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On the Isometric Extension Problem: A Survey[*,†]

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Abstract

This is a short survey on the extension of isometric mappings between the unit spheres of Banach spaces. Some important results in the development of this problem are outlined in section 1. In the section 2, we raise the approximate isometric extension problem and non-surjective isometric extension problem for the future research. In the last part, we first discuss the metric properties of isometries between the unit spheres of general Banach spaces, which is a new idea to work on this topic. And the classical Mazur-Ulam theorem has been independently generalized in several directions by Th.M. Rassias, D. Tingley, A. Vogt in a number of papers.

Keywords and Phrases: *Extension, isometric mapping, Unit sphere.*

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1. Introduction

Recall that if (E, d_E) and (F, d_F) are metric spaces then a mapping V from E into F is said to be an isometry if

$$d_F(Vx_1, Vx_2) = d_E(x_1, x_2) \text{ for all } x_1, x_2 \in E.$$

It is well known that every surjective isometry between normed spaces must be linear [31, 46]. Later, P. Mankiewicz [30] studied the extension of isometries defined on an open connected subset, and he proved that an isometric mapping from an open connected subset of a normed space E onto an open connected subset of another normed space F can be extended to an affine isometry from E onto F. So far, the classical Mazur-Ulam theorem has been independently generalized in several directions by Th.M. Rassias [38]-[42], D. Tingley [45], A. Vogt [47] in a number of papers (e.g., see [21], [22], [32], [34]-[37]). Furthermore, D. Tingley [45] raised the following isometric extension problem:

Problem 1.1 (Isometric extension problem). Let E and F be normed spaces with the unit spheres $S_1(E)$ and $S_1(F)$ respectively. Assume that V_0 is an isometry from $S_1(E)$ onto $S_1(F)$. Does there exist an affine isometry V from E onto F such that V_0 is a restriction of V?

But D. Tingley only proved that for finite dimensional spaces E and F if V_0 is an isometric mapping from the unit sphere $S_1(E)$ onto $S_1(F)$ then $V_0(-x) = -V_0(x)$ for all $x \in S_1(E)$. By the similar method Y. Ma [29] obtained the same conclusion for the infinite dimensional strictly convex normed spaces and for the space ℓ^1. Actually, without the assumption of surjectivity, we can immediately give a negative answer from the following two classical counterexamples:

Example 1.2. Define an isometric mapping V_0 from $S_1(\ell^1_{(2)})$ into $S_1(\ell^1_{(3)})$ as follows:

$$V_0(\xi_1, \xi_2) = \begin{cases} (0, \xi_1, \xi_2), & \text{if } \xi_1 < 0; \\ (\xi_1, 0, \xi_2), & \text{if } \xi_1 \geq 0. \end{cases} \qquad \forall\, (\xi_1, \xi_2) \in S_1(\ell^1_2).$$

Then V_0 can be isometrically extended to the whole space by the natural way, however, it can not have any linear extension.

Example 1.3. Let an operator T_0 from $S_1(\ell^\infty_{(2)})$ into $S_1(\ell^\infty_{(3)})$ be defined by

$$T_0(x) = (\sin \xi_1, \xi_1, \xi_2), \quad \forall\, x = (\xi_1, \xi_2) \in S_1(\ell^\infty_{(2)}).$$

It is easy to see that T_0 is an isometry with $T_0(-x) = -T_0(x)$ for all $x \in S_1(\ell^\infty_{(2)})$, but it can not be linearly extended.

R. Wang studied the surjective isometries between unit spheres of $C_0^{(n)}(X)$ [53], $C_0(\Omega, E)$ [49, 50], $C_0(\Omega, \sigma)$ [52] and ℓ^p-sum of $C_0(\Omega, E)$ spaces for $1 \le p < \infty$ [51, 54], and he got the positive answer for the isometric extension problem. D. Zhan [61] discussed the isometric extension problem of the unit spheres of $L^p(\Omega, H)$ type spaces, where H is a Hilbert space, and got an affirmative answer. Y. Xiao and R. Wang [56] gave the representation theorem of any isometry V_0 from the unit sphere of an AM-space onto the unit sphere of another AM-space, and then there exists a real linear isometry V such that V_0 is a restriction of V. S. Banach [4] got some representation theorem for isometric mappings $V \in \mathcal{B}(c)$ and $V \in \mathcal{B}(\ell^p)$ for $p \ge 1$. G. Ding [7, 9, 10] gave the representation theorem of isometric operators between unit spheres of ℓ^p for any $1 \le p \le \infty$ and $p \ne 2$ as follows:

Theorem 1.4. *Suppose that V_0 is a surjective isometric mapping from the unit sphere $S_1[\ell^p(\Gamma)]$ onto the unit sphere $S_1[\ell^p(\Delta)]$ for $1 \le p \le \infty$ and $p \ne 2$, then there exists a permutation π from Δ onto Γ and a set of real numbers $\{\theta_\delta\}_{\delta\in\Delta}$ with $|\theta_\delta| = 1$ for all $\delta \in \Delta$ such that*

$$V_0(x) = \sum_{\delta\in\Delta} \theta_\delta \cdot \xi_{\pi(\delta)} d_\delta, \quad \forall\, x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma \in S_1[\ell^p(\Gamma)].$$

Furthermore, V_0 can be extended to a linear isometry from $\ell^p(\Gamma)$ onto $\ell^p(\Delta)$.

All of above works only considered the surjective mappings between the unit spheres of two normed spaces of the same type. However, G. Ding [8] discussed the extension of isometries between unit spheres of different type. If E is a normed space such that $sm.[S_1(E)]$ is dense in $S_1(E)$ and V_0 is a surjective isometry from $S_1(E)$ onto $S_1[C(\Omega)]$ which satisfies:

$$\|V_0(x) - |\lambda| V_0(x_0)\| \le \|x - |\lambda| x_0\|, \forall\, x \in S_1(E), \lambda \in \mathbb{R}, x_0 \in sm.\,[S_1(E)],$$

where $sm.[S_1(E)]$ is the set of all the smooth points of $S_1(E)$, then V_0 can be extended to be an isometric mapping defined on the whole space. Moreover,

G. An [1] and L. Li [25] generalized Ding's result. Later, X. Fang [16] and X. Yang [59] proved that for any normed spaces E and F if V_0 is a surjective isometry from $S_1(E)$ onto $S_1(F)$ and V_0 satisfies

$$\|V_0(x) - |\lambda|V_0(y)\| \leq \|x - |\lambda|y\|, \quad \forall\, x, y \in S_1(E), \lambda \in \mathbb{R}, \tag{1.1}$$

then V_0 can be extended to a linear isometry. At the same time, X. Fang and J. Wang [17] showed that any surjective isometry from the unit sphere of a normed space E onto the unit sphere of $C(\Omega)$ for a compact metric space Ω can always be linearly isometrically extended to the whole space.

Moreover, G. Ding [6] first discussed the isometric extension problem on Hilbert spaces without the assumption of the surjectivity of V_0, and he showed that a 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space. X. Fu [19] generalized Ding's result to a local 1-Lipschitz (that is, there exists $\delta_0 > 0$ such that $\|V_0(x_1) - V_0(x_2)\| \leq \|x_1 - x_2\|$ when $\|x_1 - x_2\| \leq \delta_0$). On the other hand, L. Zhang [62] gave the following counterexample such that the isometry between the unit spheres can not be extended to a linear isometry.

Example 1.5. L. Zhang's construction is an isometry V_0 from the unit sphere of $\ell^\infty_{(2)}$ into the unit sphere of $\ell^\infty_{(3)}$ defined by:

$$V_0(x) = \begin{cases} (1, \frac{3}{4}\xi_2, \xi_2), & \text{if} \quad \xi_1 = 1, \xi_2 \geq 0; \\ (-1, \xi_2, \frac{3}{4}\xi_2), & \text{if} \quad \xi_1 = -1, \xi_2 \geq 0; \\ (\xi_1, 1 - \frac{1}{4}\xi_1, 1), & \text{if} \quad \xi_1 \geq 0, \xi_2 = 1; \\ (\xi_1, 1, 1 + \frac{1}{4}\xi_1), & \text{if} \quad \xi_1 < 0, \xi_2 = 1; \\ (\xi_1, \xi_2, \xi_2), & \text{if} \quad \xi_2 < 0; \end{cases} \qquad \forall\, x = (\xi_1, \xi_2) \in \ell^\infty_{(2)}.$$

G. Ding [11] and J. Wang [48] found a condition under which an isometry between unit spheres of atomic AL^p-spaces can be linearly isometrically extended for $0 < p < \infty$ and $p \neq 2$. And their main results are:

Theorem 1.6. *If V_0 is an isometric mapping from the unit sphere $S_1[\ell^p(\Gamma)]$ into $S_1[\ell^p(\Delta)]$ for $p > 1$, then V_0 can be extended to a linear isometry if and only if for each $x \in S_1[\ell^p(\Gamma)]$ and $\gamma \in \Gamma$ there exists a real number η_γ such that $V_0(x)|_{supp.\,(V_0(e_\gamma))} = \eta_\gamma V_0(e_\gamma)$.*

For the general AL^p-spaces, Z. Hou and H. Xu [23] gave the representation theorem of the surjective isometric operator between the unit spheres of the

L^p type spaces for $p \geq 1$, and then the isometric extension probelm is an easy consequence; and moreover, Z. Hou and L. Zhang [24] generalized the main result of [23] to into mappings. More precisely, X. Yang [57] introduce a new class of mappings, named p-affinely orthogonal mappings, which is somewhat close to the linear operators between L^p spaces. From this definition, Yang showed that any into isometric mapping between the unit sphere of $L^p(\mu)$ and the unit sphere of $L^p(\nu, H)$ (where, $1 < p \neq 2$ and H is a Hilbert space) can be extended to be a real linear isometry on the whole space. When $p = 1$, G. Ding [14] studied the isometries from the unit sphere of L^1 into the unit sphere of any Banach space E and got a beautiful theorem.

Theorem 1.7. *Let V_0 be an isometric mapping from the unit sphere $S_1[L^1(\mu)]$ into the unit sphere $S_1(E)$ of a Banach space E, then V_0 can be extended to a (real) linear isometry defined on the whole space $L^1(\mu)$ if and only if the following condition holds:*

(*) For every x_1 and x_2 in $S_1[L^1(\mu)]$ and any ξ_1 and ξ_2 in $\mathbb{R}$, we have

$$\|\xi_1 V_0(x_1) + \xi_2 V_0(x_2)\| = 1 \quad \Longrightarrow \quad \xi_1 V_0(x_1) + \xi_2 V_0(x_2) \in V_0(S_1[L^1(\mu)]).$$

On the other hand, G. Ding and other people studied the ℓ^∞ type spaces. G. Ding [12] studied the extension of isometries from the unit sphere of $\ell^\infty_{(2)}$ into the unit sphere of $L^1(\mu)$ and give some sufficient conditions such that the isometric extension problem is true. R. Liu [28] proved that if V_0 is an isometry from the unit sphere $S_1[c_0(\Gamma)]$ into $S_1[\ell^\infty(\Delta)]$ and $\dfrac{\lambda_1 V_0(x_1) + \lambda_2 V_0(x_2)}{\|\lambda_1 V_0(x_1) + \lambda_2 V_0(x_2)\|} \in V_0(S_1[c_0(\Gamma)])$ whenever $x_1, x_2 \in S_1[c_0(\Gamma)]$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 V_0(x_1) + \lambda_2 V_0(x_2) \neq 0$, then V_0 can be linearly extended to be a isometry defined on the whole space. Furthermore, G. Ding defined a class of sequence spaces, named $\mathcal{L}^\infty(\Gamma)$-spaces, which contains all $\ell^\infty(\Gamma), c(\Gamma)$ and $c_0(\Gamma)$. G. Ding [13] got the following theorem:

Theorem 1.8. *Let E be a Banach space and V_0 be an isometric mapping from the unit sphere $S_1[\mathcal{L}^\infty(\Gamma)]$ into the unit sphere $S_1(E)$. Then V_0 can be extended to a linear isometry defined on the whole space $\mathcal{L}^\infty(\Gamma)$ if and only if the following conditions hold:*

(i) For every x_1 and x_2 in $S_1[\mathcal{L}^\infty(\Gamma)]$, λ_1 and λ_2 in $\mathbb{R}$,

$$\|\lambda_1 V_0 x_1 + \lambda_2 V_0 x_2\| = 1 \quad \Longrightarrow \quad \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \in V_0(S_1[\mathcal{L}^\infty(\Gamma)]).$$

(ii) For every mutual disjoint subsets $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ *of* Γ *if* $V_0(x) = \sum_{k=1}^{n} \lambda_k V_0(\chi_{\Gamma_k})$, *then* $x = \sum_{k=1}^{n} \lambda'_k \chi_{\Gamma_k} + x_0$, *where* $supp.x_0 \subset (\bigcup_{k=1}^{n} \Gamma_k)^c$.

