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On (ψ, γ)–Stability of Quadratic Equation on Groups *

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Abstract

In this paper, the (ψ, γ)–stability of the quadratic functional equation is considered on arbitrary groups. It is proved that every group can be embedded into a group in which the quadratic equation is (ψ, γ)-stable. Further, it is shown that the quadratic functional equation is (ψ, γ)-stable on all abelian groups and some non-abelian groups such as $UT(3, K)$, $T(3, K)$ and $T(2, K)$, where K is an arbitrary field. The results of Skof [19] and Czerwik [4] are generalized in this paper.

Keywords and Phrases: (ψ, γ)*-pseudoquadratic map,* (ψ, γ)*-quasiquadratic map, Quadratic map, Quadratic functional equation, Semidirect product of groups,* (ψ, γ)*-stability of quadratic functional equation, Wreath product of groups.*

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1. Introduction

In 1940 to the audience of the Mathematics Club of the University of Wisconsin S. M. Ulam presented a list of unsolved problems [20]. One of these problems can be considered as the starting point of a new line of investigations: the stability problem. The problem was posed as follows. If we replace a given functional equation by a functional inequality, then under what conditions we can say that the solutions of the inequality are close to the solutions of the equation. For example, given a group G_1, a metric group (G_2, d) and a positive number ε, the Ulam question is: Does there exist a $\delta > 0$ such that if the map $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? In the case of a positive answer to this problem, we say that Cauchy functional equation $f(xy) = f(x)f(y)$ is *stable* for the pair (G_1, G_2). The interested reader should refer to [20] and [13] for an account on Ulam's problem.

Hyers [12] proved the following result to give an affirmative answer to Ulam's problem. Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying

$$||f(x + y) - f(x) - f(y)|| \leq \varepsilon$$

for all x, y in X. Then there exists a unique additive map $A : X \to Y$ satisfying

$$||f(x) - A(x)|| \leq \varepsilon$$

for all x in X. This pioneer result of Hyers can be expressed in the following way: Cauchy's functional equation is stable for any pair of Banach spaces.

Aoki [1] proved a generalized version of the previous result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$||\, f(x + y) - f(x) - f(y)\, || \leq \varepsilon\, (||x||^p + ||y||^p) \qquad \text{for all} \quad x, y \in X,$$

where ε and p are constants satisfying $\varepsilon > 0$ and $0 \leq p < 1$. By making use of the direct method of Hyers [12], he proved in this case too, that there is an additive function T from X into Y given by the formula

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

such that

$$||T(x) - f(x)|| \leq k\, \varepsilon\, ||x||^p,$$

where k depends on p as well as ε. Independently, Th.M. Rassias [17] in 1978 rediscovered the above result and proved that the mapping T is not only additive, under certain conditions, it is also linear. Rassias's paper [17] provided an impetus for a lot of activities in stability theory of functional equations. The first paper to extend Rassias's result to a class nonabelian groups and semigroups was [8].

The quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) \tag{1.1}$$

where f is defined on a group G and takes its values from a vector space E, is an important equation in the theory of functional equations and it plays an important role in the characterization of inner product spaces [7]. The stability of the quadratic functional equation (1.1) was first proved by Skof [19] for functions from a normed space into a Banach space. Cholewa [2] demonstrated that Skof's theorem is also valid if the relevant domain is replaced by an Abelian group. Later, Fenyő [10] improved the bound obtained and Cholewa from $\frac{\varepsilon}{2}$ to $\frac{\varepsilon+\|f(0)\|}{3}$ (cf. [3]).

Theorem 1.1. *Let G be an Abelian group and let E be a Banach space. If a function $f : G \to E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x, y \in G$, then there exists a unique quadratic function $q : G \to E$ such that

$$\|f(x) - q(x)\| \leq \frac{1}{3}(\varepsilon + \|f(0)\|)$$

for all $x \in G$.

The above theorem can be expressed in the following way: The quadratic functional equation is stable for the pair (G, E), where G is an Abelian group and E is a Banach space. In the paper [4], the following result on Hyers-Ulam-Rassias stability of quadratic functional equation on normed space was obtained that generalized the results of Skof [19] and Cholewa [2].

Theorem 1.2. *Let E_1 be a normed space and E_2 a Banach space and let $f : E_1 \to E_2$ be a function satisfying inequality*

$$||f(x+y) + f(x-y) - 2f(x) - 2f(y)|| \leq \varphi(x, y) \tag{1.2}$$

with either

1. $\varphi(x,y) = \varepsilon + \theta(||x||^p + ||y||^p), \quad p < 2,\ x, y \in X \setminus \{0\}$ *or*

2. $\varphi(x,y) = \theta(||x||^p + ||y||^p), \quad p > 2,\ x, y \in X$

for some $\varepsilon, \theta \geq 0$. *Then there exists a unique quadratic map* $Q : E_1 \to E_2$ *such that*

$$||f(x) - Q(x)|| \leq \frac{1}{3}(\varepsilon + ||f(0)||) + \frac{2\theta}{4 - 2^p}||x||^p, \quad x \in E_1 \setminus \{0\}$$

in case 1 and

$$||f(x) - Q(x)|| \leq \frac{2\theta}{2^p - 4}||x||^p, \quad x \in E_1$$

in case 2.

Various works on stability of the quadratic functional equation can be found in Skof [19], Cholewa [2], Fenyő [10], Ger [11], Czerwik [3], [4], [5], [6], Jung [14], [15], Jung and Sahoo [16], and Rassias [18]. In all these works, the stability of the quadratic equation or a more general quadratic equation was treated for the pair (G, E) when G is an Abelian group. In the present paper, we consider the stability of the functional equation (1.1) for the pair (G, E) when G is an arbitrary group and E is a real Banach space. The Skof's result [19] is a particular case of this result. We also show that any group can be embedded into a group G such that the functional equation (1.1) is stable on G.

