$$E_4A(x,y) = \frac{1}{6}\big(8E_4f(x,y) - E_4f(2x,2y)\big) = 0$$

for all $x, y \in V \setminus \{0\}$, C is a cubic function and A is an additive function.

In the following theorem we obtain the general solution for the even case.

Theorem 2.4
Suppose that the even function $f : V \to W$ satisfies (2.1) for all $x, y \in V \setminus \{0\}$ and

$$f(2x) = 4f(x)$$

for all $x \in V$. Then f is a quadratic function.
Proof. Note that $f(0) = 0$. From (2.1) and $f(2x) = 4f(x)$, we get the equality

$$\begin{aligned} D_2f(x,y) &= -\frac{1}{12}\big(4E_4f(x,y-2x) + E_4f(y,2x-2y)\big) \\ &= f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$ with $y \neq x, 2x$. Using $4f(x) = f(2x)$ and $f(3x) + f(x) = E_4f(x,-x) + 10f(x) = 10f(x)$ for all $x \in V$, we get

$$D_2f(x,y) = 0$$

for all $x, y \in V$.

Theorem 2.5 *Suppose that the even function $f : V \to W$ satisfies (2.1) for all $x, y \in V \setminus \{0\}$. Then $f - f(0)$ is a quadratic function.*
Proof. Let $g = f - f(0)$. Then g satisfies (2.1) and $g(0) = 0$. Since $g(2x) = \frac{E_4g(x,-2x)}{2} + 4g(x) = 4g(x)$ for all $x \in V \setminus \{0\}$, by Theorem 2.4, g is a quadratic function.

Now I establish the general solution of $E_4f = 0$.

Theorem 2.6 *Suppose that the function $f : V \to W$ satisfies (2.1) for all*

$x, y \in V \setminus \{0\}$. Then there exist a cubic function $C : V \to W$, a quadratic function $Q : V \to W$, and an additive function $A : V \to W$ such that

$$f(x) = C(x) + Q(x) + A(x) + f(0)$$

for all $x \in V$. The functions $C, Q, A : V \to W$ are given by

$$\begin{aligned} C(x) : &= \frac{-1}{12}\big(2f(x) - 2f(-x) - f(2x) + f(-2x)\big) \\ Q(x) : &= \frac{f(x) + f(-x)}{2} - f(0) \\ A(x) : &= \frac{1}{12}\big(8f(x) - 8f(-x) - f(2x) + f(-2x)\big) \end{aligned}$$

for all $x \in V$.

Proof. Since $f(x) = \frac{f(x)-f(-x)}{2} + \frac{f(x)+f(-x)}{2}$, we can apply Theorem 2.3 and 2.4.

3. Stability of the Equation $E_4 f = 0$

The following lemma is seen in [13].

Lemma 3.1. *Let a be a positive real number and $\Phi : X \setminus \{0\} \to [0, \infty)$ a map. Suppose that the function $f : X \to Y$ satisfies the inequality*

$$\|f(x) - \frac{f(2x)}{a}\| \le \frac{\Phi(x)}{a} \quad \text{and} \quad f(0) = 0.$$

1. *If $\sum_{l=0}^{\infty} \frac{1}{a^{l+1}}\Phi(2^l x) < \infty$ for all $x \in X \setminus \{0\}$, then there exists a unique function $F : X \to Y$ satisfying*

$$\|f(x) - F(x)\| \le \sum_{l=0}^{\infty} \frac{1}{a^{l+1}}\Phi(2^l x)$$

   for all $x \in X \setminus \{0\}$ and F is given by $F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{a^n}$ for all $x \in X$.