More precisely, R. Liu [27] showed that Theorem 1.8 is true if condition (i) only holds. At the same paper, R. Liu considered the isometric extension problem in general case and got the following theorem:

Theorem 1.9. *Let* E *and* F *be two Banach spaces. Suppose that* V_0 *is a Lipschitz mapping from* $S_1(E)$ *into* $S_1(F)$ *with the constant* $K = 1$, *that is,* $\|V_0(x) - V_0(y)\| \leq \|x - y\|$ *for any* $x, y \in S_1(E)$. *Assume also that* V_0 *is a surjective mapping such that for any* $x, y \in S_1(E)$ *and* $r > 0$, *we have*

$$\|V_0(x) - r\,V_0(y)\| \wedge \|V_0(x) + r\,V_0(-y)\| \leq \|\,x - r\,y\| \tag{1.2}$$

and $\|V_0(x) - V_0(-x)\| = 2$. *Then* V_0 *can be extended to be a linear isometry of* E *onto* F.

Obviously, the conditions here are much weaker than (1.1). In [27], R. Liu also proved the isometric extension problem in the $C(\Omega)$-space, where Ω is a compact Hausdorff space, by showing a geometric representation of the isometries. That is, if Ω is a compact Hausdorff space and E is a real Banach space, suppose that V_0 is an isometric mapping from $S_1[C(\Omega)]$ into $S_1(E)$, then, for every $t \in \Omega$, there exists an $f_t \in E^*$ such that $\|f_t\| = 1$ and

$$f_t(V_0(x)) = x(t) \quad \text{for all } x \in S_1\big[C(\Omega)\big].$$

By this property, then it is easy to prove that V_0 can be extended to be a linear isometry if and only if V_0 satisfies the following condition:

(**) For every x_1 and x_2 in $S_1[C(\Omega)]$, any λ_1 and λ_2 in $\mathbb{R}$, we have

$$\|\lambda_1 V_0 x_1 + \lambda_2 V_0 x_2\| = 1 \quad \Longrightarrow \quad \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \in V_0(S_1[C(\Omega)]).$$

Please recall the well-known Kakutani Representation Theorem. Thus, we propose the following problem:

Problem 1.10. Let F be an $AM-$space, and E a real Banach space. Let V_0 be a surjective isometry from $S_1(F)$ onto $S_1(E)$. Is V_0 necessarily the restriction of a linear or affine transformation on F

R. Wang and A. Orihara [55] gave an representation of the surjective isometry between the unit spheres of ℓ^1-sum of strictly convex spaces, and got the affirmative answer of the isometric extension problem. And R. Liu [26] generalized their result and he discussed the extension of into isometries between the unit spheres of ℓ^β-sum of strictly convex normed spaces for $0 < \beta \leq 1$.

Furthermore, G. Ding and S. Huang [15] generalized the Mankiewicz's result to Fréchet spaces. G. An [2] gave the representation theorem of isometries between unit spheres of the metric linear space (ℓ^{β_n}). She showed that if $\{\beta_n\} \subset (0,1)$ and V_0 is a surjective isometry from the unit sphere of (ℓ^{β_n}) onto itself then there exist a sequence of signs $\{\theta_n\}$ and a permutation π of positive integers such that $\beta_n = \beta_{\pi(n)}$ for any n and such that

$$V_0(x) = \sum_{n=1}^{\infty} \theta_n \xi_{\pi(n)} e_n, \quad \forall\, x = \sum_{n=1}^{\infty} \xi_n e_n \in S_1[(\ell^{\beta_n})].$$

And X. Fu studied the isometries on s [20], and got a partial isometric extension problem and the similar Mazur-Ulam theorem in s.

2. Near-isometric extension problem and Non-surjective isometric extension problem

However, it is often difficult for us to obtain an exact isometry in the process of research, so we need to consider a more natural and useful case. Traced back to 1945, D. Hyers and S. Ulam [12] introduced the ϵ-isometries. Let E and F be metric spaces and $\epsilon \geq 0$. A mapping V from E into F is called an ϵ-isometry if

$$|d_F(Vx_1, Vx_2) - d_E(x_1, x_2)| \leq \epsilon \quad \text{for all } x_1, x_2 \in E.$$

Therefore, the corresponding extension problem for ϵ-isometries can be naturally formulated as the following approximate isometric extension problem (ref. [33], [40], [43], [44]):

Problem 2.1 (Approximate isometric extension problem). Let E and F be normed spaces and $\epsilon \geq 0$.