In this paper, we generalize Theorem 1.2 in two different ways. First, we use a more general term on the right hand side of (1.2), namely $a+\theta\,[\psi(\gamma(x))+\psi(\gamma(y))]$, where a and θ are positive constants, $\gamma : G \to (0,\infty)$ is a function satisfying some special conditions to be discussed in the next section, and $\psi : [0,\infty) \to (0,\infty)$ is an increasing subadditive function. Second, we replace the domain of the function f by some of noncommutative group G. The paper is organized as follows: In Section 2, we present some preliminary results that will be needed to prove some results in the subsequent sections of this paper. In Section 3, we prove the (ψ,γ)-stability of quadratic functional equation on abelian group, and nonabelian groups such as $UT(3,K)$, $T(2,K)$, and $T(3,K)$, where K is an arbitrary field. Among other results, we prove that any group A can be embedded into a group G such that the quadratic functional equation is (ψ,γ)-stable on G.

2. Preliminary results

We will denote the set of real numbers by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Let $\mathbb{R}_0^+ = [0,\infty)$ be the set of non-negative numbers and $\mathbb{R}^+ = (0,\infty)$ be the set of positive numbers. Let G be an arbitrary group. Throughout this paper, the function $\psi : \mathbb{R}_0^+ \to \mathbb{R}^+$ is considered to be an increasing and subadditive function, that is ψ satisfies the conditions:

1. $\psi(t_1) \le \psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$ whenever $t_1 \le t_2$, and
2. $\psi(t_1 + t_2) \le \psi(t_1) + \psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$.

Throughout this paper, by γ we will mean a function $\gamma : G \to \mathbb{R}_0^+$ satisfying

1. $\gamma(x^{-1}) = \gamma(x)$ for all $x \in G$, and
2. $\gamma(xy) \le \gamma(x) + \gamma(y) + d$ for all $x, y \in G$

for some nonnegative real number d. It is clear that for any $x \in G$ and any $m \in \mathbb{N}$ the following inequalities hold

$$\psi(\gamma(x^m)) \le \psi(m\gamma(x) + md) \le m\psi(\gamma(x) + d) \le m\psi(\gamma(x)) + m\psi(d). \quad (2.1)$$

Definition 2.1. *Let G be a group and E a Banach space. The function $f : G \to E$ is said to be a (ψ,γ)–quasiquadratic mapping if there are nonnegative numbers a and θ such that for any $x, y \in G$*

$$\|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)\| \le a + \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))] \quad (2.2)$$

holds. The set of all (ψ,γ)–quasiquadratic mappings will be denoted by the set $KQ_{\psi,\gamma}(G,E)$.

Clearly, the set of all (ψ,γ)–quasiquadratic mappings, $KQ_{\psi,\gamma}(G,E)$, is a linear space.

Lemma 2.2. *Let $f \in KQ_{\psi,\gamma}(G,E)$ be a (ψ,γ)-quasiquadratic mapping. Then for any $m \ge 2$ there are nonnegative numbers c_m and θ_m such that*

$$\|f(x^m) - m^2 f(x)\| \le c_m + \theta_m\,\psi(\gamma(x)), \quad \forall x \in G. \quad (2.3)$$

Proof. We will prove this lemma by induction on m. By letting $y = x$ in (2.2), we obtain

$$\|f(x^2) + f(1) - 4f(x)\| \leq a + 2\theta\,\psi(\gamma(x)) \qquad \forall\, x \in G.$$

Therefore

$$\|f(x^2) - 4f(x)\| \leq a + \|f(1)\| + 2\theta\,\psi(\gamma(x)) \qquad \forall\, x \in G.$$

If we put $c_2 = a + \|f(1)\|$ and $\theta_2 = 2\theta$ in the last inequality, then we get

$$\|f(x^2) - 4f(x)\| \leq c_2 + \theta_2\,\psi(\gamma(x)) \qquad \forall\, x \in G.$$

Replacing x by x^m and y by x in (2.2), we obtain

$$\|f(x^{m+1}) + f(x^{m-1}) - 2f(x^m) - 2f(x)\| \leq a + \theta\,[\psi(\gamma(x^m)) + \psi(\gamma(x))]$$

for all $x \in G$. Using (2.1) in the last inequality, we see that

$$\|f(x^{m+1}) + f(x^{m-1}) - 2f(x^m) - 2f(x)\| \leq a + m\,\theta\,\psi(d) + \theta\,[m\,\psi(\gamma(x)) + \psi(\gamma(x))]$$

which is

$$\begin{aligned} &\|f(x^{m+1}) + f(x^{m-1}) - 2f(x^m) - 2f(x)\| \\ \leq &a + m\,\theta\,\psi(d) + \theta\,[m+1]\,\psi(\gamma(x)) \end{aligned} \tag{2.4}$$

for all $x \in G$. Suppose that (2.3) has been already established for $2 \leq m \leq k$. Let us check it for $k + 1$. From (2.4), we have

$$\|f(x^{k+1}) + f(x^{k-1}) - 2f(x^k) - 2f(x)\| \leq a + k\theta\psi(d) + \theta[k+1]\psi(\gamma(x)).$$

Using (2.3) in the last inequality, we see that

$$\begin{aligned} &\|f(x^{k+1}) + (k-1)^2 f(x) - 2k^2 f(x) - 2f(x)\| \\ \leq &a + k\theta\psi(d) + \theta[k+1]\psi(\gamma(x)) + c_{k-1} + \theta_{k-1}\psi(\gamma(x)) + 2c_k + 2\theta_k\,\psi(\gamma(x)). \end{aligned}$$

Therefore

$$\begin{aligned} &\|f(x^{k+1}) - (k+1)^2 f(x)\| \\ \leq &a + k\,\theta\,\psi(d) + c_{k-1} + 2c_k + [\theta(k+1) + \theta_{k-1} + 2\theta_k]\,\psi(\gamma(x)). \end{aligned}$$