2. *If $\sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty$ for all $x \in X \setminus \{0\}$, then there exists a unique function $F : X \to Y$ satisfying*

$$\|f(x) - F(x)\| \le \sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty$$

for all $x \in X \setminus \{0\}$ *and* F *is given by* $F(x) = \lim_{n\to\infty} a^n f(\frac{x}{2^n})$ *for all* $x \in X$*.*

Theorem 3.2 *Let* $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ *be a mapping satisfying the condition*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{2^i} < \infty. \tag{3.1}$$

If a function $f : V \to Y$ *satisfies*

$$\|E_4 f(x, y)\| \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in V\setminus\{0\}$*, then there exists a unique generalized polynomial function* $F : V \to Y$ *of degree 3 with* $f(0) = F(0)$ *such that*

$$\|f(x) - F(x)\| \leq \sum_{j=0}^{\infty} \Big(\frac{\psi(2^j x)}{3 \cdot 2^{j-2}} + \frac{\psi(2^j x)}{3 \cdot 8^j} + \frac{\varphi(2^j x, -2^{j+1} x)}{2^{2j+3}}\Big) \tag{3.3}$$

for all $x \in V \setminus \{0\}$*, where*

$$\psi(x) = \frac{1}{128}\big(11\varphi(x, -x) + \varphi(2x, -3x) + \varphi(x, x) + 11\varphi(-x, x) + \varphi(-2x, 3x) + \varphi(-x, -x)\big)$$

for all $x \in V \setminus \{0\}$*. In particular,* F *is represented by*

$$\begin{aligned} F(x) &= \lim_{n\to\infty} \Big(\frac{1}{3 \cdot 2^{n-1}} + \frac{1}{2 \cdot 4^n} - \frac{1}{6 \cdot 8^n}\Big) f(2^n x) \\ &\quad + \Big(\frac{-1}{3 \cdot 2^{n-1}} + \frac{1}{2 \cdot 4^n} + \frac{1}{6 \cdot 8^n}\Big) f(-2^n x) \\ &\quad - \frac{1}{12}\Big(\frac{1}{2^n} - \frac{1}{8^n}\Big) f(2^{n+1} x) + \frac{1}{12}\Big(\frac{1}{2^n} - \frac{1}{8^n}\Big) f(-2^{n+1} x)\Big) + f(0) \end{aligned}$$

for all $x \in V$*.*

Proof. Note that if $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ satisfies the condition (3.1) then φ satisfies the condition $\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{4^i} < \infty$ and $\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{8^i} < \infty$. From (3.2), we get the inequalities

$$\begin{aligned} &\Big\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x)}{4} - \frac{1}{8}\Big(\frac{f(2x) - f(-2x)}{2} - \frac{f(4x) - f(-4x)}{4}\Big)\Big\| \\ =\ & \frac{1}{128}\|11E_4 f(x, -x) + E_4 f(2x, -3x) - E_4 f(x, x) \\ & -11E_4 f(-x, x) - E_4 f(-2x, 3x) + E_4 f(-x, -x)\| \\ \leq\ & \psi(x), \end{aligned}$$

$$\begin{aligned} &\|\frac{f(x)-f(-x)}{2}-\frac{f(2x)-f(-2x)}{16}-\frac{1}{2}\big(\frac{f(2x)-f(-2x)}{2}-\frac{f(4x)-f(-4x)}{16}\big)\| \\ \leq\ & \psi(x), \end{aligned}$$

$$\begin{aligned} &\|\frac{f(x)+f(-x)}{2}-f(0)-\frac{1}{4}\big(\frac{f(2x)+f(-2x)}{2}-f(0)\big)\| \\ =\ & \|\frac{E_4f(x,-2x)}{8}\| \leq \frac{\varphi(x,-2x)}{8} \end{aligned}$$

for all $x \in V\setminus\{0\}$. By Lemma 3.1, there exist functions $C_0, A_0, Q : V \to Y$ defined by

$$C_0(x) := \lim_{n\to\infty} \frac{2f(2^n x)-2f(-2^n x)-f(2^{n+1}x)+f(-2^{n+1}x)}{2^{3n+2}},$$

$$A_0(x) := \lim_{n\to\infty} \frac{8f(2^n x)-8f(-2^n x)-f(2^{n+1}x)+f(-2^{n+1}x)}{2^{n+4}},$$

