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Superstability of Higher Derivations in Multi-Banach Algebras *

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Abstract

Let $\mathcal{A}$ be an algebra and $n_0 \in \{0, 1, \ldots, \} \cup \{\infty\}$. A sequence $(d_j)_{j=1}^{n_0}$ of linear mappings on $\mathcal{A}$ is called a (strongly) higher derivation of rank n_0 if (d_0 is the identity on $\mathcal{A}$ and) for each $0 \leq j \leq n_0$,

$$d_j(ab) = \sum_{\ell=0}^{j} d_\ell(a) d_{j-\ell}(b) \qquad (a, b \in \mathcal{A}).$$

In this paper, we define the notion of an approximate higher derivation in multi-Banach algebras and investigate the superstability of strongly higher derivations.

Keywords and Phrases: *Hyers–Ulam–Rassias stability, Multi-Banach algebra, Cauchy functional equation; Derivation; higher derivation.*

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1. Introduction and preliminaries

One of essential questions in the theory of functional equations giving the notion of stability is “When is it true that the solution of an equation differing slightly from a given one, must be close to the solution of the given equation?” The equation is called *superstable* if each its approximate solution is an exact solution (see [5] for another notion of superstability namely *superstability modulo the bounded functions*). In 1940, S.M. Ulam [28] posed the first stability problem. In the next year, D.H. Hyers [12] gave an affirmative answer to the question of Ulam. In 1950, T. Aoki [2] extended Hyers' theorem for additive mappings. In 1978, Th.M. Rassias [24] extended Hyers' theorem by obtaining a unique linear mapping under certain continuity assumption when the Cauchy difference is allowed to be unbounded (see [21]). The paper of Th.M. Rassias [24] has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to the monographs [6, 13, 15, 25].

The notion of multi-normed space was introduced by H.G. Dales and M.E. Polyakov in [9]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach latices. Motivations for the study of multi-normed spaces and many examples are given in [9].

Let $(E, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by $\mathfrak{S}_k$ the group of permutations on k symbols. Following notation and terminology of [9],

Definition 1.1. *A* multi-norm *on* $\{E^k : k \in \mathbb{N}\}$ *is a sequence*

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ *is a norm on* E^k *for each* $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ *for each* $x \in E$, *and the following axioms are satisfied for each* $k \in \mathbb{N}$ *with* $k \geq 2$*:*

(MN1)

$$\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\right\|_k = \|(x_1, \ldots, x_k)\|_k \quad (\sigma \in \mathfrak{S}_k,\, x_1, \ldots, x_k \in E)\,;$$

(MN2)

$$\|(\alpha_1 x_1, \ldots, \alpha_n x_k)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|)\,\|(x_1, \ldots, x_k)\|_k \quad (\alpha_1, \ldots, \alpha_k \in \mathbb{C},\, x_1, \ldots, x_k \in E)\,;$$

(MN3)

$$\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \quad (x_1, \ldots, x_{k-1} \in E)\,;$$

(MN4)

$$\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \quad (\,x_1, \ldots, x_{k-1} \in E)\,.$$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ *is a* multi-normed space*.*

Suppose that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space, and take $k \in \mathbb{N}$. It is easy to see that (cf. [9])

(a) $\|(x, \ldots, x)\|_k = \|x\| \quad (x \in E)$.

(b) $\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\|$ $(x_1, \ldots, x_k \in E)$.

It follows from (b) that, if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *multi-Banach space*.

Example 1.2. Let E be an arbitrary normed space. The sequence $(\|\cdot\|_k : k \in \mathbb{N})$ on $\{E^k : k \in \mathbb{N}\}$ defined by

$$\|(x_1, \ldots, x_k)\|_k := \max_{i\in\mathbb{N}_k} \|x_i\| \quad (x_1, \ldots, x_k \in E)$$

is a multi-norm called *the minimum multi-norm*. The terminology 'minimum' is justified by property (b). □

Lemma 1.3. *Suppose that* $k \in \mathbb{N}$ *and* $(x_1, \cdots, x_k) \in E^k$*. For each* $j \in \{1, \cdots, k\}$*, let* $(x_n^j)_{n=1,2,\cdots}$ *be a sequence in* E *such that* $\lim_{n\to\infty} x_n^j = x^j$*. Then for each* $(y_1, \cdots, y_k) \in E^k$ *we have*

$$\lim_{n\to\infty} (x_n^1 - y_1, \cdots, x_n^k - y_k) = (x_1 - y_1, \cdots, x_k - y_k)\,.$$