Letting $c_{k+1} = a + k\theta\psi(d) + c_{k-1} + 2c_k$ and $\theta_{k+1} = \theta(k+1) + \theta_{k-1} + 2\theta_k$ we obtain the asserted inequality (2.3) and the proof of the lemma is now complete. □Let

$$\rho_m(k) = \sum_{i=0}^{k-1} \left(\frac{1}{m^2}\right)^i \qquad \text{and} \qquad \pi_m(k) = \sum_{i=0}^{k-1} \left(\frac{1}{m}\right)^i. \tag{2.5}$$

Lemma 2.3. *Let $f \in KQ_{\psi,\gamma}(G,E)$ and $m \in \mathbb{N}$ with $m \geq 2$. For any $k \in \mathbb{N}$ the inequality*

$$\begin{aligned} &\left\| \frac{1}{m^{2k}} f(x^{m^k}) - f(x) \right\| \\ \leq &\frac{c_m}{m^2} \rho_m(k) + \psi(d)\, \theta_m\, r_m(k) + \frac{\theta_m}{m^2} \pi_m(k)\, \psi(\gamma(x)) \end{aligned} \tag{2.6}$$

holds. Here c_m and θ_m are nonnegative numbers, $\pi_m(k)$ and $\rho_m(k)$ are numbers as defined in (2.5)*, and $0 \leq r_m(n) < 1$.*

Proof. From Lemma 2.2 it follows that

$$\left\| \frac{1}{m^2} f(x^m) - f(x) \right\| \leq \frac{c_m}{m^2} + \frac{\theta_m}{m^2} \psi(\gamma(x)). \tag{2.7}$$

So, $r_m(1) = 0$. Suppose that (2.6) has been already established for $k = 1, 2, \ldots, n$. Let us check it for $k = n+1$. Using the induction hypothesis, we have

$$\begin{aligned} &\left\| \frac{1}{m^{2n}} f(x^{m^n}) - f(x) \right\| \\ \leq &\frac{c_m}{m^2} \rho_m(n) + \psi(d)\, \theta_m\, r_m(n) + \frac{\theta_m}{m^2} \pi_m(n)\, \psi(\gamma(x)). \end{aligned}$$

Substituting x^m for x, we get

$$\begin{aligned} &\left\| \frac{1}{m^{2n}} f(x^{m^{n+1}}) - f(x^m) \right\| \\ \leq &\frac{c_m}{m^2} \rho_m(n) + \psi(d)\, \theta_m\, r_m(n) + \frac{\theta_m}{m^2} \pi_m(n)\, \psi(\gamma(x^m)). \end{aligned}$$

Hence using (2.1), we obtain

$$\begin{aligned}&\left\|\frac{1}{m^{2(n+1)}}f(x^{m^{n+1}})-\frac{1}{m^2}f(x^m)\right\|\\ \leq&\frac{c_m}{m^4}\,\rho_m(n)+\psi(d)\,\theta_m\,r_m(n)\frac{1}{m^2}+\frac{\theta_m}{m^4}\,\pi_m(n)\,m\,\psi(\gamma(x)+d).\end{aligned}$$

From the last inequality and (2.7), we obtain

$$\begin{aligned}&\left\|\frac{1}{m^{2(n+1)}}f(x^{m^{n+1}})-f(x)\right\|\\ \leq&\left\|\frac{1}{m^{2(n+1)}}f(x^{m^{n+1}})-\frac{1}{m^2}f(x^m)\right\|+\left\|\frac{1}{m^2}f(x^m)-f(x)\right\|\\ \leq&\frac{c_m}{m^4}\,\rho_m(n)+\psi(d)\,\theta_m\,\frac{r_m(n)}{m^2}+\frac{\theta_m}{m^4}\,\pi_m(n)\,m\,\psi(\gamma(x)+d)+\frac{c_m}{m^2}+\frac{\theta_m}{m^2}\,\psi(\gamma(x))\\ =&\frac{c_m}{m^4}\,\rho_m(n)+\frac{c_m}{m^2}+\psi(d)\,\theta_m\,\frac{r_m(n)}{m^2}+\frac{\theta_m}{m^4}\,\pi_m(n)\,m\,\psi(\gamma(x)+d)+\frac{\theta_m}{m^2}\,\psi(\gamma(x))\\ =&\left[\frac{\rho_m(n)}{m^2}+1\right]\frac{c_m}{m^2}+\psi(d)\,\theta_m\left[\frac{r_m(n)}{m^2}+\frac{\pi_m(n)}{m^3}\right]+\left[\frac{\pi_m(n)}{m}+1\right]\frac{\theta_m}{m^2}\,\psi(\gamma(x))\\ =&\frac{c_m}{m^2}\,\rho_m(n+1)+\psi(d)\,\theta_m\left[\frac{r_m(n)}{m^2}+\frac{\pi_m(n)}{m^3}\right]+\pi_m(n+1)\,\frac{\theta_m}{m^2}\,\psi(\gamma(x)).\end{aligned}$$

Put $r_m(n+1)=r_m(n)\frac{1}{m^2}+\frac{1}{m^3}\,\pi_m(n)$ then it is clear that $0\leq r_m(n+1)<1$ and the proof of the lemma is complete. □

Lemma 2.4. *Let $f\in KQ_{\psi,\gamma}(G,E)$ be a (ψ,γ)-quasiquadratic mapping. For any $m\geq 2$ and any $x\in G$, the sequence $\left\{\frac{1}{m^{2k}}f(x^{m^k})\right\}_{k=1}^{\infty}$ is a Cauchy sequence with*

$$f_m(x)=\lim_{k\to\infty}\frac{1}{m^{2k}}\,f\left(x^{m^k}\right). \tag{2.8}$$

Proof. Let

$$\alpha_m=\sum_{i=0}^{\infty}\frac{1}{m^{2i}}\qquad\text{and}\qquad\beta_m=\sum_{i=0}^{\infty}\left(\frac{1}{m}\right)^i. \tag{2.9}$$