$$Q(x) = \frac{1}{2}\lim_{n\to\infty}\frac{f(2^n x)+f(-2^n x)}{4^n}$$

for all $x \in V$ and the functions C_0, A_0, Q satisfy the inequalities

$$\|\frac{f(x)-f(-x)}{2}-\frac{f(2x)-f(-2x)}{4}-C_0(x)\| \leq \sum_{j=0}^{\infty}\frac{\psi(2^j x)}{8^j}, \tag{3.4}$$

$$\|\frac{f(x)-f(-x)}{2}-\frac{f(2x)-f(-2x)}{16}-A_0(x)\| \leq \sum_{j=0}^{\infty}\frac{\psi(2^j x)}{2^j}, \tag{3.5}$$

$$\|\frac{f(x)+f(-x)}{2}-f(0)-Q(x)\| \leq \sum_{j=0}^{\infty}\frac{\varphi(2^j x, -2^{j+1}x)}{2^{2j+3}} \tag{3.6}$$

for all $x \in V\setminus\{0\}$. From (3.1) and (3.2), we obtain

$$\begin{aligned} E_4C_0(x,y) &= \lim_{n\to\infty}\big(\frac{2E_4f(2^n x,2^n y)-2E_4f(-2^n x,-2^n y)}{2^{3n+2}} \\ &\quad -\frac{E_4f(2^{n+1}x,2^{n+1}y)-E_4f(-2^{n+1}x,-2^{n+1}y)}{2^{3n+2}}\big) = 0, \\ E_4A_0(x,y) &= \lim_{n\to\infty}\big(\frac{8E_4f(2^n x,2^n y)-8E_4f(-2^n x,-2^n y)}{2^{n+4}} \\ &\quad -\frac{E_4f(2^{n+1}x,2^{n+1}y)-E_4f(-2^{n+1}x,-2^{n+1}y)}{2^{n+4}}\big) = 0, \\ E_4Q(x,y) &= \lim_{n\to\infty}\frac{E_4f(2^n x,2^n y)+E_4f(-2^n x,-2^n y)}{2^{2n+1}} = 0 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$. Since $C_0(2x) = 8C_0(x)($ $A_0(2x) = 2A_0(x)$ and $Q(2x) = 4Q(x)$, respectively), C_0 is a cubic function(A_0 is an additive function and Q is a quadratic function, respectively) by Theorem 2.2(Theorem 2.1 and Theorem 2.4, respectively). From (3.4), (3.5), (3.6) and the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{3}\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x)}{4} - C_0(x)\| \\ &+\frac{4}{3}\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x)}{16} - A_0(x)\| \\ &+\|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\| \end{aligned} \tag{3.7}$$

for all $x \in V \setminus \{0\}$, we get the inequality (3.3), where $F = -\frac{C_0(x)}{3} + Q(x) + \frac{4A_0(x)}{3} + f(0)$. Now, let F' be another generalized polynomial function of degree 3 satisfying (3.3) with $F'(0) = f(0)$. Then there are cubic functions $C, C' : V \to Y$, quadratic functions $Q, Q' : V \to Y$ and additive functions $A, A' : V \to Y$ such that $F(x) = C(x) + Q(x) + A(x) + f(0)$ and $F'(x) = C'(x) + Q'(x) + A'(x) + f(0)$. Since $C, C' : V \to Y$ are cubic functions, we get

$$\begin{aligned} \|C(x) - C'(x)\| &= \frac{1}{8^n}\|C(2^n x) - C'(2^n x)\| \\ &\leq \frac{1}{8^n}\|f(2^n x) - C(2^n x) - Q(2^n x) - A(2^n x) - f(0)\| \\ &+\frac{1}{8^n}\|f(2^n x) - C'(2^n x) - Q'(2^n x) - A'(2^n x) - f(0)\| \\ &+\frac{1}{8^n}\|Q(2^n x) - Q'(2^n x)\| + \frac{1}{8^n}\|A(2^n x) - A'(2^n x)\| \\ &\leq \frac{2}{8^n}\sum_{j=0}^{\infty}\left(\frac{4}{3}\frac{\psi(2^{j+n}x)}{2^j} + \frac{1}{3}\frac{\psi(2^{j+n}x)}{8^j} + \frac{\varphi(2^{j+n}x, -2^{j+n+1}x)}{2^{2j+3}}\right) \\ &+\frac{1}{2^n}\|Q(x) - Q'(x)\| + \frac{1}{4^n}\|A(x) - A'(x)\| \end{aligned}$$

for all $x \in V \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $C(x) = C'(x)$ for all $x, y \in V$. Hence