Definition 1.4. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. A sequence (x_n) in E is a* multi-null sequence *if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{k\in\mathbb{N}} \|(x_n, \ldots, x_{n+k-1})\|_k < \varepsilon \quad (n \geq n_0).$$

Let $x \in E$. We say that the sequence (x_n) is multi-convergent *to x in E and write*

$$\operatorname*{Lim}_{n\to\infty} x_n = x$$

if $(x_n - x)$ is a multi-null sequence.

Definition 1.5. *Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra such that $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. Then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed algebra if*

$$\|(a_1b_1, \ldots, a_kb_k)\|_k \leq \|(a_1, \ldots, a_k)\|_k \, \|(b_1, \ldots, b_k)\|_k$$

for $k \in \mathbb{N}$ and $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathcal{A}$. Further, the multi-normed algebra $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra if $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 1.6. Let p, q with $1 \leq p \leq q < \infty$, and $\mathcal{A} = \ell^p$. The algebra $\mathcal{A}$ is a Banach sequence algebra with respect to coordinatewise multiplication of sequences, see [7, Example 4.1.42]. Let $(\|\cdot\|_k : k \in \mathbb{N})$ be the standard (p,q)-multi-norm on $\{\mathcal{A}^k : n \in \mathbb{N}\}$ (see [9]). Then $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra.

Let $\mathcal{A}$ be an algebra and $n_0 \in \{0, 1, \ldots,\} \cup \{\infty\}$. A sequence $(d_j)_{j=1}^{n_0}$ of linear mappings on $\mathcal{A}$ is called a higher derivation of rank n_0 if for each $0 \leq j \leq n_0$,

$$d_j(ab) = \sum_{\ell=0}^{j} d_\ell(a) d_{j-\ell}(b) \qquad (a, b \in \mathcal{A}).$$

It is obvious that d_0 is a homomorphism and d_1 is a d_0-derivation in the sense of [18]. If d_0 is the identity operator $id_{\mathcal{A}}$ on $\mathcal{A}$, then $(d_j)_{j=0}^{n_0}$ is called a *strongly higher derivation*. The notion of higher derivation was introduced by Hasse and Schmidt [10]. This notion closely concerns the concept of homomorphisms [7]. In [27] higher derivations are applied to study generic solving of higher differential equations. A standard example of a strongly higher derivation is $(\frac{D^j}{j!})_{j=0}^{\infty}$, where D is a derivation on an algebra $\mathcal{A}$. The interested reader is referred to [11, 14, 17, 26] and the references therein for more information about higher derivations.

The stability of derivations was studied by C.-G. Park in [22, 23]. A discussion of stability of the so-called generalized derivations is given in [1, 3, 19, 20]. In this paper, using some ideas from [4, 7, 20], we investigate the superstability of higher derivations in multi-Banach algebras. We should noticed that our main result is a generalization of [16, Corollary 2.5].

2. Superstability of Higher Derivations

We start our work by providing a proof for the following theorem by using the direct method (see also [7, Corollary 3.4] in which we used a fixed point approach).

Lemma 2.1. *Let $(E, \|\cdot\|)$ be a normed space, and let $((F^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$, $\alpha \geq 0$, and let $f : E \to F$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|\left(f(\mu x_1 + y_1) - \mu f(x_1) - f(y_1), \ldots, f(\mu x_k + y_k) - \mu f(x_k) - f(y_k)\right)\right\|_k \leq \alpha \tag{2.1}$$

for all $\mu \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\left\|\left(f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k)\right)\right\|_k \leq \alpha$$

for all $x_1, \ldots, x_k \in E$.

