Tamsui Oxford Journal of Mathematical Sciences **24(4)** (2008) 381-388
Aletheia University

On the Hyers-Ulam Stability of a System of Euler Differential Equations of First Order *

Soon-Mo Jung†, Byungbae Kim‡
Mathematics Section, College of Science and Technology
Hong-Ik University, 339-701 Chochiwon, Korea

and

Themistocles M. Rassias§
Department of Mathematics, National Technical University of Athens
Zografou Campus, 15780 Athens, Greece

Received May 26, 2008, Accepted May 26, 2008.

Abstract

In this paper, we prove the Hyers-Ulam stability of a special type of systems of Euler differential equations of first order.

Keywords and Phrases: *Hyers-Ulam stability, Euler differential equation, Matrix method.*

---

*2000 *Mathematics Subject Classification.* Primary: 26D10; secondary: 34A40, 39B82.
†E-mail: smjung@hongik.ac.kr
‡E-mail: bkim@hongik.ac.kr
§E-mail: trassias@math.ntua.gr

1. Introduction

Assume that X is a normed space over a scalar field $\mathbb{K}$ and that I is an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Let a_0, a_1, …, $a_n : I \to \mathbb{K}$ be given continuous functions, let $g : I \to X$ be a given continuous function, and let $y : I \to X$ be any n times continuously differentiable function satisfying the inequality

$$\|a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) + g(t)\| \le \varepsilon$$

for all $t \in I$ and for a given $\varepsilon > 0$. If there exists a function $y_0 : I \to X$ satisfying

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) + g(t) = 0$$

and $\|y(t) - y_0(t)\| \le K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [2, 3, 4, 7, 9].

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of differential equations: They proved in [1] that if a differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality $|y'(t) - y(t)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a differentiable function $y_0 : I \to \mathbb{R}$ satisfying $y_0'(t) = y_0(t)$ and $|y(t) - y_0(t)| \le 3\varepsilon$ for any $t \in I$.

This result of Alsina and Ger has been generalized by Takahasi, Miura and Miyajima. They proved in [8] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$. Miura, Miyajima and Takahasi [6] investigated the Hyers-Ulam stability of nth order linear differential equation with complex coefficients. Furthermore, they proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function.

Now, suppose we are given a system of first order Euler differential equations as follows:

$$\begin{cases} ty_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + \cdots + a_{1n}y_n(t) + b_1(t), \\ ty_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + \cdots + a_{2n}y_n(t) + b_2(t), \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ ty_n'(t) = a_{n1}y_1(t) + a_{n2}y_2(t) + \cdots + a_{nn}y_n(t) + b_n(t). \end{cases} \tag{1}$$

This system may be written in a simple matrix notation

$$t\vec{y}\,'(t) = \mathbf{A}\vec{y}(t) + \vec{b}(t)$$

if we set

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \vec{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

In this paper, we will adopt the idea of [5] and prove the Hyers-Ulam stability of the system (1) of Euler differential equations of first order. More precisely, we prove that if a continuously differentiable vector function $\vec{y} : (0,\infty) \to \mathbb{C}^n$ satisfies

$$\|t\vec{y}\,'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t)\|_n \leq \varepsilon$$

for all $t \in (0,\infty)$, where $\|\cdot\|_n$ is a norm on $\mathbb{C}^n$, then there exists a differentiable vector function $\vec{y}_0 : (0,\infty) \to \mathbb{C}^n$ and a constant $K > 0$ such that

$$t\vec{y}_0'(t) = \mathbf{A}\vec{y}_0(t) + \vec{b}(t) \quad \text{and} \quad \|\vec{y}(t) - \vec{y}_0(t)\|_n \leq K\varepsilon$$

for all $t > 0$.

2. Main Result

Let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We notice that for $t \in \mathbb{R}$ and $\mathbf{A} \in \mathbb{C}^{n\times n}$ we use the notation $\mathbf{A}t$ (instead of $t\mathbf{A}$) for the scalar multiplication of t and $\mathbf{A}$. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, i.e., both norms obey

$$\|\mathbf{AB}\|_{n\times n} \leq \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n}, \quad \|\mathbf{A}\vec{x}\|_n \leq \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{2}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

Using the notations and assumptions given in the preceding section, recently Jung [5] proved the Hyers-Ulam stability of the system of first order linear differential equations with constant coefficients.