Then by (2.7) and (2.9), we have

$$\left\|\frac{1}{m^{2n}}f(x^{m^n})-f(x)\right\|\leq\frac{c_m}{m^2}\,\alpha_m+\psi(d)\,\theta_m+\frac{\theta_m}{m^2}\,\beta_m\,\psi(\gamma(x)).$$

Substituting x^{m^k} for x in the last inequality, we get

$$\begin{aligned}
&\left\| \frac{1}{m^{2n}} f(x^{m^{n+k}}) - f(x^{m^k}) \right\| \\
\leq &\frac{c_m}{m^2}\, \alpha_m + \psi(d)\, \theta_m + \frac{\theta_m}{m^2}\, \beta_m\, \psi(\gamma(x^{m^k})) \\
\leq &\frac{c_m}{m^2}\, \alpha_m + \psi(d)\, \theta_m + \frac{\theta_m}{m^2}\, \beta_m\, m^k\, \psi(d) + \frac{\theta_m}{m^2}\, \beta_m\, m^k\, \psi(\gamma(x))
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left\| \frac{1}{m^{2(n+k)}} f(x^{m^{n+k}}) - \frac{1}{m^{2k}} f(x^{m^k}) \right\| \\
\leq &\frac{\alpha_m}{m^{2k}} \frac{c_m}{m^2} + \frac{\psi(d)\, \theta_m}{m^{2k}} + \frac{\beta_m}{m^{2k}} \frac{\theta_m}{m^2}\, m^k\, \psi(d) + \frac{\beta_m}{m^{2k}} \frac{\theta_m}{m^2}\, m^k\, \psi(\gamma(x)) \\
\leq &\frac{c_m}{m^{2k+2}}\, \alpha_m + \psi(d) \theta_m \frac{1}{m^{2k}} + \frac{\theta_m}{m^{k+2}}\, \beta_m\, \psi(d) + \frac{\theta_m}{m^{k+2}}\, \beta_m\, \psi(\gamma(x)).
\end{aligned}$$

From the latter relation it follows that the sequence $\left\{ \frac{1}{m^{2k}} f(x^{m^k}) \right\}_{k=1}^{\infty}$ is a Cauchy sequence, and therefore has a limit which we denote by $f_m(x)$. This completes the proof of the lemma. □

Let

$$a_m = \frac{c_m}{m^2}\, \alpha_m + \psi(d)\, \theta_m \qquad \text{and} \qquad b_m = \frac{\theta_m}{m^2}\, \beta_m. \tag{2.10}$$

Let $\delta_m(x) = a_m + b_m\, \psi(\gamma(x))$. Then

$$\begin{aligned}
\delta_m(xy) &= a_m + b_m\, \psi(\gamma(xy)) \\
&\leq a_m + b_m\, \psi(\gamma(x)) + b_m\, \psi(\gamma(y)) + b_m\, \psi(d) \\
&\leq \delta_m(x) + \delta_m(y) + b_m\, \psi(d) \\
&= 2\, a_m + b_m\, \psi(d) + b_m\, [\psi(\gamma(x)) + \psi(\gamma(y))].
\end{aligned}$$

Similarly

$$\delta_m(xy^{-1}) \leq 2\, a_m + b_m\, \psi(d) + b_m\, [\psi(\gamma(x)) + \psi(\gamma(y))].$$

From (2.6) it follows that

$$\left\| \frac{1}{m^{2k}} f(x^{m^k}) - f(x) \right\| \leq a_m + b_m\, \psi(\gamma(x)), \tag{2.11}$$

and letting $k \to \infty$, we have

$$\|f_m(x) - f(x)\| \leq \delta_m(x).$$

Lemma 2.5. *For any $m \geq 2$, the function f_m, defined in* (2.8)*, belongs to the set $KQ_{\psi,\gamma}(G,E)$.*

Proof. Indeed, for any $x, y \in G$, we have

$$\begin{aligned}
&\left\|f_m(xy) + f_m(xy^{-1}) - 2f_m(x) - 2f_m(y)\right\| \\
\leq &\|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)\| + \|f_m(xy) - f(xy)\| \\
&+ \|f_m(xy^{-1}) - f(xy^{-1})\| + 2\|f_m(x) - f(x)\| + 2\|f_m(y) - f(y)\| \\
\leq &a + \theta[\psi(\gamma(x)) + \psi(\gamma(y))] + \delta_m(xy) + \delta_m(xy^{-1}) + 2\delta_m(x) + 2\delta_m(y) \\
\leq &a + \theta[\psi(\gamma(x)) + \psi(\gamma(y))] + 2a_m + b_m\psi(d) + b_m[\psi(\gamma(x)) + \psi(\gamma(y))] \\
&+ 2a_m + b_m\psi(d) + b_m[\psi(\gamma(x)) + \psi(\gamma(y))] \\
&+ 2a_m + 2b_m\psi(\gamma(x)) + 2a_m + 2b_m\psi(\gamma(y)). \\
\leq &a + 8a_m + 2b_m\psi(d) + (\theta + 4b_m)[\psi(\gamma(x)) + \psi(\gamma(y))].
\end{aligned}$$

Since $a + 8a_m + 2b_m\psi(d)$ and $\theta + 4b_m$ are nonnegative, $f_m \in KQ_{\psi,\gamma}(G,E)$. This completes the proof of the lemma. □

Lemma 2.6. *The function f_m, defined in* (2.8)*, satisfies $f_m = f_2$ for all $m > 2$.*

Proof. From the definition of f_m it follows that for any $k \in \mathbb{N}$ and $x \in G$ the relations $f_m(x^{m^k}) = m^{2k} f_m(x)$ and $f_2(x^{2^k}) = 2^{2k} f_2(x)$ hold. From Lemma 2.5, we see that $f_m, f_2 \in KQ_{\psi,\gamma}(G,E)$. Hence, by Lemma 2.4, function

$$\phi(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} f_m\left(x^{2^n}\right)$$

is well defined and belongs to the set $KQ_{\psi,\gamma}(G,E)$. Let c and d be nonnegative numbers such that

$$\|\phi(x) - f_m(x)\| \leq c + d\,\psi(\gamma(x)), \qquad \forall x \in G. \tag{2.12}$$

