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Coincidences and Fixed Points of Hybrid Contractions*

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Abstract

In this paper we study the existence of coincidences and fixed points of generalized hybrid contractions involving single-valued and multivalued maps in generalized metric spaces. Some special cases are also discussed.

Keywords and Phrases: *Coincidence point;Fixed point; Hybrid contraction.*

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1. Introduction

Hybrid fixed point theory is a recent development is the ambit of fixed point theorems for contracting single-valued and multivalued maps in metric spaces. Indeed, the study of such maps was initiated during 1980-83 by Bhaskaran and Subrahmanyam [2], Hadzic [10], Kaneko [14], Kulshrestha [18], Kubiak [19], Naimpally et al. [25] and Singh and Kulshrestha [35]. For a history of the fundamental work on this line, refer to Singh and Mishra [37], and for more recent work on this line Beg and Azam [1], Jungck and Rhoades [12], Kamran [13], Kaneko [15], Kaneko and Sessa [16], Liu, Wu, and Li [20], Mishra, Singh and Talwar [22], Naidu [24], Pathak et al. [26], Popa [27], Rhoades et al. [28], Shahzad [30], and Singh et al.[31, 33, 34, 36-40]. Hybrid fixed point theory has potential applications in functional inclusions, optimization theory, fractal graphics and discrete dynamics for set-valued operators.

The following fundamental coincidence theorem for a pair of multivalued and single-valued maps is essentially due to Singh and Kulshrestha [35] (see also [18] and [37]).

Theorem 1.1 ([35]). *Let X be a metric space and (CL(X), H) the Hausdorff metric space induced by d, where CL(X) is the collection of all nonempty closed subsets of X. Let $P : X \to CL(X)$ and $f : X \to X$ be such that $P(X) \subseteq f(X)$ and*

$$H(Px, Py) \le q.\max\{d(fx, fy), d(fx, Px), d(fy, Py), [d(fx, Py) + d(fy, Px)]/2\} \qquad \text{(SK)}$$

for all $x, y \in X$, where $0 \le q < 1$. If f(X) [or P(X)] is a complete subspace of X, then P and f have a coincidence, i.e., there exists a point $z \in X$ such that $fz \in Pz$.

We remark that under the conditions of Theorem 1.1, f and P need not have a common fixed point even if f and P are commuting (cf. Def. 2.3) and continuous as the following example shows (see also [25, 33, 37-40]).

Example 1.1 ([25]). Let $X = [0, \infty)$ be endowed with the usual metric,

$Px = [1 + x, \infty)$ and $fx = 2x$. Then $P(X) \subseteq f(X) = X$. Further

$$H(Px, Py) \le qd(fx, fy), \quad x, y \in X, \quad 1/2 \le q < 1. \qquad \text{(NSW)}$$

Thus P and f satisfy all the requirements of Theorems 1.1, since (NSW) implies (SK).

Evidently, P and f have a coincidence point z (≥ 1), i.e., $fz \in Pz$ for any $z \ge 1$. Notice that P and f have no common fixed points. Moreover, P is not a multivalued contraction in the sense of Nadler, Jr. [23], since $H(Px, Py) = d(x, y)$, $x, y \in X$. (Recall that Nadler's multivalued contraction is (NSW) with f = the identity map on X, wherein $0 \le q < 1$.)

Theorem 1.1 has been generalized and extended on various settings (see, for instance, [1, 20, 24, 27, 28, 34, 36-40]). In this paper, we obtain a few generalizations and extensions of Theorem 1.1 and other similar results (cf. [15] and [31]). Using these coincidence theorems, we obtain a few fixed point theorems, wherein continuity of maps is not needed, completeness of the space is relaxed to the completeness of a subspace, and the commutativity requirement is tight and minimal.

2. Preliminaries

Consistent with [7] and [32], we use the following notations and definitions.

Definition 2.1 ([7]). *Let X be (nonempty) a set and $s \geq 1$ a given real number. A function*
$d : X \times X \rightarrow R^{+}$ (nonnegative real numbers) is called a b-metric provided that, for all $x, y, z \in X$,

$$d(x, y) = 0 \text{ iff } x = y, \qquad \text{(bm-1)}$$

$$d(x, y) = d(y, x), \qquad \text{(bm-2)}$$

$$d(x, z) \leq s[d(x, y) + d(y, z)]. \qquad \text{(bm-3)}$$

The pair (X, d) is called a b-metric space.

