Tamsui Oxford Journal of Mathematical Sciences **24(4)** (2008) 445-456
Aletheia University

Ger Type Stability of the First Order Linear Differential Equation $y'(t) = h(t)y(t)^*$

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Received June 17, 2008, Accepted June 17, 2008.

Abstract

Let $0 \in I \subseteq \mathbb{R}$ be an open interval and let $C^1(I, \mathbb{C}^\times)$ be the set of all continuously differentiable functions from I to $\mathbb{C}^\times$, where $\mathbb{C}^\times$ is the set of all non-zero complex numbers. If $h\colon I \to \mathbb{C}^\times$ is a continuous

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[*]2000 *Mathematics Subject Classification.* 34K20, 26D10.
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function with $M = \sup_{t\in I} |\int_0^t |h(s)|\,ds| < \infty$, then for each $\varepsilon \geq 0$ and $f \in C^1(I, \mathbb{C}^\times)$ satisfying

$$\left|\frac{f'(t)}{h(t)f(t)} - 1\right| \leq \varepsilon \qquad (\forall t \in I)$$

there exists $f_0 \in C^1(I, \mathbb{C}^\times)$ such that ${f_0}'(t) = h(t)f_0(t)$ and that

$$\max\left\{\left|\frac{f(t)}{f_0(t)} - 1\right|, \left|\frac{f_0(t)}{f(t)} - 1\right|\right\} \leq e^{M\varepsilon} - 1$$

for all $t \in I$. We give an example that the constant $e^{M\varepsilon} - 1$ can not be improved in general. We also prove that the assumption $\sup_{t\in I} |\int_0^t |h(s)|\,ds| < \infty$ is essential for Ger type stability.

Keywords and Phrases: *Exponential functions, Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ger type stability*

1. Introduction

It seems that the stability problem of functional equations had been first raised by S. M. Ulam (cf. [24, Chapter VI]). "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism? (An ε-automorphism of G means a transformation f of G into itself such that $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all $x, y \in G$.)"

D. H. Hyers [6] gave an affirmative answer to the problem as follows.

Theorem A. *Suppose that E_1 and E_2 are two real Banach spaces and $f\colon E_1 \to E_2$ is a mapping. If there exists $\varepsilon \geq 0$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear, where $\mathbb{R}$ is the real number field.

This result is called the *Hyers-Ulam stability* of the *additive* Cauchy equation $g(x + y) = g(x) + g(y)$. Here we note that Hyers [6] calls any solution of this equation a "linear" function or transformation. Hyers considered only *bounded* Cauchy difference $f(x + y) - f(x) - f(y)$. T. Aoki [2] introduced unbounded one and generalized a result [6, Theorem 1] of Hyers obtaining the stability of additive mapping. Th.M. Rassias [16], who independently introduced the unbounded Cauchy difference, was the first to prove the stability of the linear mapping between Banach spaces. The concept of the Hyers-Ulam-Rassias stability was originated from Rassias' paper [16] for the stability of the linear mapping. Rassias [16] generalized Hyers' result as follows:

Theorem B. *Suppose that E_1 and E_2 are two real Banach spaces and $f\colon E_1 \to E_2$ is a mapping. If there exist $\varepsilon \geq 0$ and $0 \leq p < 1$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there is a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\,\|x\|^p$$

for all $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear.

This result is, what is called, the *Hyers-Ulam-Rassias stability* of the additive Cauchy equation $g(x + y) = g(x) + g(y)$. The result of Hyers is just the case where $p = 0$. So, the result of Rassias is a generalization to the case where $0 \leq p < 1$: It should be mentioned that it allows Cauchy difference to be unbounded. During the 27th International Symposium on Functional Equations, Rassias raised the problem whether a similar result holds for $1 \leq p$. Z. Gajda [3, Theorem 2] proved that Theorem B is valid for $1 < p$; In the same paper [3, Example], he also gave an example to show that a similar result to the above does not hold for $p = 1$. Later, Th.M. Rassias and P. Šemrl [17, Theorem 2] gave another counter example for $p = 1$. Note that if $p < 0$, then $\|0\|^p$ is obviously meaningless. However, if we assume that $\|0\|^p$ means ∞, then with minor changes in the proof given in [16], we can prove that the result is also valid for $p < 0$. Thus, the Hyers-Ulam-Rassias stability of the additive Cauchy equation holds for all $p \in \mathbb{R} \setminus \{1\}$.