From (2.11) it follows

$$\|\phi(x) - f(x)\| \leq c + a_m + (d + b_m)\,\psi(\gamma(x)), \qquad \forall\, x \in G. \tag{2.13}$$

Taking into account relation

$$\|f_2(x) - f(x)\| \leq a_2 + b_2\,\psi(\gamma(x)), \qquad \forall\, x \in G$$

we get

$$\|\phi(x) - f_2(x)\| \le c + a_m + a_2 + (d + b_m + b_2)\,\psi(\gamma(x)), \qquad \forall\, x \in G. \quad (2.14)$$

Therefore

$$\|f_m(x) - f_2(x)\| \le a_m + a_2 + (b_m + b_2)\,\psi(\gamma(x)), \qquad \forall x \in G.$$

It is clear that for any $\ell \in \mathbb{N}$ we have

$$\phi(x^{m^\ell}) = m^{2\ell}\phi(x), \qquad \phi(x^{2^\ell}) = 2^{2\ell}\phi(x),$$

hence, from (2.12) we have

$$\begin{aligned}
\|\phi(x^{2^\ell}) - f_2(x^{2^\ell})\| &\le c + a_m + a_2 + (d + b_m + b_2)\,\psi(\gamma(x^{2^\ell})),\\
2^{2\ell}\|\phi(x) - f_2(x)\| &\le c + a_m + a_2 + (d + a_m + a_2)\,2^\ell\,\psi(\gamma(x) + d),\\
\|\phi(x) - f_2(x)\| &\le \frac{c + a_m + a_2}{2^{2\ell}} + (d + b_m + b_2)\,\frac{2^\ell}{2^{2\ell}}\,\psi(\gamma(x) + d),
\end{aligned}$$

and we see that $\phi \equiv f_2$. Similarly we check that $\phi \equiv f_m$. Therefore $f_m \equiv f_2$. This completes the proof. □

Denote by $\widehat{f}$ a function defined by the formula

$$\widehat{f}(x) = \lim_{k\to\infty} \frac{1}{4^k}\, f(x^{2^k}). \quad (2.15)$$

Definition 2.7. *A (ψ,γ)–quasiquadratic mapping $\phi : G \to E$ is said to be (ψ,γ)–pseudoquadratic mapping if ϕ satisfies $\varphi(x^n) = n^2\varphi(x)$ for all $x \in G$ and all $n \in \mathbb{N}$. The set of all (ψ,γ)–pseudoquadratic mappings will be denoted by the set $PQ_{\psi,\gamma}(G,E)$.*

From Lemma 2.6 we obtain the following corollary.

Corollary 2.8. *The function $\widehat{f}$, defined by* (2.15)*, is a (ψ,γ)–pseudoquadratic mapping and satisfies the following relation*

$$\left\|\frac{1}{m^{2k}} f(x^{m^k}) - f(x)\right\| \le a_m + b_m\,\psi(\gamma(x)). \quad (2.16)$$

Definition 2.9. *By $B_{\psi,\gamma}(G,E)$ we donote the set of all functions f such that if f belongs to $B_{\psi,\gamma}(G,E)$, then there are nonnegative numbers a and b such that*

$$\|f(x)\| \le a + b\,\psi(\gamma(x)) \quad (2.17)$$

for all $x \in G$.

Theorem 2.10. *The linear space $KQ_{\psi,\gamma}(G,E)$ of all (ψ,γ)-quasiquadratic mappings can be decomposed as the direct sum of $PQ_{\psi,\gamma}(G,E)$ and $B_{\psi,\gamma}(G,E)$, that is $KQ_{\psi,\gamma}(G,E) = PQ_{\psi,\gamma}(G,E) \oplus B_{\psi,\gamma}(G,E)$.*

Proof. It is easy to see that $PQ_{\psi,\gamma}(G,E)$ and $B_{\psi,\gamma}(G,E)$ are linear subspaces of $KQ_{\psi,\gamma}(G,E)$. Let us show that $PQ_{\psi,\gamma}(G,E) \cap B_{\psi,\gamma}(G,E) = \{0\}$. Indeed, if $f \in PQ_{\psi,\gamma}(G,E) \cap B_{\psi,\gamma}(G,E)$, then using (2.17) we have for any $k \in \mathbb{N}$

$$\|f(x^{2^k})\| \leq a + b\,\psi(\gamma(x^{2^k}))$$

which by (2.1) and the fact that f is (ψ,γ)-pseudoquadratic implies

$$4^k\,\|f(x)\| \leq a + b\,2^k\,\psi(\gamma(x) + d).$$

Rewriting the last inequality, we have

$$\|f(x)\| \leq \frac{a}{4^k} + \frac{b}{2^k}\,\psi(\gamma(x) + d),$$

and taking the limit as $k \to \infty$ we see that $f(x) = 0$.

Let f be an arbitrary element of $KQ_{\psi,\gamma}(G,E)$, then by Corollary 2.8, $\widehat{f} \in PQ_{\psi,\gamma}(G,E)$. Again from Corollary 2.8 we see that $f - \widehat{f} \in BQ_{\psi,\gamma}(G,E)$. Now the proof is complete. □

3. Stability

Definition 3.1. *Let $\psi : \mathbb{R}_0^+ \to \mathbb{R}^+$ and $\gamma : G \to \mathbb{R}_0^+$ be the functions as stated in the beginning of Section 2, and let $f : G \to E$. The quadratic equation*

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0 \tag{3.1}$$

is said to be (ψ,γ)-stable if for any function φ satisfying condition

$$\|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y)\| \leq a + b\,[\psi(\gamma(x)) + \psi(\gamma(y))]$$

there exists a solution g of the equation (3.1), such that

$$\|\varphi(x) - g(x)\| \leq c + d\,\psi(\gamma(x))$$

for some nonnegative numbers c and d and any $x \in G$.

