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On the Stability of Orthogonal Functional Equations *

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Abstract

We prove the generalized Hyers-Ulam stability of the orthogonal functional equations

$$2f(\frac{x+y}{2}) = f(x)+f(y), \tag{0.1}$$

$$2f(\frac{x+y}{2})+2f(\frac{x-y}{2}) = f(x)+f(y) \tag{0.2}$$

for all $x, y \quad (x \perp y)$, where $\perp$ is the orthogonality in the sense of Rätz.

Keywords and Phrases: *Generalized Hyers-Ulam stability, Orthogonally additive functional equation, Orthogonally quadratic functional equation, Orthogonality space, Jensen additive mapping, Jensen quadratic mapping.*

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1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space. This phenomenon may show the significance of study of orthogonal Cauchy equation.

G. Pinsker [16] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [24] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [10]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [21] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [22] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [21].

Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(O2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;

(O3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space equipped with a norm.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. It is remarkable to note that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space (see [1] and [2]).

In the recent decades, stability of functional equations have been investigated by many mathematicians (see [15]). The first author treating the stability of the Cauchy equation was D.H. Hyers [11] by proving that if f is a mapping from a normed space X into a Banach space satisfying $\|f(x+y) - f(x) - f(y)\| \leq \epsilon$ for some $\epsilon > 0$, then there is a unique additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \epsilon$.

R. Ger and J. Sikorska [9] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a function from an orthogonality space X into a real Banach space Y and $\|f(x + y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in X$ with $x \perp y$ and some $\epsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\epsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [23] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon$ for some $\epsilon > 0$, then there is a unique quadratic function $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\epsilon}{2}$. P. W. Cholewa [3] extended Skof's theorem by replacing X by an abelian group G. Skof's result was later generalized by S. Czerwik [4] in the spirit of Hyers-Ulam. The stability problem of functional equations has been extensively investigated by some mathematicians (see [5], [6], [12], [17]–[20], [26]).

The orthogonally quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ x \perp y$$

was first investigated by F. Vajzović [27] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [7], M. Fochi [8], M. Moslehian [13, 14] and G. Szabó [25] generalized this result.

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real and nonnegative real numbers, respectively.

Let X be an orthogonality space and Y a real Banach space. A mapping $f : X \to Y$ is called *orthogonally Jensen additive* if it satisfies the so-called orthogonally Jensen additive functional equation

$$2f(\frac{x+y}{2}) = f(x) + f(y) \tag{1.1}$$

for all $x, y \in X$ with $x \perp y$. A mapping $f : X \to Y$ is called *orthogonally Jensen quadratic* if it satisfies the so-called orthogonally Jensen quadratic functional equation

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y) \tag{1.2}$$

for all $x, y \in X$ with $x \perp y$.

In this paper, we prove the generalized Hyers-Ulam stability of the orthogonally Jensen additive functional equation (1.1) and of the orthogonally Jensen quadratic functional equation (1.2).

2. Stability of the orthogonally Jensen additive functional equation

Applying some ideas from [9] and [12], we deal with the conditional stability problem for

$$2f(\frac{x+y}{2}) = f(x) + f(y)$$

for all $x, y \in X$ with $x \perp y$.

Throughout this section, $(X, \perp)$ denotes an orthogonality normed space with norm $\| \cdot \|_X$ and $(Y, \| \cdot \|_Y)$ is a Banach space.

Theorem 2.1. *Let θ and p $(p < 1)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p) \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \frac{2^p\theta}{2 - 2^p}\|x\|_X^p \tag{2.2}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.1), we get

$$\|2f(\frac{x}{2}) - f(x)\|_Y \leq \theta\|x\|_X^p \tag{2.3}$$

for all $x \in X$, since $x \perp 0$. So

$$\|f(x) - \frac{1}{2}f(2x)\|_Y \leq \frac{2^p\theta}{2}\|x\|_X^p$$

for all $x \in X$. Hence

$$\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\|_Y \leq \frac{2^p\theta}{2}\sum_{k=n}^{m-1}\frac{2^{pk}}{2^k}\|x\|_X^p \tag{2.4}$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (2.4), we get the inequality (2.2).

It follows from (2.1) that

$$\begin{aligned}\|2T(\frac{x+y}{2}) - T(x) - T(y)\|_Y &= \lim_{n\to\infty}\frac{1}{2^n}\|2f(2^{n-1}(x+y)) - f(2^n x) - f(2^n y)\|_Y \\ &\leq \lim_{n\to\infty}\frac{2^{pn}\theta}{2^n}(\|x\|_X^p + \|y\|_X^p) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$2T(\frac{x+y}{2}) - T(x) - T(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $T : X \to Y$ is an orthogonally Jensen additive mapping.