In connection with the stability of exponential functions, C. Alsina and R. Ger [1] remarked that the differential equation $y' = y$ has the Hyers-Ulam stability: more explicitly, if I is an open interval, $\varepsilon > 0$ and $f\colon I \to \mathbb{R}$ is a differentiable function satisfying $|f'(t) - f(t)| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $f_0\colon I \to \mathbb{R}$ such that ${f_0}'(t) = f_0(t)$ and that $|f(t) - f_0(t)| \le 3\varepsilon$ for all $t \in I$. S.-E. Takahasi, T. Miura and S. Miyajima [23] considered the Banach space valued differential equation $y' = \lambda y$, where λ is a complex constant. Then they proved the Hyers-Ulam stability of $y' = \lambda y$ under the condition that $\operatorname{Re}\lambda \neq 0$. This result is generalized by Miura, Miyajima and Takahasi [9]. They considered the Banach space valued n-th order linear differential equation with constant coefficients.

Let $\mathbb{C}$ be the complex number field and let $\mathbb{C}^\times$ the set of all non-zero complex numbers. Taking the group structure of $\mathbb{C}^\times$ into account, R. Ger and P. Šemrl [4] considered the following inequality

$$\left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \le \varepsilon \qquad (\forall x, y \in S)$$

for a mapping $f\colon S \to \mathbb{C}^\times$, where $(S, +)$ is a semigroup. If $0 \le \varepsilon < 1$ and if $(S, +)$ is a cancellative abelian semigroup, then they proved that there is a unique function $f_0\colon S \to \mathbb{C}^\times$ such that $f_0(x+y) = f_0(x)f_0(y)$ for all $x, y \in S$ and that

$$\max\left\{\left|\frac{f(x)}{f_0(x)} - 1\right|, \left|\frac{f_0(x)}{f(x)} - 1\right|\right\} \le \sqrt{1 + \frac{1}{(1-\varepsilon)^2} - 2\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}}$$

for all $x \in S$. The stability phenomena of this kind is called *Ger type stability.*

Ger type stability of first order linear differential equation $y' = \lambda y$ for entire functions was studied in [11], where $\lambda \in \mathbb{C}^\times$. In this paper, we will consider Ger type stability of the first order linear differential equation $y' = hy$ for a continuous function h from an open interval I into $\mathbb{C}^\times$. Just for the sake of simplicity, we only consider the case when $0 \in I$.

2. Main Results

Lemma 2.1. *Let $f\colon I \to \mathbb{C}$ be a differentiable function, and let $\varphi\colon I \to \mathbb{C}$ be a continuous function. Each of the following conditions are equivalent.*

(i) $f'(t) = \varphi(t)f(t)$ *for every* $t \in I$.

(ii) $f(t) = f(0) \exp \int_0^t \varphi(s)\, ds$ *for every* $t \in I$*.*

Proof. (i) $\Rightarrow$ (ii) Set $\tilde{\varphi}(t) = \exp \int_0^t \varphi(s)\, ds$ for each $t \in I$. We see that $\tilde{\varphi}'(t) = \varphi(t)\tilde{\varphi}(t)$. Thus, we have, for each $t \in I$, that

$$\left\{ \frac{f(t)}{\tilde{\varphi}(t)} \right\}' = \frac{f'(t)\tilde{\varphi}(t) - f(t)\tilde{\varphi}'(t)}{\tilde{\varphi}^2(t)} = \frac{f'(t) - f(t)\varphi(t)}{\tilde{\varphi}(t)} = 0,$$

and so $f(t)/\tilde{\varphi}(t)$ is a constant function. This implies that $f(t) = f(0)\tilde{\varphi}(t)$ for every $t \in I$.