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \frac{1}{4^n}\|Q(2^n x) - Q'(2^n x)\| \\ &\leq \frac{1}{4^n}\|f(2^n x) - C(2^n x) - Q(2^n x) - A(2^n x) - f(0)\| \\ &\quad + \frac{1}{4^n}\|f(2^n x) - C(2^n x) - Q'(2^n x) - A'(2^n x) - f(0)\| \\ &\quad + \frac{1}{4^n}\|A(2^n x) - A'(2^n x)\| \\ &\leq \frac{2}{4^n}\sum_{j=0}^{\infty}\big(\frac{4}{3}\frac{\psi(2^{j+n}x)}{2^j} + \frac{1}{3}\frac{\psi(2^{j+n}x)}{8^j} + \frac{\varphi(2^{j+n}x, -2^{j+n+1}x)}{2^{2j+3}}\big) \\ &\quad + \frac{1}{2^n}\|A(x) - A'(x)\|\end{aligned}$$

for all $x \in V \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $Q(x) = Q'(x)$ for all $x, y \in V$. Similarly, we get $A(x) = A'(x)$ for all $x, y \in V$ as we desired.

By the similar method in the proof of Theorem 3.2, I can prove the following theorem.

Theorem 3.3 *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ be a mapping satisfying the condition*

$$\sum_{i=0}^{\infty} 8^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \tag{3.8}$$

for all $x, y \in V \setminus \{0\}$. If a function $f : V \to Y$ satisfies (3.2) for all $x, y \in V \setminus \{0\}$, then there exists a unique generalized polynomial function $F : V \to Y$ of degree 3 with $f(0) = F(0)$ such that

$$\|f(x) - F(x)\| \leq \sum_{j=1}^{\infty}\big(\frac{8^j + 2^{j+2}}{3}\psi(\frac{x}{2^j}) + 2^{2j-3}\varphi(\frac{x}{2^j}, -\frac{x}{2^{j-1}})\big) \tag{3.9}$$

for all $x \in V \setminus \{0\}$, where

$$\begin{aligned}\psi(x) &= \frac{1}{128}\big(11\varphi(x, -x) + \varphi(2x, -3x) + \varphi(x, x) + 11\varphi(-x, x) \\ &\quad + \varphi(-2x, 3x) + \varphi(-x, -x)\big)\end{aligned}$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$\begin{aligned} F(x) &= \lim_{n\to\infty} \big((\frac{8^n - 2^n}{4\cdot 3}) f(\frac{x}{2^{n-1}}) + (\frac{-8^n + 2^n}{4\cdot 3}) f(\frac{-x}{2^{n-1}}) \\ &\quad + (\frac{2^{n+2} - 8^n}{2\cdot 3} + 2^{2n-1}) f(\frac{x}{2^n}) + (\frac{8^n - 2^{n+2}}{2\cdot 3} + 2^{2n-1}) f(\frac{-x}{2^n}) + (1 - 4^n) f(0) \big) \end{aligned}$$

for all $x \in V$.

Proof. Note that if $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ satisfies the condition (3.8) then φ satisfies the condition $\sum_{i=0}^{\infty} 4^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$ and $\sum_{i=0}^{\infty} 2^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$. From (3.2), we get the inequalities

$$\begin{aligned} &\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x))}{4} - 8\big(\frac{f(\frac{x}{2}) - f(-\frac{x}{2})}{2} - \frac{f(x) - f(-x)}{4})\big)\| \\ \leq\ & 8\psi(\frac{x}{2}), \\ &\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x))}{16} - 2\big(\frac{f(\frac{x}{2}) - f(-\frac{x}{2})}{2} - \frac{f(x) - f(-x)}{16})\big)\| \\ \leq\ & 2\psi(\frac{x}{2}), \\ &\|\frac{f(x) + f(-x)}{2} - f(0) - 4\big(\frac{f(\frac{x}{2}) + f(-\frac{x}{2})}{2} - f(0))\big)\| \leq \frac{\varphi(\frac{x}{2}, -x)}{2} \end{aligned}$$