The set of all solutions of the quadratic functional equation (3.1) will be denoted by $Q(G,E)$. Clearly, $Q(G,E)$ is a linear space.

Proposition 3.2. *The quadratic equation* (3.1) *is* (ψ,γ)*-stable if and only if* $PQ_{\psi,\gamma}(G,E)=Q(G,E)$.

Proof. The proof follows from Theorem 2.10. □

Lemma 3.3. *The quadratic equation* (3.1) *is* (ψ,γ)*-stable for any abelian group* G.

Proof. Let G be an abelian group. Thus $(xy)^p = x^p y^p$ for any $p\in\mathbb{N}$ and for any $x,y\in G$. Let $f\in PQ_{\psi,\gamma}(G,E)$. Then we have

$$\begin{aligned}&\| f((xy)^p)+f((xy^{-1})^p)-2f(x^p)-2f(y^p)\|\\ =&\| f(x^p y^p)+f(x^p(y^{-1})^p)-2f(x^p)-2f(x^p)\|\\ \leq& a+b\,[\psi(\gamma(x^p))+\psi(\gamma(y^p))]\\ \leq& a+b\,[p\,\psi(\gamma(x)+d)+p\,\psi(\gamma(y)+d)].\end{aligned}$$

Therefore

$$\begin{aligned}&p^2\,\| f(xy)+f(xy^{-1})-2f(x)-2f(y)\|\\ \leq& a+b\,p\,[\psi(\gamma(x)+d)+\psi(\gamma(y)+d)]\end{aligned}$$

which is

$$\begin{aligned}&\| f(xy)+f(xy^{-1})-2f(x)-2f(y)\|\\ \leq&\frac{a}{p^2}+\frac{b}{p^2}\,[\psi(\gamma(x)+d)+\psi(\gamma(y)+d)].\end{aligned}$$

Letting $p\to\infty$ in the last inequality, we have

$$f(xy)+f(xy^{-1})-2f(x)-2f(y)=0.$$

Hence $f\in Q(G,E)$. By Proposition 3.2 the equation (3.1) is (ψ,γ)-stable and the proof of the lemma is complete. □

Lemma 3.4. *Let* $f\in PQ_{\psi,\gamma}(G,E)$. *Then for any* $x,y\in G$ *the following relations hold:*

1. $f(x^{-1})=f(x)$,
2. $f(xy)=f(yx)$.

Proof. 1. Since $f \in PQ_{\psi,\gamma}(G,E)$, we have

$$\| f(y) + f(y^{-1}) - 2f(1) - 2f(y) \| \leq a + b\,\psi(\gamma(1)) + b\,\psi(\gamma(y)).$$

Hence

$$\| f(y^{-1}) - f(y) \| \leq a + 2\,\|f(1)\| + b\,\psi(\gamma(1)) + b\,\psi(\gamma(y))$$

Therefore, for any $n \in \mathbb{N}$, we have

$$\| f(y^{-n}) - f(y^n) \| \leq a + 2\,\|f(1)\| + b\,\psi(\gamma(1)) + b\,\psi(\gamma(y^n))$$

and

$$\| f(y^{-1}) - f(y) \| \leq \frac{a + 2\,\|f(1)\| + b\,\psi(\gamma(1))}{n^2} + b\,\frac{n}{n^2}\,\psi(\gamma(y) + d).$$

Letting $n \to \infty$ in the last inequality, we see that $f(y^{-1}) = f(y)$.

2. Since $f \in PQ_{\psi,\gamma}(G,E)$, we obtain

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \leq a + b[\psi(\gamma(x)) + \psi(\gamma(y))].$$

Interchanging x with y in the last inequality, we get

$$\| f(yx) + f(yx^{-1}) - 2f(y) - 2f(x) \| \leq a + b\,[\psi(\gamma(x)) + \psi(\gamma(y))].$$

Using the fact that $f(x^{-1}) = f(x)$ for all $x \in G$, we get from the last inequality

$$\| f(yx) + f(xy^{-1}) - 2f(y) - 2f(x) \| \leq a + b\,[\psi(\gamma(x)) + \psi(\gamma(y))].$$

Therefore

$$\| f(xy) - f(yx) \| \leq 2a + 2b\,[\psi(\gamma(x)) + \psi(\gamma(y))].$$

Changing x by $y^{-1}x$, we get

$$\begin{aligned} \| f(y^{-1}xy) - f(x) \| &\leq 2a + 2b\,[\psi(\gamma(y^{-1}x)) + \psi(\gamma(y))] \\ &\leq 2a + 2b\,[\psi(\gamma(y^{-1}) + \gamma(x) + d) + \psi(\gamma(y))] \\ &\leq 2a + 2\psi(d) + 4b\,[\psi(\gamma(x)) + \psi(\gamma(y))]. \end{aligned}$$

Therefore for any $n \in \mathbb{N}$, replacing x by x^n we have

$$\| f(y^{-1}x^n y) - f(x^n) \| \leq 2a + 2\psi(d) + 4b\,[\psi(\gamma(x^n)) + \psi(\gamma(y^n))]$$

which simplifies to

$$n^2 \| f(y^{-1}xy) - f(x) \| \leq 2a + 2\psi(d) + 4bn\,[\psi(\gamma(x) + d) + \psi(\gamma(y) + d)].$$

Hence

$$\| f(y^{-1}xy) - f(x) \| \leq \frac{2a}{n^2} + \frac{2\psi(d)}{n^2} + \frac{4b}{n}\,[\psi(\gamma(x) + d) + \psi(\gamma(y) + d)].$$

Taking $n \to \infty$ in the last inequality, we see that $f(y^{-1}xy) = f(x)$ which is $f(xy) = f(yx)$, and the proof of the lemma is now complete. □