Theorem 1. *Let* $\vec{b} : \mathbb{R} \to \mathbb{C}^n$ *and* $\vec{y} : \mathbb{R} \to \mathbb{C}^n$ *be a continuous vector function and a continuously differentiable vector function, respectively. Assume that a continuous vector function* $\vec{v} : \mathbb{R} \to \mathbb{C}^n$ *defined by*

$$\vec{v}(t) = \vec{y}\,'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t)$$

satisfies

$$\|\vec{v}(t)\|_n \leq \varepsilon$$

for all $t \in \mathbb{R}$ *and for some* $\varepsilon > 0$. *Suppose that there exists a positive number* C *such that*

$$\int_{-\infty}^{\infty} |t-s|^k \left| [\mathbf{N}^{-1}\vec{v}(s)]_i \right| ds \leq k! C\varepsilon \|\mathbf{N}^{-1}\|_{n\times n}$$

for all $t \in \mathbb{R}$, *any* $k \in \{0, 1, \ldots, n\}$ *and any* $i \in \{1, \ldots, n\}$, *where* $\mathbf{N}$ *is a nonsingular matrix such that* $\mathbf{N}^{-1}\mathbf{A}\mathbf{N}$ *is a Jordan form matrix and* $[\mathbf{N}^{-1}\vec{v}(s)]_i$ *denotes the* i*-th component of* $\mathbf{N}^{-1}\vec{v}(s)$. *Then there exists a differentiable vector function* $\vec{y}_0 : \mathbb{R} \to \mathbb{C}^n$ *such that*

$$\vec{y}_0\,'(t) = \mathbf{A}\vec{y}_0(t) + \vec{b}(t) \quad \textit{and} \quad \|\vec{y}(t) - \vec{y}_0(t)\|_n \leq \varepsilon \|\mathbf{N}\|_{n\times n} \|\mathbf{N}^{-1}\|_{n\times n} \|\mathbf{B}\vec{e}\,\|_n$$

for all $t \in \mathbb{R}$, *where* $\vec{e} = (1, 1, \ldots, 1)^{tr} \in \mathbb{C}^n$ *and* $\mathbf{B}$ *is some matrix constructed by the eigenvalues of* $\mathbf{A}$.

Using the above theorem we can prove the Hyers-Ulam stability of the system (1) of first order linear differential equations with constant coefficients.

Theorem 2. *Let* $\vec{b} : \mathbb{R}^+ \to \mathbb{C}^n$ *and* $\vec{y} : \mathbb{R}^+ \to \mathbb{C}^n$ *be a continuous vector function and a continuously differentiable vector function, respectively. Assume that a continuous vector function* $\vec{v} : \mathbb{R}^+ \to \mathbb{C}^n$ *defined by*

$$\vec{v}(t) = t\vec{y}\,'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t)$$

satisfies

$$\|\vec{v}(t)\|_n \le \varepsilon \tag{3}$$

for all $t > 0$ and for some $\varepsilon > 0$. Suppose that there exists a positive number C such that

$$\int_{-\infty}^{\infty} |t-s|^k \left| [\mathbf{N}^{-1}\vec{v}(e^s)]_i \right| ds \le k! C\varepsilon \|\mathbf{N}^{-1}\|_{n\times n} \tag{4}$$

for all $t \in \mathbb{R}$, any $k \in \{0, 1, \ldots, n\}$ and any $i \in \{1, \ldots, n\}$, where $\mathbf{N}$ is a nonsingular matrix such that $\mathbf{N}^{-1}\mathbf{A}\mathbf{N}$ is a Jordan form matrix and $[\mathbf{N}^{-1}\vec{v}(s)]_i$ denotes the i-th component of $\mathbf{N}^{-1}\vec{v}(s)$. Then there exists a differentiable vector function $\vec{y}_0 : \mathbb{R}^+ \to \mathbb{C}^n$ such that

$$t\vec{y}_0'(t) = \mathbf{A}\vec{y}_0(t) + \vec{b}(t) \quad \textit{and} \quad \|\vec{y}(t) - \vec{y}_0(t)\|_n \le \varepsilon \|\mathbf{N}\|_{n\times n} \|\mathbf{N}^{-1}\|_{n\times n} \|\mathbf{B}\vec{e}\|_n$$

for all $t > 0$, where $\vec{e} = (1, 1, \ldots, 1)^{tr} \in \mathbb{C}^n$ and $\mathbf{B}$ is some matrix constructed by the eigenvalues of $\mathbf{A}$.

Proof. Let $t = e^\tau$ and $\vec{z} : \mathbb{R} \to \mathbb{C}^n$ given by $\vec{z}(\tau) = \vec{y}(e^\tau)$. Then

$$\vec{z}\,'(\tau) = \frac{d\vec{z}(\tau)}{d\tau} = e^\tau \frac{d\vec{y}}{dt}(e^\tau) = t\vec{y}\,'(t)$$

and

$$\vec{z}\,'(\tau) - \mathbf{A}\vec{z}(\tau) - \vec{b}(e^\tau) = t\vec{y}\,'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t) = \vec{v}(t) = \vec{v}(e^\tau)$$

and from the assumption (3),

$$\|\vec{v}(e^\tau)\|_n = \|\vec{v}(t)\|_n \le \varepsilon$$

for all $\tau \in \mathbb{R}$ and for some $\varepsilon > 0$.

By assumption, there exists a positive number C such that the inequality (4) holds for all $t \in \mathbb{R}$, any $k \in \{0, 1, \ldots, n\}$ and any $i \in \{1, \ldots, n\}$, where $\mathbf{N}$ is a nonsingular matrix such that $\mathbf{N}^{-1}\mathbf{A}\mathbf{N}$ is a Jordan form matrix and $[\mathbf{N}^{-1}\vec{v}(s)]_i$ denotes the i-th component of $\mathbf{N}^{-1}\vec{v}(s)$.