Proof. Let $x_1, \ldots, x_k \in E$. Replacing $y_1, \cdots, y_k$ by $x_1, \cdots, x_k$ in (2.1) we obtain

$$\sup_{k\in\mathbb{N}} \left\|\left(f(2x_1) - 2f(x_1), \ldots, f(2x_k) - 2f(x_k)\right)\right\|_k \leq \alpha\,. \tag{2.2}$$

Replacing $x_1, \ldots, x_k$ by $2^n x_1, \ldots, 2^n x_k$ and dividing by 2^{n+1} from relation (2.2) one gets

$$\sup_{k\in\mathbb{N}} \left\|\left(\frac{f(2^{n+1}x_1)}{2^{n+1}} - \frac{f(2^n x_1)}{2^n}, \ldots, \frac{f(2^{n+1}x_k)}{2^{n+1}} - \frac{f(2^n x_k)}{2^n}\right)\right\|_k \leq \frac{\alpha}{2^{n+1}}\,. \tag{2.3}$$

It follows from (2.3)that

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f(2^{n+m}x_1)}{2^{n+m}}-\frac{f(2^n x_1)}{2^n},\dots,\frac{f(2^{n+m}x_k)}{2^{n+m}}-\frac{f(2^n x_k)}{2^n}\right)\right\|_k\le\alpha\Big(\frac{1}{2^{n+1}}+\cdots+\frac{1}{2^{n+m}}\Big) \tag{2.4}$$

for $n,m\in\mathbb{N},\ m\ge 1$. This implies that $\left(\frac{f(2^n x)}{2^n}\right)$ is Cauchy for each fixed x. Hence this sequence is convergent in the complete multi-norm F. Set

$$T(x):=\operatorname*{Lim}_{n\to\infty}\frac{f(2^n x)}{2^n}.$$

Hence for each $\varepsilon>0$ there is n_0 such that

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f(2^n x)}{2^n}-T(x),\dots,\frac{f(2^{n+k-1}x)}{2^{n+k-1}}-T(x)\right)\right\|_k<\varepsilon$$

for all $n\ge n_0$. In particular, by property (b) of multi-norm, we have

$$\lim_{n\to\infty}\left\|\frac{f(2^n x)}{2^n}-T(x)\right\|=0 \qquad (x\in E). \tag{2.5}$$

Next put $n=0$ in (2.4) to get

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f(2^m x_1)}{2^m}-f(x_1),\dots,\frac{f(2^m x_k)}{2^m}-f(x_k)\right)\right\|_k\le\alpha\,.$$

Letting m tend to infinity and using Lemma 1.3 and (2.5) we obtain

$$\sup_{k\in\mathbb{N}}\|(T(x_1)-f(x_1),\dots,T(x_k)-f(x_k))\|_k\le\alpha\,.$$

Let $x,y\in E$. Put $x_1=\cdots=x_k=2^n x, y_1=\cdots=y_k=2^n y$ in (2.1) and divide both sides by 2^n to obtain

$$\|(2^{-n}f(2^n(x+y))-2^{-n}f(2^n x)-2^{-n}f(2^n y)\|\le 2^{-n}\alpha$$

taking the limit as $n\to\infty$ we get $T(x+y)=T(x)+T(y)$. Hence T is additive. If T' is another required mapping, then

$$\begin{aligned}\|T'(x)-T(x)\| &\le \frac{1}{2^n}\|T'(2^n x)-T(2^n x)\|\\ &\le \frac{1}{2^n}\|T'(2^n x)-f(2^n x)\|+\frac{1}{2^n}\|f(2^n x)-T(2^n x)\|\\ &\le \frac{1}{2^n}\Big(\frac{\alpha}{2-s}+\frac{\alpha}{2-s}\Big) \quad \text{(by property (a) of multi-norm)}\end{aligned}$$

Hence $T' = T$. This proves the uniqueness assertion. The homogenous property of T can be proved in a standard fashion. □

Definition 2.2. *Let $\alpha > 0$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a control function such that*

$$\varphi(2^n a, 2^m b) \leq \beta^{n+m} \varphi(a, b) \tag{2.6}$$

for some $0 < \beta < 2$, all nonnegative numbers m, n and all $a, b \in \mathcal{A}$. Let $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. An (α, φ)-approximate strongly higher derivation of rank n_0 is a sequence $(f_j)_{j=0}^{n_0}$ of mappings $f_j : \mathcal{A} \to \mathcal{A}$ with $f_j(0) = 0$ and such that $f_0 = id_{\mathcal{A}}$,

$$\sup_{k\in\mathbb{N}} \|(f_j(\mu x_1 + y_1) - \mu f_j(x_1) - f_j(y_1), \ldots, f_j(\mu x_k + y_k) - \mu f_j(x_k) - f_j(y_k))\|_k \leq \alpha$$

for all $0 \leq j \leq n_0$, all $\mu \in \mathbb{T}$ and all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$, and

$$\|f_j(ab) - \sum_{\ell=0}^{j} f_\ell(a) f_{j-\ell}(b)\| \leq \varphi(a, b) \tag{2.7}$$

for all $0 \leq j \leq n_0$ and all $a, b \in \mathcal{A}$.