1. Does there exist a constant $K > 0$ such that, for any bijective ϵ-isometry $V_0 : S_1(E) \to S_1(F)$, there always exists a bijective $(K \cdot \epsilon)$-isometry $V : B_1(E) \to B_1(F)$ such that $V|_{S_1(E)} = V_0$?

2. If the answer to the previous question is affirmative, then find the optimal value of K.

In 2004, the following non-surjective isometric extension problem appeared in [48]:

Problem 2.2 (Non-surjective isometric extension problem)*.* Let $S_1(E)$ be the unit sphere of a Banach space E, and let $S_1(F)$ be the unit sphere of another Banach space F. Suppose that V_0 is an isometry from $S_1(E)$ into $S_1(F)$. Does there exist an isometry V from E into F such that $V|_{S_1(E)} = V_0$?

Recall that G. Ding actually first discussed this problem on Hilbert spaces in [6], where he showed that an isometry between the unit spheres of two Hilbert spaces can be isometrically extended to the whole space. X. Yang in [57] extended that any isometry between the unit spheres of $L^p(\mu)$ and $L^p(\nu, H)$ $(1 < p \neq 2, H$ is a Hilbert space) can be extended to be an isometry on the whole space, which gives an affirmative answer to the above problem.

3. Metric structure and extension properties of isometric operators between unit spheres

Notice that it is the first time to discuss the metric structure on isometries between unit spheres with respect to the extension property.

Throughout this section, we always assume that X and Y are real Banach spaces and denote by $\mathfrak{I}(X,Y)$ the set of all isometric operators from $S_1(X)$ into $S_1(Y)$, and define the natural distance of two elements $T_1, T_2 \in \mathfrak{I}$ by

$$d(T_1, T_2) = \sup_{x \in S_1(X)} \|T_1(x) - T_2(x)\|_Y. \tag{3.1}$$

Obviously, $(\mathfrak{I}(X,Y), d)$ is a complete metric space.

For any nonempty subsets $\mathfrak{I}_1, \mathfrak{I}_2 \subset \mathfrak{I}(X,Y)$, define

$$d(\mathfrak{I}_1, \mathfrak{I}_2) = \inf_{T_1 \in \mathfrak{I}_1, T_2 \in \mathfrak{I}_2} d(T_1, T_2). \tag{3.2}$$

Now we give the notations of some special subsets of $\mathfrak{I}$. The set of all isometric operators which can be extended to be a linear isometry of X into Y is denoted by $\mathfrak{I}_l$. The set of all isometric operators which can be extended to be an isometry $\widetilde{T}$ from X into Y with $\widetilde{T}(\theta_X) = \theta_Y$ is denoted by $\mathfrak{I}_i$. The set of all non-surjective isometries is denoted by $\mathfrak{I}_n$, and the set of all surjective isometric operators is denoted by $\mathfrak{I}_s$.

Proposition 3.1. *$\mathfrak{I}_l$ is closed in $\mathfrak{I}(X,Y)$.*

Proof. Since $\mathfrak{I}(X,Y)$ is a complete metric space, then suppose $(T_n)_{n=1}^{\infty} \subset \mathfrak{I}_l$ converge to some T in $\mathfrak{I}(X,Y)$, and $(\widetilde{T_n})_{n=1}^{\infty}$ are their corresponding linear isometric extension. It is easy to see that

$$\begin{aligned} d(T_n, T_m) &= \sup_{x \in S_1(X)} \|T_n(x) - T_m(x)\| \\ &= \sup_{x \in S_1(X)} \|\widetilde{T_n}(x) - \widetilde{T_m}(x)\| \\ &= \sup_{x \in S_1(X)} \|(\widetilde{T_n} - \widetilde{T_m})(x)\| \\ &= \|\widetilde{T_n} - \widetilde{T_m}\|_{B(X,Y)}, \end{aligned}$$

then the Cauchy sequence $(\widetilde{T_n})_{n=1}^{\infty}$ convergent to some linear isometry $\widetilde{T}$. Clearly T belongs to $\mathfrak{I}_l$ since it is the restriction of $\widetilde{T}$. □