for all $x \in V \setminus \{0\}$. By Lemma 3.1, there exist functions $C_0, A_0, Q : V \to Y$ defined by

$$\begin{aligned} C_0(x) :&= \lim_{n\to\infty} 8^n (\frac{f(\frac{x}{2^n}) - f(-\frac{x}{2^n})}{2} - \frac{f(\frac{x}{2^{n-1}}) + f(-\frac{x}{2^{n-1}})}{4}), \\ A_0(x) :&= \lim_{n\to\infty} 2^{n-1} (f(\frac{x}{2^n}) - f(-\frac{x}{2^n}) - \frac{f(\frac{x}{2^{n-1}}) + f(-\frac{x}{2^{n-1}})}{8}), \\ Q(x) &= \lim_{n\to\infty} 2^{2n-1} (f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) - 2f(0)) \end{aligned}$$

for all $x \in V$ and the functions C_0, A_0, Q satisfy the inequalities

$$\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x))}{4} - C_0(x)\| \leq \sum_{j=1}^{\infty} 8^j \psi(\frac{x}{2^j}), \qquad (3.10)$$

$$\|\frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x))}{16} - A_0(x)\| \leq \sum_{j=1}^{\infty} 2^j \psi(\frac{x}{2^j}), \qquad (3.11)$$

$$\|\frac{f(x)+f(-x)}{2}-f(0)-Q(x)\|\le\sum_{j=1}^{\infty}2^{2j-3}\varphi(\frac{x}{2^j},-\frac{x}{2^{j-1}}) \tag{3.12}$$

for all $x\in V\backslash\{0\}$. From (3.2) and (3.8), we obtain $E_4C_0(x,y)=0$, $E_4A_0(x,y)=0$, $E_4Q(x,y)=0$ for all $x,y\in V\setminus\{0\}$. Since $C_0(2x)=8C_0(x)($ $A_0(2x)=2A_0(x)$ and $Q(2x)=4Q(x)$, respectively), C_0 is a cubic function(A_0 is an additive function and Q is a quadratic function, respectively) by Theorem 2.2(Theorem 2.1 and Theorem 2.4, respectively). From (3.8), (3.10), (3.11) and (3.12), we get the inequality (3.9), where $F=\frac{-C_0(x)}{3}+Q(x)+\frac{4A_0(x)}{3}+f(0)$. Now, let F' be another generalized polynomial function of degree 3 satisfying (3.9) with $F'(0)=f(0)$. Then there are cubic functions $C,C':V\to Y$, quadratic functions $Q,Q':V\to Y$ and additive functions $A,A':V\to Y$ such that $F(x)=C(x)+Q(x)+A(x)+f(0)$ and $F'(x)=C'(x)+Q'(x)+A'(x)+f(0)$. Since $A,A':V\to Y$ are additive functions, we get

$$\begin{aligned}\|A(x)-A'(x)\| &= 2^n\|A(\frac{x}{2^n})-A'(\frac{x}{2^n})\|\\ &\le 2^n\|f(\frac{x}{2^n})-C(\frac{x}{2^n})-Q(\frac{x}{2^n})-A(\frac{x}{2^n})-f(0)\|\\ &\quad +2^n\|f(\frac{x}{2^n})-C'(\frac{x}{2^n})-Q'(\frac{x}{2^n})-A'(\frac{x}{2^n})-f(0)\|\\ &\quad +2^n\|C(\frac{x}{2^n})-C'(\frac{x}{2^n})\|+2^n\|Q(\frac{x}{2^n})-Q'(\frac{x}{2^n})\|\\ &\le 2^{n+1}\sum_{j=1}^{\infty}\big(\frac{8^j+2^{j+1})}{3}\psi(\frac{x}{2^{j+n+1}})+2^{2j-3}\varphi(\frac{x}{2^{j+n}},-\frac{x}{2^{j+n-1}})\big)\\ &\quad +\frac{1}{4^n}\|C(x)-C'(x)\|+\frac{1}{2^n}\|Q(x)-Q'(x)\|\end{aligned}$$