We remark that a metric space is evidently a b-metric space. However, Czerwik [6, 7] has shown that a b-metric on X need not be a metric on X (see also [8, 9, 32]).

Definition 2.2 ([7]). *Let (X, d) be a b-metric space. The Hausdorff b-metric H on CL(X), the collection of all nonempty closed subsets of (X, d) is defined as follows:*

$$H(A, B) := \begin{cases} \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}, & \text{if the maximum exists}, \\ \infty, & \text{otherwise}. \end{cases}$$

In all that follows Y is an arbitrary nonempty set and (X, d) a b-metric space unless otherwise specified. For the following definition in a metric space, one may refer to Itoh and Takahashi [11] and Singh and Mishra [39].

Definition 2.3. *Let Y be a nonempty set, $f : Y \rightarrow Y$ and $P : Y \rightarrow 2^{Y}$, the collection of all nonempty subsets of Y. Then the hybrid pair (P, f) is (IT)-commuting at $x \in Y$ if $fPx \subseteq Pfx$ for each $x \in Y$.*

We cite the following lemmas from Czerwik [7-9] and Singh et al. [31, 32].

Lemma 2.1. *For any $A, B, C \in CL(X)$,*

(i) $d(x, B) \le d(x, y)$ *for any* $y \in B$,

(ii) $d(A, B) \le H(A, B)$,

(iii) $d(x, B) \le H(A, B), \quad x \in A$

(iv) $H(A, C) \le s[H(A, B) + H(B, C)]$,

(v) $d(x, A) \le sd(x, y) + sd(y, A), \quad x, y \in X$.

Lemma 2.2. *Let A and $B \in CL(X)$. Then for any $x \in A$ and for some $0 < q, \ k < 1$, there exists a $y \in B$ such that*

$$d^2(x, y) \le q^{-k} H^2(A, B).$$

For an excellent collection of such results in metric spaces, one may refer to Rus [29].
Lemma 2.2 in a metric space is essentially due to Nadler, Jr. [23] (see also [3] and [5]).

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3. Coincidence Theorems

We begin with the following result.

Lemma 3.1. *Let (X, d) be a b-metric space and $\{y_n\}$ a sequence in X such that*

$$d(y_{n+1}, y_{n+2}) \le qd(y_n, y_{n+1}), \qquad n = 0, 1, \cdots,$$

where $0 \le q < 1$. Then the sequence $\{y_n\}$ is Cauchy sequence in X provided that $sq < 1$.

Proof. For any n,

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\le qd(y_n, y_{n+1}) \\ &\le q^2 d(y_{n-1}, y_n) \le \cdots \le q^{n+1} d(y_0, y_1). \end{aligned}$$

For $n < m$, by the triangle inequality (cf. Def. 2.1 (bm-3)),

$$\begin{aligned} d(y_n, y_m) &\le sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + \cdots + s^{m-n-2}[d(y_{m-2}, y_{m-1}) + d(y_{m-1}, y_m)] \\ &< sq^n(1 + sq + s^2q^2 + \cdots)d(y_0, y_1) \\ &= [sq^n/(1 - sq)]d(y_0, y_1) \to 0 \text{ as } n \to \infty, \end{aligned}$$

and $\{y_n\}$ is Cauchy. □

Following Liu et al. [21], Singh et al. [33, 35] and Tan et al. [41], we consider the following conditions for $f : Y \to X$ and $P, Q : Y \to CL(X)$:

$$H(Px, Qy) \leq q.\max\{d(fx, fy), d(fx, Px), d(fy, Qy), [d(fx, Qy) + d(fy, Px)]/2\}, x, y \in X, \quad (1)$$

where $q \in (0, 1)$; and

$$H^2(Px, Py) \leq q.\max m(x, y), \; x, y \in X, \quad (2)$$

where $q \in (0, 1)$ and

$$m(x, y): = \max\{d^2(fx, fy), d(fx, fy).d(fx, Px), d(fx, fy).d(fy, Py), d(fx, fy).[d(fx, Py) + d(fy, Px)]/2, d(fx, Px).d(fy, Py), d(fx, Px).[d(fx, Py) + d(fy, Px)]/2, d(fy, Py).[d(fx, Py) + d(fy, Px)]/2, d(fx, Py).d(fy, Px)\}.$$

We remark that (1) with $P = Q$ and $Y = X$, a metric space is (SK), while the main condition studied in [31] is based on the work of [21] and [41], and is a particular case of (2).