Proposition 3.2. *If Y is a strictly convex Banach space, then $\mathfrak{I}_l = \mathfrak{I}_i$.*

Proof. Firstly, by the above definition, we have that $\mathfrak{I}_l \subseteq \mathfrak{I}_i$. On the other hand, if $T \in \mathfrak{I}_i$, since Y is strictly convex, then by Baker Theorem [3], the corresponding extension isometry $\widetilde{T}$ with $\widetilde{T}(\theta_X) = \theta_Y$ must be linear, and hence $\mathfrak{I}_i \subseteq \mathfrak{I}_l$. □

Proposition 3.3. *Let $T_1 \in \mathfrak{I}_s$ and $T_2 \in \mathfrak{I}_l \cap \mathfrak{I}_n$, then $d(T_1, T_2) \geq 1$. In particular, whenever both $\mathfrak{I}_s$ and $\mathfrak{I}_l \cap \mathfrak{I}_n$ are non-empty, we have $d(\mathfrak{I}_s, \mathfrak{I}_l \cap \mathfrak{I}_n) \geq 1$.*

Proof. Suppose, on the contrary, that there exist $T_1 \in \mathfrak{I}_s$ and $T_2 \in \mathfrak{I}_l \cap \mathfrak{I}_n$ such that $d(T_1, T_2) < 1$. By the hypothesis, T_2 can be extended to be a linear isometric operator $\widetilde{T}_2$ on the whole space X. It is easy to prove that $\widetilde{T}_2(X)$ is a closed subspace of Y. And since $T_2 \in \mathfrak{I}_n$, so

$$S_1(\widetilde{T}_2(X)) = \widetilde{T}_2(S_1(X)) = T_2(S_1(X)) \neq S_1(Y),$$

which implies that $\widetilde{T}_2(X)$ is a proper subspace of Y. Then, by Riesz Lemma, there is a $y_1 \in S_1(Y)$ such that

$$d(y_1, \widetilde{T}_2(X)) = \inf_{y \in \widetilde{T}_2(X)} \|y_1 - y\| > d(T_1, T_2).$$

Then, from $T_1 \in \mathfrak{I}_s$, we know that there must be some $x_1 \in S_1(X)$ such that $T_1(x_1) = y_1$. Thus,

$$\begin{aligned} d(T_1, T_2) &< d(y_1, \widetilde{T}_2(X)) = \inf_{y \in \widetilde{T}_2(X)} \|y_1 - y\| \\ &\leq \|y_1 - \widetilde{T}_2(x_1)\| = \|T_1(x_1) - T_2(x_1)\| \\ &\leq d(T_1, T_2), \end{aligned}$$

which is a contradiction. □

Proposition 3.4. *If* $\dim X = \dim Y < \infty$*, then* $\mathfrak{I}_n \cap \mathfrak{I}_i = \emptyset$.

Proof. Assume that $T \in \mathfrak{I}_n \cap \mathfrak{I}_i$, then T has a corresponding extension isometry $\widetilde{T}$. Since X and Y are complete and $\widetilde{T}$ is isometric, $\widetilde{T}(X)$ is closed. Clearly $\widetilde{T}$ is injective and continuous, then by Brouwer's Domain Invariant Theorem [5], we can get that $\widetilde{T}$ is an open map, so $\widetilde{T}(X)$ is also open, hence it is a clopen set. Y is connected, so $\widetilde{T}(X) = Y$. ¿From the definition $\widetilde{T}(\theta_X) = \theta_Y$ and $\widetilde{T}$ is linear we have that

$$T(S_1(X)) = \widetilde{T}(S_1(X)) = S_1(\widetilde{T}(X)) = S_1(Y),$$

which is a contradiction since $T \in \mathfrak{I}_n$. □

Proposition 3.5. *Suppose that* $T_1 \in \mathfrak{I}_s$ *and* $T_2 \in \mathfrak{I}_n \cap (\mathfrak{I}_i \setminus \mathfrak{I}_l)$*, then* $d(T_1, T_2) \geq 1$*. In particular, whenever both* $\mathfrak{I}_s$ *and* $\mathfrak{I}_n \cap (\mathfrak{I}_i \setminus \mathfrak{I}_l)$ *are non-empty, we have* $d(\mathfrak{I}_s, \mathfrak{I}_n \cap (\mathfrak{I}_i \setminus \mathfrak{I}_l)) \geq 1$.