Theorem 2.3. *Every (α, φ)-approximate strongly higher derivation in a multi-Banach algebra is a higher derivation.*

Proof. Let $((\mathcal{A}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra and let $(f_j)_{j=0}^{n_0}$ be an (α, φ)-approximate higher derivation. By Lemma 2.1, for each $0 \leq j \leq n_0$, there is a linear mapping D_j defined by $D_j(a) := \lim_{n\to\infty} \frac{f_j(2^n a)}{2^n}$ such that

$$\|D_j(a) - f_j(a)\| \leq \alpha$$

for all $a \in \mathcal{A}$. It follows from (2.6) and (2.7) that

$$\left\| \frac{f_j(4^n ab)}{4^n} - \sum_{\ell=0}^{j} \frac{f_\ell(2^n a)}{2^n} \frac{f_{j-\ell}(2^n b)}{2^n} \right\| \leq \frac{\varphi(2^n a, 2^n b)}{4^n} \leq \left(\frac{\beta}{2}\right)^{2n} \varphi(a, b)$$

from which we infer that (D_j) is a higher derivation. Further,

$$\begin{aligned}
& \|2^n(f_1(2^m a) - 2^m f_1(a))\| \\
\leq\ & \|2^n 1 f_1(2^m a) - f_1(2^n 1)2^m a - f_1((2^n 1)(2^m a))\| \\
& +\|f_1((2^n 1)(2^m a)) - f_1(2^n 1)2^m a - 2^{n+m} 1 f_1(a)\| \\
\leq\ & \beta^{n+m}\varphi(1,a) + \|f_1((2^n 1)(2^m a)) - f_1(2^n 1)2^m a - 2^{n+m} 1 f_1(a)\| \\
\leq\ & \beta^{n+m}\varphi(1,a) + \|f_1((2^n 1)(2^m a)) - D_1((2^n 1)(2^m a))\| \\
& +\|D_1((2^n 1)(2^m a)) - 2^{n+m} 1 f_1(a) - f_1(2^n 1)2^m a\| \\
\leq\ & \beta^{n+m}\varphi(1,a) + \alpha \\
& +\|D_1((2^n 1)(2^m a)) - 2^{n+m} 1 f_1(a) - f_1(2^n 1)2^m a\| \\
\leq\ & \beta^{n+m}\varphi(1,a) + \alpha + 2^m\|D_1(2^n 1a) - f_1(2^n 1a)\| \\
& +2^m\|f_1(2^n 1a) - 2^n 1 f_1(a) - f_1(2^n 1)a\| \\
\leq\ & \beta^{n+m}\varphi(1,a) + \alpha + 2^m\alpha + 2^m\varphi(2^n 1, a) \\
\leq\ & (\beta^{n+m} + 2^m\beta^n)\varphi(1,a) + (1+2^m)\alpha,
\end{aligned}$$

for all nonnegative integers m, n and all $a \in \mathcal{A}$. Fix m and let n tend to ∞ in the following inequality

$$\|f_1(2^m a) - 2^m f_1(a)\| \leq \frac{(\beta^{n+m} + 2^m\beta^n)\varphi(1,a) + (1+2^m)\alpha}{2^n}.$$

Then $f_1(2^m a) = 2^m f_1(a)$ for all m and all $a \in \mathcal{A}$. Therefore $D_1(a) = \lim_{m\to\infty} \frac{f_1(2^m a)}{2^m} = f_1(a)$ for all $a \in \mathcal{A}$.
By utilizing induction on j and applying

$$\|f_j(ab) - af_j(b) - \sum_{\ell=1}^{j-1} f_\ell(a) f_{j-\ell}(b) - f_j(a)b\| \leq \varphi(a,b)$$

we conclude that $D_j = f_j$ for all $0 \leq j \leq n_0$. □

Remark 2.4. A typical example of the control function φ is $\varphi(a,b) = \beta\varepsilon(\|x\|^p + \|y\|^q) + \delta\|x\|^p\,\|y\|^q$, where $\varepsilon, \delta \geq 0$ and $p, q \in [0,1)$. A general example is the function $\varphi(a,b) = \psi(\|a\|) + \psi(\|b\|)$, where $\psi : [0,\infty) \to [0,\infty)$ is a function with $\psi(2) < 2$.