for all $x\in V\setminus\{0\}$ and $n\in\mathbb{N}$. As $n\to\infty$, we may conclude that $A(x)=A'(x)$ for all $x,y\in V$. Similarly, we get $Q(x)=Q'(x)$ and $C(x)=C'(x)$ for all $x,y\in V$ as we desired.

Theorem 3.4 *Let $\varphi:V\setminus\{0\}\times V\setminus\{0\}\to[0,\infty)$ be a mapping satisfying the conditions $\sum_{i=0}^{\infty}\frac{\varphi(2^ix,2^iy)}{8^{i+1}}$ and $\sum_{i=0}^{\infty}4^i\varphi(\frac{x}{2^{i+1}},\frac{y}{2^{i+1}})<\infty$ for all $x,y\in V\setminus\{0\}$. If a function $f:V\to Y$ satisfies (3.2) for all $x,y\in V\setminus\{0\}$, then there exists a generalized polynomial function $F:V\to Y$ of degree 3 with $f(0)=F(0)$ such that*

$$\|f(x)-F(x)\|\le\sum_{j=0}^{\infty}\frac{\psi(2^jx)}{3\cdot 8^j}+\sum_{j=1}^{\infty}\big(\frac{2^{j+2}}{3}\psi(\frac{x}{2^j})+2^{2j-3}\varphi(\frac{x}{2^j},-\frac{x}{2^{j-1}})\big) \tag{3.13}$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$\begin{aligned} F(x) &= \lim_{n\to\infty}\Big(\frac{2^{n-2}}{3}(f(\frac{-x}{2^{n-1}}) - f(\frac{x}{2^{n-1}})) + \frac{f(2^{n+1}x) - f(-2^{n+1}x)}{12\cdot 8^n} \\ &\quad +(\frac{2^{n+1}}{3} + 2^{2n-1})f(\frac{x}{2^n}) + (\frac{-2^{n+1}}{3} + 2^{2n-1})f(\frac{-x}{2^n}) + (1-4^n)f(0) \\ &\quad + \frac{f(-2^n x) - f(2^n x)}{6\cdot 8^n}\Big) \end{aligned}$$

for all $x \in V$.

Proof. Let the function C_0 be as in the proof of Theorem 3.2 and let Q, A_0 as in the proof of Theorem 3.3. We easily get C_0, Q, A_0 and the inequalities (3.4), (3.11) and (3.12) for all $x \in V \setminus \{0\}$. From (3.4), (3.11) and (3.12), we obtain (3.13), where $F = -\frac{1}{3}C_0 + \frac{4}{3}A_0 + Q + f(0)$. **Theorem 3.5** *Let $\varphi : V\setminus\{0\}\times V\setminus\{0\} \to [0,\infty)$ be a mapping satisfying the conditions $\sum_{i=0}^{\infty}\frac{\varphi(2^i x, 2^i y)}{4^{i+1}}$ and $\sum_{i=0}^{\infty} 2^i\varphi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}) < \infty$ for all $x, y \in V \setminus \{0\}$. If a function $f : V \to Y$ satisfies (3.2) for all $x, y \in V \setminus \{0\}$, then there exists a generalized polynomial function $F : V \to Y$ of degree 3 with $f(0) = F(0)$ such that*