Therefore by Theorem 1, there exists a differentiable vector function $\vec{z}_0 : \mathbb{R} \to \mathbb{C}^n$ such that

$$\vec{z}_0'(\tau) = \mathbf{A}\vec{z}_0(\tau) + \vec{b}(e^\tau)$$

with

$$\|\vec{z}(\tau) - \vec{z}_0(\tau)\|_n \leq \varepsilon \|\mathbf{N}\|_{n\times n} \|\mathbf{N}^{-1}\|_{n\times n} \|\mathbf{B}\vec{e}\|_n$$

for all $\tau \in \mathbb{R}$, where $\vec{e} = (1, 1, \ldots, 1)^{tr} \in \mathbb{C}^n$.

Then the function $\vec{y}_0(t) = \vec{z}_0(\ln t)$ satisfies

$$\vec{y}_0'(t) = \frac{1}{t}\frac{dz_0}{d\tau}(\ln t) = \frac{1}{t}[\mathbf{A}\vec{z}_0(\ln t) + \vec{b}(e^{\ln t})]$$

or

$$t\vec{y}_0'(t) = \mathbf{A}\vec{y}_0(t) + \vec{b}(t)$$

with

$$\|\vec{y}(t) - \vec{y}_0(t)\|_n = \|\vec{z}(\ln t) - \vec{z}_0(\ln t)\|_n \leq \varepsilon \|\mathbf{N}\|_{n\times n} \|\mathbf{N}^{-1}\|_{n\times n} \|\mathbf{B}\vec{e}\|_n$$

for all $t > 0$, where $\vec{e} = (1, 1, \ldots, 1)^{tr} \in \mathbb{C}^n$ and $\mathbf{B}$ is some matrix constructed by the eigenvalues of $\mathbf{A}$. □

3. Some Example

Some of the most important matrix norms are induced by p-norms. For $1 \leq p \leq \infty$, the norm induced by the p-norm,

$$\|\mathbf{A}\|_p = \sup_{\vec{x}\neq\vec{0}} \frac{\|\mathbf{A}\vec{x}\|_p}{\|\vec{x}\|_p} \quad (\mathbf{A} \in \mathbb{C}^{n\times n}),$$

is called the matrix p-norm. For example, we get

$$\|\mathbf{A}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|, \quad \|\mathbf{A}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$

It is well known that the matrix p-norm, together with the p-norm, satisfies both conditions in (2).

Example 1. We consider a system of Euler differential equations of first order in the following form

$$\begin{cases} ty_1'(t) = y_1(t) + 2y_2(t) + b_1(t), \\ ty_2'(t) = 3y_1(t) + 2y_2(t) + b_2(t). \end{cases}$$

This system can be written in a matrix notation

$$t\vec{y}\,'(t) = \mathbf{A}\vec{y}(t) + \vec{b}(t),$$

where

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \vec{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

Assume that a continuous vector function $\vec{b} : (0,\infty) \to \mathbb{C}^2$ and a continuously differentiable vector function $\vec{y} : (0,\infty) \to \mathbb{C}^2$ satisfy

$$\|t\vec{y}\,'(t) - \mathbf{A}\vec{y}(t) - \vec{b}(t)\|_\infty \le \varepsilon$$

for all $t > 0$ and for some $\varepsilon \ge 0$. Since $\mathbf{A}$ has two distinct eigenvalues -1 and 4, we can choose a nonsingular matrix $\mathbf{N}$ and a diagonal matrix $\mathbf{J}$

$$\mathbf{N} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$$

such that $\mathbf{J} = \mathbf{N}^{-1}\mathbf{A}\mathbf{N}$. Furthermore, since $d = 2$, $m_1 = m_2 = 1$ and $p_{11} = p_{21} = 1$, it follows that

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

According to Theorem 2, there exists a differentiable vector function $\vec{y}_0 : (0,\infty) \to \mathbb{C}^2$ of the form

$$\vec{y}_0(t) = e^{\mathbf{A}\ln t}\vec{k} + e^{\mathbf{A}\ln t}\int_1^t e^{-\mathbf{A}\ln s}\vec{b}(s)\,\frac{ds}{s}$$

with

$$\|\vec{y}(t) - \vec{y}_0(t)\|_\infty \le 4\varepsilon$$

for all $t > 0$, where $\vec{k} \in \mathbb{C}^2$ is a constant.

References

[1] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function, *Journal of Inequalities and Applications* **2** (1998), 373–380.

[2] D. H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences, U.S.A.* **27** (1941), 222–224.

[3] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[4] D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44** (1992), 125–153.

[5] S.-M. Jung, Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients, *Journal of Mathematical Analysis and Applications* **320** (2006), 549–561.

[6] T. Miura, S. Miyajima and S.-E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, *Mathematische Nachrichten* **258** (2003), 90–96.

[7] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society* **72** (1978), 297–300.

[8] S.-E. Takahasi, T. Miura and S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, *Bulletin of the Korean Mathematical Society* **39** (2002), 309–315.

[9] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1964.