Corollary 2.5. *Every (α,φ)-approximate strongly higher derivation in a Banach algebra is a higher derivation.*

Corollary 2.6. *Every* (α, φ)*-approximate derivation (regarded as a approximate strongly higher derivation of rank* 2*) in a multi-Banach algebra is a higher derivation.*

References

[1] M. Amyari, C. Baak and M. S. Moslehian, Nearly ternary derivations *Taiwanese J. Math.* **11** (2007), no. 5, 1417–1424.

[2] T. Aoki, the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan* **2** (1950), 64–66.

[3] C. Baak and M. S. Moslehian, On the stability of θ-derivations on JB^*-triples,*Bull. Braz. Math. Soc.* **38** (2007), no. 1, 115–127.

[4] R. Badora, On approximate derivations, *Math. Inequal. Appl.* **9** (2006), no. 1, 167–173.

[5] J. Baker, The stability of the cosine equation, *Proc. Amer. Math. Soc.* **74** (1979), 242–246.

[6] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[7] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, New Series, 24. Oxford University Press, Oxford, 2000.

[8] H. G. Dales and M. S. Moslehian, Stability of mappings on multi-normed spaces, *Glasgow Math. J.* **49** (2007), no.2, 321–332.

[9] H. G. Dales and M. E. Polyakov, *Multi-normed spaces and multi-Banach algebras*, (preprint).

[10] H. Hasse and F. K. Schmidt, Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Funktionenkörper einer Unbestimmten (German) *J. Reine Angew. Math.*, **177** (1937). 215–237.

[11] S. Hejazian and T. L. Shatery, Automatic continuity of higher derivations on JB^*-algebras, *Bull Iranian Math. Soc.*, **33** (2007), no. 1, 11–23.

[12] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.

[13] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, Basel, 1998.

[14] K. W. Jun and Y. W. Lee, The image of a continuous strong higher derivation is contained in the radical, *Bull. Korean Math.* Soc. **33** (1996), 229–232.

[15] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Florida, 2001.

[16] Y.-S. Jung and I.-S. Chang, On approximately higher ring derivations, *J. Math. Anal. Appl.* **343** (2008), 636–643.

[17] J. B. Miller, Higher derivations on Banach algebras, *Amer. J. Math.*, **92** (1970), 301-?31.

[18] M. Mirzavaziri and M. S. Moslehian, Automatic continuity of σ-derivations in C^*-algebras, *Proc. Amer. Math. Soc.* **134** (2006), no. 11, 3319–3327.

[19] T. Miura, G. Hirasawa, S.-E. Takahasi, A perturbation of ring derivations on Banach algebras, *J. Math. Anal. Appl.* **319** (2006), no. 2, 522–530.

[20] M. S. Moslehian, Hyers–Ulam–Rassias stability of generalized derivations, *Internat. J. Math. Math. Sci.* **2006** (2006), 93942, 1–8.

[21] M. S. Moslehian and Th. M. Rassias, Stability of functional equations in non-Arhimedian spaces, *Appl. Anal. Disc. Math.* **1** (2007), 325–334.

[22] C.-G. Park, Linear derivations on Banach algebras, *Nonlinear Funct. Anal. Appl.* **9** (2004), 359–368.

[23] C.-G. Park, Lie *-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras, *J. Math. Anal. Appl.* **293** (2004), 419–434.

[24] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.

[25] Th. M. Rassias (ed.), *Functional equations, inequalities and applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.

[26] P. Ribenboim, Higher derivations of rings. I. & II., *Rev. Roumaine Math. Pures Appl.*, **16** (1971), 77-11, 245-272.

[27] Y. Uchino and T. Satoh, Function field modular forms and higher derivations, *Math. Ann.* **311** (1998), 439–466.

[28] S. M. Ulam, *A collection of mathematical problems* Interscience Publ., New York, 1960. Problems in Modern Mathematics, Wiley, New York, 1964.