$$\|f(x) - F(x)\| \le \sum_{j=0}^{\infty}\Big(\frac{\psi(2^j x)}{3\cdot 8^j} + \frac{\varphi(2^j x, -2^{j+1}x)}{2^{2j+3}}\Big) + \sum_{j=1}^{\infty}\frac{2^{j+2}}{3}\psi(\frac{x}{2^j})$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$\begin{aligned} F(x) &= \lim_{n\to\infty}\Big(\frac{2^{n-2}}{3}(f(\frac{-x}{2^{n-1}}) - f(\frac{x}{2^{n-1}})) + \frac{f(2^{n+1}x) - f(-2^{n+1}x)}{12\cdot 8^n} \\ &\quad +\frac{2^{n+1}}{3}(f(\frac{x}{2^n}) - f(\frac{-x}{2^n})) + (\frac{1}{2\cdot 4^n} + \frac{1}{6\cdot 8^n})f(-2^n x) \\ &\quad +(\frac{1}{2\cdot 4^n} - \frac{1}{6\cdot 8^n})f(2^n x)\Big) + f(0) \end{aligned}$$

for all $x \in V$.

Corollary 3.6 *Let $p \neq 1, 2, 3$ and $\varepsilon > 0$. Suppose that the function $f : V \to Y$ satisfies*

$$\|E_4 f(x,y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in V \setminus \{0\}$. Then there exists a generalized polynomial function $F : V \to Y$ of degree 3 with $f(0) = F(0)$ such that

$$\|f(x) - F(x)\| \le \Big(\frac{24 + 2^p + 3^p}{24}(\frac{1}{|2-2^p|} + \frac{1}{|8-2^p|}) + \frac{1+2^p}{2|4-2^p|}\Big)\varepsilon\|x\|^p$$

for all $x \in V \setminus \{0\}$.

Proof. Applying Theorem 3.2, 3.3, 3.4 and 3.5, the following corollary can be proved easily.

Corollary 3.7 *Let* $\varepsilon > 0$. *Suppose that the function* $f : V \to Y$ *satisfies*

$$\|E_4 f(x,y)\| \le \varepsilon$$

for all $x, y \in V \setminus \{0\}$. *Then there exists a unique generalized polynomial function* $F : V \to Y$ *of degree 3 with* $f(0) = F(0)$ *such that*

$$\|f(x) - F(x)\| \le \frac{11}{14}\varepsilon$$

for all $x \in V \setminus \{0\}$.

4. Superstability of the Equation $E_4 f = 0$

Theorem 4.1 *Let* $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ *be a mapping satisfying the condition*

$$\lim_{(x,y)\to\infty} \varphi(x,y) = 0 \tag{4.1}$$

for all $x, y \in V \setminus \{0\}$. *If a function* $f : V \to Y$ *satisfies*

$$\|E_4 f(x,y)\| \le \varphi(x,y)$$

for all $x, y \in V \setminus \{0\}$, *then* f *is a generalized polynomial function of degree 3.*

Proof. Note that if $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ satisfies the condition (4.1) then φ satisfies the condition (4.1). By Theorem 3.1, there exist a unique generalized polynomial function $F : V \to Y$ of degree 3 with $f(0) = F(0)$ such that the inequality (3.3) holds for all $x \in V \setminus \{0\}$. Hence the inequality

$$\begin{aligned}
4\|f(x) - F(x)\| &\le \|E_4 f((k+1)x, -kx) - E_4 F((k+1)x, -kx)\| \\
&\quad + \|(f - F)((3k+4)x)\| + 4\|(f - F)((2k+3)x)\| \\
&\quad + 6\|(f - F)((k+2)x)\| + \|(f - F)(-kx)\| \\
&\le \varphi((k+1)x, -kx) + \Phi((3k+4)x) + 4\Phi((2k+3)x) \\
&\quad + 6\Phi((k+2)x) + \Phi(-kx))
\end{aligned}$$

for all $x \in V \setminus \{0\}$ and $k \in \mathbb{N}$, where Φ is defined by

$$\Phi(x) := \sum_{j=0}^{\infty} \Big(\frac{\psi(2^j x)}{3 \cdot 2^{j-2}} + \frac{\psi(2^j x)}{3 \cdot 8^j} + \frac{\varphi(2^j x, -2^{j+1} x)}{2^{2j+3}}\Big).$$

Taking as $k \to \infty$, we conclude $f(x) = F(x)$ for all $x \in V \setminus \{0\}$.

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