(ii) $\Rightarrow$ (i) By a simple calculation, we have $f'(t) = \varphi(t)f(t)$ for each $t \in I$, and the proof is complete. □

Theorem 2.2. *Let* $h \colon I \to \mathbb{C}^\times$ *be a continuous function with*

$$M \overset{\text{def}}{=} \sup_{t \in I} \left| \int_0^t |h(s)|\, ds \right| < \infty.$$

Then to each $\varepsilon \geq 0$ *and* $f \in C^1(I, \mathbb{C}^\times)$ *satisfying*

$$\left| \frac{f'(t)}{h(t)f(t)} - 1 \right| \leq \varepsilon \qquad (\forall t \in I) \tag{1}$$

there exists $f_0 \in C^1(I, \mathbb{C}^\times)$ *such that* ${f_0}'(t) = h(t)f_0(t)$ *and that*

$$\max \left\{ \left| \frac{f(t)}{f_0(t)} - 1 \right|, \left| \frac{f_0(t)}{f(t)} - 1 \right| \right\} \leq e^{M\varepsilon} - 1 \tag{2}$$

for all $t \in I$*.*

Proof. Take $\varepsilon \leq 0$ and $f \in C^1(I, \mathbb{C}^\times)$ with (1). We will prove that there exists $f_0 \in C^1(I, \mathbb{C}^\times)$ such that ${f_0}'(t) = h(t)f_0(t)$ and (2) holds for all $t \in I$. To do this, set

$$g(t) \overset{\text{def}}{=} \frac{f'(t)}{h(t)f(t)} - 1 \qquad (\forall t \in I)$$

so that $|g(t)| \leq \varepsilon$ for each $t \in I$. We can write

$$f'(t) = (1 + g(t))h(t)f(t) \qquad (\forall t \in I).$$

If we apply Lemma 2.1, then we have

$$\begin{aligned} f(t) &= f(0)\exp\int_0^t (1+g(s))h(s)\,ds \\ &= f(0)\exp\int_0^t h(s)\,ds\,\exp\int_0^t g(s)h(s)\,ds \end{aligned}$$

for every $t \in I$. Set, for each $t \in I$, $f_0(t) = f(0)\exp\int_0^t h(s)\,ds$. We see that

$$f_0{}'(t) = h(t)f_0(t) \qquad \text{and} \qquad f(t) = f_0(t)\exp\int_0^t g(s)h(s)\,ds.$$

for every $t \in I$. Since $|g(t)| \le \varepsilon$ for each $t \in I$, we have

$$\begin{aligned} \left|\frac{f(t)}{f_0(t)} - 1\right| &= \left|\exp\int_0^t g(s)h(s)\,ds - 1\right| = \left|\sum_{n=1}^{\infty}\frac{1}{n!}\left(\int_0^t g(s)h(s)\,ds\right)^n\right| \\ &\le \sum_{n=1}^{\infty}\frac{1}{n!}\left|\int_0^t |g(s)h(s)|\,ds\right|^n \le \sum_{n=1}^{\infty}\frac{1}{n!}\left|\int_0^t \varepsilon|h(s)|\,ds\right|^n. \end{aligned}$$

Since $|\int_0^t |h(s)|\,ds| \le M$ for each $t \in I$, we have

$$\left|\frac{f(t)}{f_0(t)} - 1\right| \le \sum_{n=1}^{\infty}\frac{1}{n!}(M\varepsilon)^n = e^{M\varepsilon} - 1 \qquad (\forall t \in I).$$

In the same way, we have, for each $t \in I$, that

$$\left|\frac{f_0(t)}{f(t)} - 1\right| \le e^{M\varepsilon} - 1.$$

We thus conclude that

$$\max\left\{\left|\frac{f(t)}{f_0(t)} - 1\right|, \left|\frac{f_0(t)}{f(t)} - 1\right|\right\} \le e^{M\varepsilon} - 1$$

for every $t \in I$, and the proof is complete. □

Remark 2.1. *Let $\mathbb{R}^\times$ be the set of all non-zero real numbers. Similar result to Theorem 2.2 holds for continuous function $h\colon I \to \mathbb{R}^\times$ and $C^1(I, \mathbb{R}^\times)$ in spite of $C^1(I, \mathbb{C}^\times)$, where $C^1(I, \mathbb{R}^\times)$ denotes the set of all continuously differentiable functions from I to $\mathbb{R}^\times$.*

Example 2.1. Set, for each $t \in \mathbb{R}$, $h(t) = 2e^{-t^2}$. Then

$$\sup_{t\in\mathbb{R}} \left| \int_0^t |h(s)|\, ds \right| = \int_0^\infty h(s)\, ds = \sqrt{\pi}.$$

For each ε with $\varepsilon > 0$, we define

$$f(t) = \exp \int_0^t (1+\varepsilon) h(s)\, ds \qquad (\forall t \in \mathbb{R}). \tag{3}$$