Proof. Suppose, on the contrary, taht $T_1 \in \mathfrak{I}_s$ and $T_2 \in \mathfrak{I}_n \cap (\mathfrak{I}_i \setminus \mathfrak{I}_l)$, but $d(T_1, T_2) < 1$. Also we can assume that $\widetilde{T}_2$ is the extension isometric operator of T_2 with $\widetilde{T}_2(\theta_X) = \theta_Y$.

Claim 1: The linear span of $\widetilde{T}_2(X)$ is dense in Y.

Indeed, suppose, on the contrary, that $\overline{span(\widetilde{T}_2(X))}$ is a proper subspace of Y, then by Riesz Lemma, there is a $y_0 \in Y \setminus \overline{span(\widetilde{T}_2(X))}$ such that $\|y_0\| = 1$ and $d(y_0, \overline{span(\widetilde{T}_2(X))}) > d(T_1, T_2)$. Then similar to the method in Proposition 3.3, by the hypothesis T_1 is surjective (that is, $y_0 = T_1(x_0)$ for some $x_0 \in S_1(X)$), so there must be some $x_1 \in S_1(X)$ such that $T_1(x_1) = y_1$. Therefore,

$$\begin{aligned} d(T_1, T_2) &< d(y_0, \overline{span(\widetilde{T}_2(X))}) \leq d(y_0, T_2(X)) \\ &\leq \|y_0 - \widetilde{T}_2(x_0)\| = \|T_1(x_0) - T_2(x_0)\| \\ &\leq d(T_1, T_2), \end{aligned}$$

which is a contradiction.

Claim 2: There exists a $\widetilde{y} \in S_1(Y)$ such that

$$d(\widetilde{y}, \widetilde{T_2}(S_1(X))) \geq 1.$$

Indeed, by the famous Theorem of T. Figiel [18], we can get that there is a linear continuous operator G from $\overline{span(\widetilde{T_2}(X))}$ into X satisfying $\|G\| = 1$ and $G(\widetilde{T_2}(x)) = x$ for all $x \in X$. Then from the hypothesis $T_2 \in \mathfrak{I}_n \cap (\mathfrak{I}_i \setminus \mathfrak{I}_l)$ and $\widetilde{T_2}$ is not linear, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in X$ such that

$$\widetilde{T_2}(\lambda_1 x_1 + \lambda_2 x_2) \neq \lambda_1 \widetilde{T_2}(x_1) + \lambda_2 \widetilde{T_2}(x_2).$$

Let $y_1 = \widetilde{T_2}(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 \widetilde{T_2}(x_1) - \lambda_2 \widetilde{T_2}(x_2)$, then by $G \circ \widetilde{T_2} = id_X$ we have that

$$\begin{aligned} G(y_1) &= G(\widetilde{T_2}(\lambda_1 x_1 + \lambda_2 x_2)) - \lambda_1 G(\widetilde{T_2}(x_1)) - \lambda_2 G(\widetilde{T_2}(x_2)) \\ &= \lambda_1 x_1 + \lambda_2 x_2 - \lambda_1 x_1 - \lambda_2 x_2 = \theta_Y. \end{aligned}$$

When we choose a real number $r > 0$ such that $\|ry_1\| = 1$, for any $x \in S_1(X)$, we have taht

$$\begin{aligned} \|ry_1 - \widetilde{T_2}(x)\| &\geq \|G(ry_1 - \widetilde{T_2}(x))\| \\ &= \|G(ry_1) - G(\widetilde{T_2}(x))\| \\ &= \|x\| = 1, \end{aligned}$$

which implies that $d(ry_1, \widetilde{T_2}(S_1(X))) \geq 1$. Therefore, by the above claims and the assumption that T_1 is surjective, we obtain that

$$\begin{aligned} 1 > d(T_1, T_2) &= \sup_{x \in S(X)} \|T_1(x) - T_2(x)\| \\ &\geq \|T_1(T_1^{-1}(ry_1)) - T_2(T_1^{-1}(ry_1))\| \\ &= \|ry_1 - T_2(T_1^{-1}(ry_1))\| \geq 1, \end{aligned}$$

which is a contradiction. □

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