Let $L : X \to Y$ be another orthogonally Jensen additive mapping satisfying (2.2). Then

$$\begin{aligned} \|T(x) - L(x)\|_Y &= \frac{1}{2^n}\|T(2^n x) = L(2^n x)\|_Y \\ &\leq \frac{1}{2^n}(\|f(2^n x) - T(2^n x)\|_Y + \|f(2^n x) - L(2^n x)\|_Y) \\ &\leq \frac{2^{p+1}\theta}{2 - 2^p} \cdot \frac{2^{pn}}{2^n}\|x\|_X^p, \end{aligned}$$

which tends to zero for all $x \in X$. So we have $T(x) = L(x)$ for all $x \in X$. This proves the uniqueness of T.

Theorem 2.2. *Let θ and $p \quad (p > 1)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{2.5}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \frac{2^p\theta}{2^p - 2}\|x\|_X^p \tag{2.6}$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$\|2^n f(\frac{x}{2^n}) - 2^m f(\frac{x}{2^m})\|_Y \leq \theta \sum_{k=n}^{m-1} \frac{2^k}{2^{pk}}\|x\|_X^p \tag{2.7}$$

for all nonnegative integers n, m with $n < m$. Thus $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (2.7), we get the inequality (2.6).

The rest of the proof is simialr to the proof of Theorem 2.1.

3. Stability of the orthogonally Jensen quadratic functional equation

In this section, we deal with the conditional stability problem for

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y)$$

for all $x, y \in X$ with $x \perp y$.

Throughout this section, $(X, \perp)$ denotes an orthogonality normed space with norm $\| \cdot \|_X$ and $(Y, \| \cdot \|_Y)$ is a Banach space.

Theorem 3.1. *Let θ and p $(p < 2)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{3.1}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{2^p\theta}{4 - 2^p}\|x\|_X^p \tag{3.2}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (3.1), we get

$$\|4f(\frac{x}{2}) - f(x)\|_Y \leq \theta\|x\|_X^p \tag{3.3}$$

for all $x \in X$, since $x \perp 0$. So

$$\|f(x) - \frac{1}{4}f(2x)\|_Y \leq \frac{2^p\theta}{4}\|x\|_X^p$$

for all $x \in X$. Hence

$$\|\frac{1}{4^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\|_Y \leq \frac{2^p\theta}{4}\sum_{k=n}^{m-1}\frac{2^{pk}}{4^k}\|x\|_X^p \tag{3.4}$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $Q : X \to Y$ defined by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (3.4), we get the inequality (3.2).

It follows from (3.1) that

$$\begin{aligned}
&\|2Q(\frac{x+y}{2}) + 2Q(\frac{x-y}{2}) - Q(x) - Q(y)\|_Y \\
&= \lim_{n\to\infty} \frac{1}{4^n} \|2f(2^{n-1}(x+y)) + 2f(2^{n-1}(x-y)) - f(2^n x) - f(2^n y)\|_Y \\
&\le \lim_{n\to\infty} \frac{2^{pn}\theta}{4^n} (\|x\|_X^p + \|y\|_X^p) = 0
\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$2Q(\frac{x+y}{2}) + 2Q(\frac{x-y}{2}) - Q(x) - Q(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $Q : X \to Y$ is an orthogonally Jensen quadratic mapping.

Let $L : X \to Y$ be another orthogonally Jensen quadratic mapping satisfying (3.2). Then

$$\begin{aligned}
\|Q(x) - L(x)\|_Y &= \frac{1}{2^n} \|Q(2^n x) = L(2^n x)\|_Y \\
&\le \frac{1}{4^n} (\|f(2^n x) - Q(2^n x)\|_Y + \|f(2^n x) - L(2^n x)\|_Y) \\
&\le \frac{2^{p+1}\theta}{4 - 2^p} \cdot \frac{2^{pn}}{4^n} \|x\|_X^p,
\end{aligned}$$

which tends to zero for all $x \in X$. So we have $Q(x) = L(x)$ for all $x \in X$. This proves the uniqueness of Q.

Theorem 3.2. *Let θ and p $(p > 2)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p) \tag{3.5}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{2^p\theta}{2^p - 4}\|x\|_X^p \tag{3.6}$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$\|4^n f(\frac{x}{2^n}) - 4^m f(\frac{x}{2^m})\|_Y \leq \theta \sum_{k=n}^{m-1} \frac{4^k}{2^{pk}}\|x\|_X^p \tag{3.7}$$

for all nonnegative integers n, m with $n < m$. Thus $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $Q : X \to Y$ defined by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (3.7), we get the inequality (3.6).

The rest of the proof is simialr to the proof of Theorem 3.1.

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