Theorem 3.5. *Let E_1 and E_2 be Banach spaces. Then quadratic equation* (3.1) *is (ψ,γ)-stable for the pair $(G; E_1)$ if and only if it is (ψ,γ)-stable for the pair $(G; E_2)$.*

Proof. Let E be a real Banach space and $\mathbb{R}$ be the set of reals. Suppose that equation (3.1) is (ψ,γ)-stable for pair $(G; E)$ and it is not (ψ,γ)-stable for pair $(G,\mathbb{R})$. Then there is nontrivial $(\psi,\gamma)-$pseudoquadratic mapping f on G. By nontrivial $(\psi,\gamma)-$pseudoqudratic mapping we mean an element of $PQ_{\psi,\gamma}(G,E)$ which in not quadratic mapping. Therefore for some $a, b \geq 0$ we have

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \leq a + b\,[\psi(\gamma(x)) + \psi(\gamma(y))]$$

for all $x, y \in G$. Let $e \in E$ with $\|e\| = 1$. Consider a function $\varphi : G \to E$ defined by the formula $\varphi(x) = f(x) \cdot e$. It is clear that φ is nontrivial $(\psi,\gamma)-$pseudoquadratic E–valued mapping. Therefore we come to a contradiction. Now suppose that the equation (3.1) is (ψ,γ)-stable for the pair $(G,\mathbb{R})$, that is $PQ_{\psi,\gamma}(G;\mathbb{R}) = Q(G,\mathbb{R})$. Denote by E^* the space of linear bounded functionals on E with norm topology. It is clear that for any $\varphi \in PQ_{\psi,\gamma}(G;E)$ and any $\lambda \in E^*$ function $\lambda \circ \varphi$ belongs $PQ_{\psi,\gamma}(G,\mathbb{R})$. Indeed, let for nonnegative numbers a, b and any $x, y \in G$ the following relation is fulfilled

$$\|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y)\| \leq a + b\,[\psi(\gamma(x)) + \psi(\gamma(y))].$$

Then

$$\begin{aligned}
&|\lambda \circ \varphi(xy) + \lambda \circ \varphi(xy^{-1}) - 2\lambda \circ \varphi(x) - 2\lambda \circ \varphi(y)| \\
=&|\lambda(\varphi(xy) + \lambda\varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y))| \\
\leq&\|\lambda\|(a + b\,[\psi(\gamma(x)) + \psi(\gamma(y))]) \\
=&\|\lambda\|\, a + \|\lambda\|\, b\,[\psi(\gamma(x)) + \psi(\gamma(y))].
\end{aligned}$$

It is clear that $\lambda \circ \varphi(x^{2^n}) = 4^n \lambda \circ \varphi(x)$ for any $x \in G$ and any $n \in \mathbb{N}$. Therefore the function $\lambda \circ \varphi$ belongs to $PQ_{\psi,\gamma}(G,\mathbb{R})$. Let $f : G \to E$ be a nontrivial (ψ, γ)–pseudoquadratic mapping. Then there are $x, y \in G$ such that $f(xy)+f(xy^{-1})-2f(x)-2f(y) \neq 0$. By Hahn-Banach theorem there is $\ell \in E^*$ such that $\ell(f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)) \neq 0$. Hence, $\ell \circ f$ is a nontrivial (ψ, γ)–pseudoquadratic function on G. Thus we come to a contradiction and the proof is complete. □

Due to Theorem 3.5 we can simply say that equation (3.1) is (ψ, γ)-stable or not (ψ, γ)-stable on the group G, without mentioning a Banach space. From now on in the case $E = \mathrm{R}$, we denote spaces $KQ_{\psi,\gamma}(G, \mathrm{R})$, $PQ_{\psi,\gamma}(G, \mathrm{R})$ and $Q(G, \mathrm{R})$ by $KQ_{\psi,\gamma}(G)$, $PQ_{\psi,\gamma}(G)$ and $Q(G)$, respectively.

3.1 G=UT(3,K)

Let K be an arbitrary field and K^* its multiplicative group. Denote by G the group $UT(3, K)$ consisting of matrices of the form

$$\begin{bmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}; \quad x,\, y,\, t \in K\,.$$

Now our goal is to establish (ψ, γ)-stability of (3.1) on the group $UT(3, K)$. To establish (ψ, γ)-stability of (3.1) we need to show that $PQ_{\psi,\gamma}(G, E) = Q(G)$. Denote by A, B, C subgroups of G, consisting of matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,\, b,\, c \in \mathbb{R}$$

respectively. Denote by H a subgroup of G generated by B and C.

Proposition 3.6. *If $\varphi \in PQ_{\psi,\gamma}(G,E)$, then φ has presentation of the form $\varphi(x) = q(\tau(x))$, where $\tau : G \to K \times K$ is a homomorphism defined by the formula*

$$\tau : \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \to (a,b)$$

and $q \in Q(K \times K)$. Therefore $PQ_{\psi,\gamma}(G) = Q(G)$ and equation (3.1) is (ψ,γ)-stable on G.

Proof. Let $\varphi \in PQ_{\psi,\gamma}(G)$. By Lemma 3.4, the function φ is invariant with respect inner automorphisms of G. Hence, from relation

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b & ba_1 - b_1 a + c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

it follows

$$\varphi\left(\begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 & b & ba_1 - b_1 a + c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}\right). \tag{3.3}$$

Let us check that $\varphi\big|_C \equiv 0$. Let a and b be nonnegative numbers, such that

$$|\,\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y)\,| \le a + b\,[\psi(\gamma(x)) + \psi(\gamma(x))]$$

for all $x, y \in G$. A subgroup of G generated by B and C is an abelian group. Therefore for any $\beta \in B$ and $\alpha \in C$ we have

$$\varphi(\alpha\beta^2) + \varphi(\alpha) - 2\varphi(\alpha\beta) - 2\varphi(\beta) = 0. \tag{3.4}$$