Assume that $\beta : = sq^{1-k}[1 + \sqrt{(1 + 8q^{-1+k}s^{-1})}]/4$, where $0 < q, \; k < 1$.

Theorem 3.1. *Let Y be an arbitrary nonempty set and (X, d) a b-metric space. Let*

$P : Y\to CL(X)$ and $f : Y\to X$ be such that $P(Y) \subseteq f(Y)$ and (2) holds for all $x, y \in Y$. If $sq^{1-k} < 1$, $\beta s < 1$ and one of P(Y) or f(Y) is a complete subspace of X, then $fx \in Px$ has a solution, that is P and f have a coincidence. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that

(I) $fx_{n+1} \in Px_n$, $n = 0, 1, 2, \cdots$;

(II) the sequence $\{fx_n\}$ converges to fz for some $z \in Y$, and $fz \in Pz$, that is, P and f have a coincidence at z; and

(III) $d(fx_n, fz) \leq [s\beta^n/(1 - s\beta)]d(fx_0, fx_1)$.

Proof. Pick $x_0 \in Y$. Let k be a positive number such that $k < 1$. Following Kulshrestha [18] and Singh and Kulshrestha [35], we construct sequences $\{x_n\} \subseteq Y$ and $\{fx_n\} \subseteq X$ in the following manner. Since $P(Y) \subseteq f(Y)$, we may choose a point $x_1 \in Y$ such that $fx_1 \in Px_0$.

If $Px_0 = Px_1$ then $x_1 = z$ is a coincidence point of P and f, and we are done. So assume that $Px_0 \neq Px_1$.

Now the condition $P(Y) \subseteq f(Y)$ and Lemma 2.2 allow us to choose a point $x_2 \in Y$ such that $fx_2 \in Px_1$ and

$$d^2(fx_1, fx_2) \leq q^{-k} H^2(Px_0, Px_1).$$

If $Px_1 = Px_2$, then x_2 becomes a coincidence point of P and f. If not, continue the process. In general, if $Px_n \neq Px_{n+1}$, we choose $fx_{n+2} \in Px_{n+1}$ such that

$$d^2(fx_{n+1}, fx_{n+2}) \leq q^{-k} H^2(Px_n, Px_{n+1}).$$

Then by (2),

$$\begin{aligned} d^2(fx_{n+1}, fx_{n+2}) \leq\ & q^{-k} H^2(Px_n, Px_{n+1}), \\ \leq\ & q^{1-k}.\max\{d^2(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}).d(fx_n, Px_n), \\ & d(fx_n, fx_{n+1}).d(fx_{n+1}, Px_{n+1}), \\ & d(fx_n, fx_{n+1}).[d(fx_n, Px_{n+1}) + d(fx_{n+1}, Px_n)]/2, \\ & d(fx_n, Px_n).d(fx_{n+1}, Px_{n+1}), \\ & d(fx_n, Px_n).[d(fx_n, Px_{n+1}) + d(fx_{n+1}, Px_n)]/2, \\ & d(fx_{n+1}, Px_{n+1}).[d(fx_n, Px_{n+1}) + d(fx_{n+1}, Px_n)]/2, \\ & d(fx_n, Px_{n+1}).d(fx_{n+1}, Px_n)\}. \end{aligned}$$

For the sake of simplicity, we take $y_n := fx_n$, $d_n := d(y_n, y_{n+1})$ and $\lambda := q^{1-k}$.