It is easy to see that

$$\frac{f'(t)}{h(t)f(t)} - 1 = \varepsilon \qquad (\forall t \in \mathbb{R}).$$

By the proof of Theorem 2, we see that $f_0(t) = \exp \int_0^t h(s)\, ds$ is a solution of the equation $y' = hy$ with

$$\max\left\{ \left| \frac{f(t)}{f_0(t)} - 1 \right|, \left| \frac{f_0(t)}{f(t)} - 1 \right| \right\} \le e^{\sqrt{\pi}\varepsilon} - 1$$

for every $t \in \mathbb{R}$. Recall, by Lemma 2.1, that the solution in $C^1(\mathbb{R}, \mathbb{R}^\times)$ of the equation $y' = hy$ is of the form $y(t) = cf_0(t)$ ($t \in \mathbb{R}$) for some constant $c \in \mathbb{R}^\times$. We will prove that if $c \in \mathbb{R}^\times$ satisfies the condition (4) below, then $c = 1$ holds.

$$\max\left\{ \left| \frac{f(t)}{cf_0(t)} - 1 \right|, \left| \frac{cf_0(t)}{f(t)} - 1 \right| \right\} \le e^{\sqrt{\pi}\varepsilon} - 1 \qquad (\forall t \in \mathbb{R}). \tag{4}$$

To do this, let $c \in \mathbb{R}^\times$ satisfy (4). Set

$$\tilde{h}(t) = \exp \int_0^t \varepsilon h(s)\, ds \qquad (\forall t \in \mathbb{R}).$$

Then, by (3), $\tilde{h} \colon \mathbb{R} \to \mathbb{R}$ satisfies $f(t) = f_0(t)\tilde{h}(t)$ for every $t \in \mathbb{R}$,

$$\lim_{t\to-\infty} \tilde{h}(t) = e^{-\sqrt{\pi}\varepsilon} \quad \text{and} \quad \lim_{t\to\infty} \tilde{h}(t) = e^{\sqrt{\pi}\varepsilon}. \tag{5}$$

By (4), we have

$$\max\left\{ \left| \frac{\tilde{h}(t)}{c} - 1 \right|, \left| \frac{c}{\tilde{h}(t)} - 1 \right| \right\} \le e^{\sqrt{\pi}\varepsilon} - 1 \qquad (\forall t \in \mathbb{R}). \tag{6}$$

It follows from (6) that

$$2 - e^{\sqrt{\pi}\varepsilon} \leq \frac{\tilde{h}(t)}{c} \leq e^{\sqrt{\pi}\varepsilon}$$

for every $t \in \mathbb{R}$. By letting $t \to \infty$, we have by (5) that

$$2 - e^{\sqrt{\pi}\varepsilon} \leq \frac{e^{\sqrt{\pi}\varepsilon}}{c} \leq e^{\sqrt{\pi}\varepsilon},$$

which proves that

$$2e^{-\sqrt{\pi}\varepsilon} - 1 \leq \frac{1}{c} \leq 1. \tag{7}$$

By (6), we also have

$$2 - e^{\sqrt{\pi}\varepsilon} \leq \frac{c}{\tilde{h}(t)} \leq e^{\sqrt{\pi}\varepsilon} \qquad (\forall t \in \mathbb{R}).$$

Letting $t \to -\infty$, it follows from (5) that

$$2e^{-\sqrt{\pi}\varepsilon} - 1 \leq c \leq 1. \tag{8}$$

We prove $c > 0$. For if $c < 0$, then we have by (8) that $2e^{-\sqrt{\pi}\varepsilon} - 1 < 0$. It follows from (7) and (8) that

$$2e^{-\sqrt{\pi}\varepsilon} - 1 \leq c \leq \frac{1}{2e^{-\sqrt{\pi}\varepsilon} - 1}.$$

Since $2e^{-\sqrt{\pi}\varepsilon} - 1 < 0$, we have $(2e^{-\sqrt{\pi}\varepsilon} - 1)^2 \geq 1$, which shows

$$4e^{-\sqrt{\pi}\varepsilon}(e^{-\sqrt{\pi}\varepsilon} - 1) \geq 0.$$

We now reach a contradiction since $0 < e^{-\sqrt{\pi}\varepsilon} < 1$. Thus, we have $c > 0$.