Then the above inequality, after simplification, yields

$$d^2_{n+1} \leq \lambda.\max\{d^2_n, d_n d_{n+1}, d_n[d(y_n, y_{n+2})]/2, d_{n+1}[d(y_n, y_{n+2})]/2)\},$$

that is

$$d^2_{n+1} \leq \lambda.\max\{d^2_n, d_n d_{n+1}, s(d_n[d_n + d_{n+1}]/2), s(d_{n+1}[d_n + d_{n+1}]/2)\}. \tag{3}$$

We remark that in the construction of sequences $\{x_n\}$ and $\{fx_n\}$, x_n (for each n) is not a coincidence point of P and f. This together with $Px_n \neq Px_{n+1}$ means that $fx_n \neq fx_{n+1}$. Indeed, if at any stage $fx_n = fx_{n+1}$ then $fx_n \in Px_n$ and $\{x_n\}$ is a coincidence point of P and f. Therefore, according to our construction of the sequences, $d_n \neq 0$. Hence the inequality (3) implies one of the following:

$$d^2_{n+1} \leq \lambda d^2_n$$

that is

$$d_{n+1} \leq \sqrt{\lambda} d_n;$$

$$d^2_{n+1} \leq \lambda d_n d_{n+1} \text{ implies } d_{n+1} \leq \lambda d_n;$$

$d^2_{n+1} \leq \lambda s(d_n[d_n + d_{n+1}]/2)$ being a quadratic inequality in d_{n+1} gives

$$d_{n+1} \leq [\lambda s/4 + \sqrt{((\lambda^2 s^2/16) + \lambda s/2)}]d_n$$
$$= \{\lambda s[1+ \sqrt{(1+ 8(\lambda s)^{-1})}]/4\}d_n;$$

$d^2_{n+1} \leq \lambda s(d_{n+1}[d_n + d_{n+1}]/2)$ implies $d_{n+1} \leq [\lambda s/(2 - \lambda s)]d_n$.

These four outcomes together imply

$$d_{n+1} \leq \max\{\sqrt{\lambda}, \lambda, \lambda s[1+ \sqrt{(1+ 8(\lambda s)^{-1})}]/4, \lambda s/(2 - \lambda s)\}d_n = \beta d_n,$$

where $\beta := \lambda s[1+ \sqrt{(1+ 8(\lambda s)^{-1})}]/4$. Notice that $0 < \beta < 1$ and $\beta s < 1$. So, by Lemma 3.1, $\{fx_n\}$ is a Cauchy sequence. Now let $f(Y)$ be a complete subspace of X. Then the sequence $\{fx_n\}$ has a limit in $f(Y)$. Call it u. Hence, there exists a point $z \in Y$ such that $fz = u$. Since $\{fx_n\}$ converges to fz,

$$d(fx_n, Px_n) \leq d(fx_n, fx_{n+1}) \text{ implies that } d(fx_n, Px_n) \to 0 \text{ as } n \to \infty.$$

By Lemma 2.1 (iii) and (2),

$$\begin{aligned} d^2(fx_{n+1}, Pz) &\leq q^{-k} H^2(Px_n, Pz) \\ &\leq q^{1-k}.\max\{d^2(fx_n, fz), d(fx_n, fz).d(fx_n, Px_n), d(fx_n, fz).d(fz, Pz), \\ &\quad d(fx_n, fz).[d(fx_n, Pz) + d(fz, Px_n)]/2, d(fx_n, Px_n).d(fz, Pz), \\ &\quad d(fx_n, Px_n).[d(fx_n, Pz) + d(fz, Px_n)]/2, \\ &\quad d(fz, Pz).[d(fx_n, Pz) + d(fz, Px_n)]/2, d(fx_n, Pz).d(fz, Px_n)\}. \end{aligned}$$

Making $n \to \infty$, $d(fz, Pz) \leq \lambda d(fz, Pz)$.

This yields $fz \in Pz$, since Pz is closed and $\lambda < 1$. This argument applies to the case when $P(Y)$ is a complete subspace of X, since $P(Y) \subseteq f(Y)$.

This proves (I) and (II).

For $n < m$,

$$\begin{aligned} d(fx_n, fx_m) &\leq sd(fx_n, fx_{n+1}) + s^2 d(fx_{n+1}, fx_{n+2}) + \cdots + s^{m-n-2}[d(fx_{m-2}, fx_{m-1}) \\ &\quad + d(fx_{m-1}, fx_m)] \\ &< s\beta^n(1 + s\beta + s^2\beta^2 + \cdots)d(fx_0, fx_1) \\ &= [s\beta^n/(1 - s\beta)]d(fx_0, fx_1). \end{aligned}$$

This in the limit $(m \to \infty)$ yields (III). □

Now we extend Theorem 3.1 to the setting of a pair of multivalued maps and a single-valued map on Y with values in a b-metric space X.

Theorem 3.2. *Let $P, Q : Y \to CL(X)$ and $f : Y \to X$ such that $P(Y) \cup Q(Y) \subseteq f(Y)$ and the following holds for all $x, y \in Y$:*

$$H^2(Px, Qy) \le q.\max\{d^2(fx, fy), d(fx, fy).d(fx, Px), d(fx, fy).d(fy, Qy), d(fx, fy).[d(fx, Qy) + d(fy, Px)]/2, d(fx, Px).d(fy, Qy), d(fx, Px).[d(fx, Qy) + d(fy, Px)]/2, d(fy, Qy).[d(fx, Qy) + d(fy, Px)]/2, d(fx, Qy).d(fy, Px)\},$$

where $0 < q < 1$. If one of $P(Y)$, $Q(Y)$ or $f(Y)$ is a complete subspace of X, then $fx \in Px \cap Qx$ has a solution. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that

(I) $fx_{2n+1} \in Px_{2n}$, $fx_{2n+2} \in Qx_{2n+1}$, $n = 0, 1, \cdots;$

(II) the sequence $\{fx_n\}$ converges to fz for some $z \in Y$, and $fz \in Pz \cap Qz$;

(III) $d(fx_n, fz) \le [s\beta^n/(1 - s\beta)]d(fx_0, fx_1)$.

Proof. It may be completed following the proofs of Theorems 3.1 and 3.3. □

Assume that $0 < q, k < 1$ and $\alpha := \max\{q^{1-k}, sq^{1-k}/(2 - sq^{1-k})\}$.

Theorem 3.3. *Let Y be an arbitrary nonempty set and (X, d) a b-metric space. Let $P, Q : Y \to CL(X)$ and $f : Y \to X$ such that $P(Y) \cup Q(Y) \subseteq f(Y)$ and the condition (1) for all $x, y \in Y$. If $sq^{1-k} < 1$, $\alpha s < 1$, and one of $P(Y)$, $Q(Y)$ or $f(Y)$ is a complete subspace of X, then $fx \in Px \cap Qx$ has a solution. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that*

(I) $fx_{2n+1} \in Px_{2n}$ and $fx_{2n+2} \in Qx_{2n+1}$, $n = 0, 1, \cdots;$

(II) the sequence $\{fx_n\}$ converges to fz for some $z \in Y$, and $fz \in Pz \cap Qz$;

(III) $d(fx_n, fz) \le [s\alpha^n/(1 - s\alpha)]d(fx_0, fx_1)$.

Proof. Pick $x_0 \in Y$. Notice that $q^{-k} > 1$ since $0 < q, k < 1$. We construct sequences $\{x_n\}$ in Y and $\{fx_n\}$ in X in the following manner. Since $P(Y) \subseteq f(Y)$, we can find a point

$x_1 \in Y$ such that $fx_1 \in Px_0$. Noting that $Q(Y)$ is also a subspace of $f(Y)$, we, for a suitable point $x_2 \in Y$, can choose a point $fx_2 \in Qx_1$ such that

$$d(fx_1, fx_2) \le q^{-k} H(Px_0, Qx_1).$$

We remark that such a choice is possible by Lemma 2.2. In general, we can choose a sequence $\{x_n\}$ in Y such that

$$fx_{2n+1} \in Px_{2n},\ fx_{2n+2} \in Qx_{2n+1},\ fx_{2n+3} \in Px_{2n+2}$$

and

$$d(fx_{2n+1}, fx_{2n+2}) \le q^{-k} H(Px_{2n}, Qx_{2n+1}),$$
$$d(fx_{2n+2}, fx_{2n+3}) \le q^{-k} H(Qx_{2n+1}, Px_{2n+2}).$$

Taking $y_n := fx_n$, $d_n := d(y_n, y_{n+1})$ and $\lambda := q^{1-k}$, by (1),

$$d_{2n+1} = d(fx_{2n+1}, fx_{2n+2}) \le \lambda.\max\{d_{2n}, d_{2n}, d_{2n+1}, [d(y_{2n}, y_{2n+2}) + 0]/2\}$$
$$\le \lambda.\max\{d_{2n}, d_{2n+1}, s[d_{2n} + d_{2n+1}]/2\},$$

giving $d_{2n+1} \le \alpha d_{2n}$, where $\alpha = \max\{\lambda, \lambda s/(2 - \lambda s)\}$.