Finally, we show that $c = 1$. Indeed, we have $c \leq 1$ by (8). On the other hand, since $c > 0$, it follows from (7) that $1 \leq c$. We thus conclude $c = 1$.

Remark 2.2. *Let f and f_0 be from Example 2.1. By a simple calculation, we see that*

$$\sup_{t\in\mathbb{R}} \left| \frac{f(t)}{f_0(t)} - 1 \right| = \sup_{t\in\mathbb{R}} \left| \exp \int_0^t \varepsilon h(s)\, ds - 1 \right| = e^{\sqrt{\pi}\varepsilon} - 1.$$

Since (4) holds only for $c = 1$, the constant $e^{\sqrt{\pi}\varepsilon} - 1$ is best possible. Therefore, the constant $e^{M\varepsilon} - 1$ is Theorem 2.2 can not be improved in general.

In Theorem 2.2, we proved Ger type stability of the first order linear differential equation $y' = hy$ for continuous function $h\colon I \to \mathbb{C}^\times$ with $\sup_{t\in I} |\int_0^t |h(s)|\, ds < \infty$. The above condition is essential in the following sense.

Theorem 2.3. *Let $h\colon I \to \mathbb{C}^\times$ be a continuous function. Suppose that there exists a constant $K \geq 0$ with the following condition:*

($*$) *to each $f \in C^1(I, \mathbb{C}^\times)$ satisfying*

$$\left|\frac{f'(t)}{h(t)f(t)} - 1\right| \leq 1 \qquad (\forall t \in I)$$

there corresponds $f_0 \in C^1(I, \mathbb{C}^\times)$ such that ${f_0}'(t) = h(t)f_0(t)$ and that

$$\max\left\{\left|\frac{f(t)}{f_0(t)} - 1\right|, \left|\frac{f_0(t)}{f(t)} - 1\right|\right\} \leq K$$

for all $t \in I$.

Then we have

$$\sup_{t\in I}\left|\int_0^t |h(s)|\, ds\right| < \infty.$$

Proof. Suppose that there exists a constant $K \geq 0$ with the condition ($*$). Set, for each $t \in I$,

$$f(t) = \exp\int_0^t (|h(s)| + h(s))\, ds. \tag{9}$$

It is obvious that

$$f'(t) = (|h(t)| + h(t))f(t) \qquad (\forall t \in I).$$

It follows that

$$\left|\frac{f'(t)}{h(t)f(t)} - 1\right| = \left|\frac{|h(t)| + h(t)}{h(t)} - 1\right| = \left|\frac{|h(t)|}{h(t)}\right| = 1$$

for each $t \in I$. By the condition ($*$), there exist $f_0 \in C^1(I, \mathbb{C}^\times)$ such that ${f_0}'(t) = h(t)f_0(t)$ and that

$$\max\left\{\left|\frac{f(t)}{f_0(t)} - 1\right|, \left|\frac{f_0(t)}{f(t)} - 1\right|\right\} \leq K \tag{10}$$

for every $t \in I$. Since ${f_0}' = hf_0$, it follows from Lemma 2.1 that f_0 is of the form

$$f_0(t) = f_0(0) \exp \int_0^t h(s)\, ds \qquad (\forall t \in I). \tag{11}$$

Note that $f_0(0) \neq 0$ since $f_0 \in C^1(I, \mathbb{C}^\times)$. Set, for each $t \in I$,

$$\psi(t) = \frac{1}{f_0(0)} \exp \int_0^t |h(s)|\, ds.$$

Then, by (9) and (11), we have

$$f(t) = f_0(t)\psi(t) \qquad (\forall t \in I). \tag{12}$$

It follows from (10) and (12), that

$$\max \left\{ |\psi(t) - 1|, \left| \frac{1}{\psi(t)} - 1 \right| \right\} \leq K \tag{13}$$

for every $t \in I$. Therefore, we have

$$|\psi(t)| \leq K + 1 \quad \text{and} \quad \left| \frac{1}{\psi(t)} \right| \leq K + 1 \qquad (\forall t \in I).$$

By the definition of ψ, we have

$$\frac{|f_0(0)|}{K+1} \leq \exp \int_0^t |h(s)|\, ds \leq (K+1)|f_0(0)| \qquad (\forall t \in I).$$

We thus conclude that $\sup_{t \in I} | \int_0^t |h(s)|\, ds| < \infty$. This completes the proof. □

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