Similarly, by (1),

$$d_{2n+2} \le q^{-k} H(Px_{2n+2}, Qx_{2n+1})$$
$$\le \lambda.\max\{d_{2n+1}, d_{2n+2}, d_{2n+1}, [0 + d(y_{2n+1}, y_{2n+3})]/2\},$$
$$\le \lambda.\max\{d_{2n+1}, d_{2n+2}, s[d_{2n+1} + d_{2n+2}]/2\},$$

giving $d_{2n+2} \le \alpha d_{2n+1}$.

Thus, in general, $d_{n+1} \le \alpha d_n$, $n = 0, 1, \cdots$.

Note that $0 < \alpha < 1$, and by hypothesis $\alpha s < 1$. So, by Lemma 3.1, $\{y_n\}$ is a Cauchy sequence. If we assume that $f(Y)$ is a complete subspace of X, then the sequence $\{y_n\}$ and its subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ have a limit in $f(Y)$. Call it u. Then there exists a point $z \in Y$ such that $fz = u$. By (1),

$$d(fx_{2n+2}, Pz) \le H(Qx_{2n+1}, Pz) = H(Pz, Qx_{2n+1})$$
$$\le q.\max\{d(fz, fx_{2n+1}), d(fz, Pz), d(fx_{2n+1}, Qx_{2n+1}), [d(fz, Qx_{2n+1}) + d(fx_{2n+1}, Pz)]/2\}$$
$$\le q.\max\{d(fz, fx_{2n+1}), d(fz, Pz), d(fx_{2n+1}, fx_{2n+2}), [d(fz, fx_{2n+2}) + d(fx_{2n+1}, Pz)]/2\}.$$

Making $n \to \infty$, $d(fz, Pz) \le q d(fz, Pz)$.

This gives $fz \in Pz$, since $0 < q < 1$ and Pz is closed. Similarly $fz \in Qz$. Thus $fz \in Pz \cap Qz$.

The above argument applies to the case when P(Y) or Q(Y) is a complete subspace of X, since P(Y) and Q(Y) are contained in f(Y). This proves (I) and (II). The proof of the last part is analogous to that of Theorem 3.1 (III). □

Corollary 3.1. *Let $P : Y \to CL(X)$ and $f : Y \to X$ such that $P(Y) \subseteq f(Y)$ and (SK) (cf. Th. 1.1) holds for all $x, y \in Y$. If one of P(Y) or f(Y) is a complete subspace of X, then $fx \in Px$ has a solution. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that conclusions (I), (II) of Theorem 3.1 and the conclusion (III) of Theorem 3.3 hold.*

Proof. It comes from Theorem 3.3 when P = Q. □

We remark that Corollary 3.1 is an extension of Theorem 1.1 to b-metric spaces. Certain results of Czerwik [6, 7] and Singh et al. [32] are particular cases of the above corollary.

4. Fixed Point Theorems

We apply coincidence theorems of the previous section to study solutions of $x = fx \in Px$, $x \in Px$, $x = fx \in Px \cap Qx$ and $x \in Px \cap Qx$, for $P, Q : X \to CL(X)$ and $f : X \to X$.

Theorem 4.1. *Let all the hypotheses of Theorem 3.1 be satisfied with Y = X. If f and P are (IT)-commuting just at a coincidence point z (say) of f and P, and if u = fz is fixed point of f, then u is a common fixed point of f and P.*

Proof. It comes from Theorem 3.1 that there exist points $z, u \in X$ such that

$$u = fz \in Pz.$$

If u is a fixed point of u = fu and f, P are (IT)-commuting at z then

$$u = fu = ffz \in fPz \subseteq Pfz = Pu.$$

This completes the proof. □

Theorem 4.2. *Let all the hypotheses of Theorem 3.2 be satisfied with Y = X. If f is (IT)-commuting with each of P and Q at their common coincidence point z, and if u = fz is fixed point of f, then f, P and Q have a common fixed point, i.e.,*

$$u = fu \in Pu \cap Qu.$$

Proof. It comes from Theorem 3.2 that there exist $z, u \in X$ such that

$u = fz \in Pz$ and $u = fz \in Qz$. Since $u = fu$, the (IT)-commutativity of f and P implies that
$u = fu = ffz \in fPz \subseteq Pfz = Pu$. Similarly $u = fu \in Qu$. So $u = fu \in Pu \cap Qu$. This completes the proof. □

Theorem 4.3. *Let all the hypotheses of Theorem 3.3 be satisfied with $Y = X$. If f is (IT)-commuting with each of P and Q at one of their common coincidences z (say), and if $u = fz$ is a fixed point of f, then f, P and Q have a common fixed point, i.e., $u = fu \in Pu \cap Qu$.*

Proof. It comes from Theorem 3.3 that there exist points $z, u \in X$ such that

$u = fz \in Pz \cap Qz$. The rest part of the proof is now evident. □

Now we derive some corollaries.

Corollary 4.1. *Let (X, d) be a complete b-metric space and $P, Q : X \to CL(X)$ such that $H(Px, Qy) \leq q.\max\{d(x, y), d(x, Px), d(y, Qy), [d(x, Qy) + d(y, Px)]/2\}$ for all $x, y \in X$, where $0 < q, k < 1$, $sq^{1-k} < 1$ with $\alpha s < 1$. Then the functional inclusion $x \in Px \cap Qx$ has a solution.*

Proof. It comes from Theorem 3.3 with $Y = X$ when f = is the identity map on X. □

Corollary 4.2. *Let (X, d) be a complete b-metric space and $P, Q : X \to CL(X)$ such that*

$$\begin{aligned} H^2(Px, Qy) \leq q.\max\{&d^2(x, y), d(x, y).d(x, Px), d(x, y).d(y, Qy),\\ &d(x, y).[d(x, Qy) + d(y, Px)]/2,\\ &d(x, Px).d(y, Qy), d(x, Px).[d(x, Qy) + d(y, Px)]/2,\\ &d(y, Qy).[d(x, Qy) + d(y, Px)]/2, d(x, Qy).d(y, Px)\},\end{aligned}$$

where $0 < q, k < 1$, $sq^{1-k} < 1$ with $\beta s < 1$. Then $x \in Px \cap Qx$ has a solution.

Proof. It comes from Theorem 3.2 with $Y = X$ when f = is the identity map on X. □

The following result is an extension of the main result of Ciric [3] and Theorem 1.1 with f the identity map on X.

Corollary 4.3. *Let (X, d) be a complete b-metric space and $P : X \to CL(X)$ such that*

$$H(Px, Py) \leq q.\max\{d(x, y), d(x, Px), d(y, Py), [d(x, Py) + d(y, Px)]/2\} \quad (C\text{-}1)$$

for all $x, y \in X$, where $0 < q, k < 1, sq^{1-k} < 1$ with $\alpha s < 1$. Then $x \in Px$ has a solution.

Proof. It comes from Corollary 4.1 with P = Q. □

Ciric [3] was the first to study the contraction (C-1) in a metric space. Using a similar condition for a pair of multivalued maps in a metric space, Khan [17] obtained some interesting fixed point theorems in metric spaces. We remark that Corollary 4.3 is an improvement in respect of the statement of a main result of Singh et al. [32, Th. 4.1]. Further, the above corollaries improve and extend several fixed point theorems for multivalued maps in metric and b-metric spaces (see, for instance, [1], [5], [6, 7], [17] and [23]).

The following question merits attention: Does the Corollary 4.3 hold when (C-1) is replaced by

$$H(Px, Py) \leq q.\max\{d(x, y), d(x, Px), d(y, Py), d(x, Py), d(y, Px)\}. \quad (C\text{-}2)$$

We remark that (C-2) is the main contraction condition due to Ciric [4] when X is a metric space and P is a single-valued map on X.

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