Let

$$\beta = \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \alpha = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $b \neq 0$. Then from (3.3) it follows

$$\varphi\left(\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right), \tag{3.5}$$

for any $c \in \mathbb{R}$. So, $\varphi(\alpha\beta^2) = \varphi(\beta^2)$ and (3.4) implies

$$\varphi(\beta^2) + \varphi(\alpha) - 2\varphi(\beta) - 2\varphi(\beta) = 0.$$

Hence

$$4\varphi(\beta) + \varphi(\alpha) - 2\varphi(\beta) - 2\varphi(\beta) = 0$$

which simplifies to

$$\varphi(\alpha) = 0.$$

Therefore, $\varphi\big|_C \equiv 0$. Now from (3.3) we obtain that φ is constant on any coset of the group G by its subgroup C. Hence, there is $q \in Q(K \times K)$ such that $\varphi(x) = q(\tau(x))$ and, hence, $\varphi \in Q(G)$. The proof is complete. □

3.2 T(2,K)

Elementary computations show that

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -\frac{c}{ba} \\ 0 & b^{-1} \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} \alpha & \frac{\alpha c}{a} - \frac{\beta c}{a} \\ 0 & \beta \end{bmatrix}. \tag{3.6}$$

Lemma 3.7. *Let*

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$$

be an element of $T(2,R)$ such that $x \neq y$. Then there exist a, b, c, α, β such that

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}. \tag{3.7}$$

Proof. The proof follows from letting $\alpha = x$, $\beta = y$, $a = 1$, and $c = \frac{z}{x-y}$ in (3.6). □

Lemma 3.8. *If $f \in PQ_{\psi,\gamma}(G,E)$ and $f(g) = 0$ for any diagonal matrix g, then $f \equiv 0$.*

Proof. Let

$$u = \begin{bmatrix} x & z \\ 0 & x \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since $f \in PQ_{\psi,\gamma}(G, E)$, we have

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(x)| \leq p + q\,[\psi(\gamma(x)) + \psi(\gamma(y))].$$

for some positive numbers $p, q \geq 0$. For any $n \in \mathbb{N}$, replacing x by u^n and y by v, we obtain

$$|f(u^n v) + f(u^n v^{-1}) - 2f(u^n) - 2f(v)| \leq p + q\,[\psi(\gamma(u^n)) + \psi(\gamma(v))].$$

Since v is an element of order two and f has the property $f(z^n) = n^2 f(z)$ for any $n \in Z$ and any $z \in G$ we get $f(v) = 0$. Now by previous lemma we have $f(u^n v) = 0$. Hence

$$2|f(u^n)| \leq p + q\,[\psi(\gamma(u^n)) + \psi(\gamma(v))].$$

Since $f \in PQ_{\psi,\gamma}(G, E)$, we have $f(u^n) = n^2\, f(u)$ and hence the last inequality yields

$$2|f(u)| \leq \frac{p}{n^2} + q\,\left[\frac{n}{n^2}\psi(\gamma(u) + d) + \frac{1}{n^2}\psi(\gamma(v))\right].$$

So, letting $n \to \infty$, we have $f(u) = 0$. Taking into account the previous lemma we obtain $f \equiv 0$. The proof of the lemma is now complete. □

Theorem 3.9. *$PQ_{\psi,\gamma}(G) = Q(G)$. So, quadratic functional equation* (3.1) *is (ψ, γ)–stable on $G = T(2, K)$.*

Proof. The proof follows from two previous lemmas. □

3.3 G=T(3,K)

By some elementary computations we have

$$\begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -b & ab - c \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b_1 & c_1 + ab_1 - ba_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 & b_1 & c_1 + ab_1 - ba_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$
$$= \begin{bmatrix} 1 & x^{-1}yb_1 & x^{-1}zc_1 + x^{-1}zab_1 - x^{-1}zba_1 \\ 0 & 1 & y^{-1}za_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lemma 3.10. *Let $f \in PQ_{\psi,\gamma}(G)$, then $f\big|_{TU(3,K)} \equiv 0$.*

Proof. Let

$$g = \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us check that a class of conjugate elements containing g contains matrix g^2 too. To do this we need to show that for any a_1, b_1 and c_1 one can choose numbers x, y, z, a, b such that the equality

$$\begin{bmatrix} 1 & x^{-1}yb_1 & x^{-1}zc_1 + x^{-1}zab_1 - x^{-1}zba_1 \\ 0 & 1 & y^{-1}za_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2b_1 & a_1b_1 + 2c_1 \\ 0 & 1 & 2a_1 \\ 0 & 0 & 1 \end{bmatrix}$$

holds. Indeed, if $a_1 = b_1 = 0$ we can put $z = 2x$. If $a_1 = 0$, $b_1 \neq 0$ we can put $y = z = 2x$, $a = 0$, and if $a_1 \neq 0$, $b_1 \neq 0$ we can put $y = 2x$, $z = 4x$, $b = 0$, $a = \frac{a_1b_1 - 2c_1}{4b_1}$.

So, we see that g is conjugate to g^2. It follows that $f(g) = f(g^2) = 4f(g)$, and $f(g) = 0$. This completes the proof of the lemma. □

Arguing as in the case $G = T(2, K)$ we get the following theorem

Theorem 3.11. *$PQ_{\psi,\gamma}(G) = Q(G)$, that is, the quadratic functional equation (3.1) is (ψ, γ)–stable on $G = T(3, K)$.*

4. Embedding

Let G be an arbitrary group and $f \in PQ_{\psi,\gamma}(G)$. Hence for nonnegative numbers δ and θ and for any $x, y \in G$, we have

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \leq \delta + \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))]. \qquad (4.1)$$

Let b, c, u, v be elements of G and $x = bu$ and $y = cv$. We will use notation a^b for element $b^{-1}ab$. From (4.1), we get

$$\begin{aligned} &|f(bcu^cv) + f(bc^{-1}(uv^{-1})^{c^{-1}}) - 2f(bu) - 2f(cv)| \\ =&\,|f(bucv) + f(buv^{-1}c^{-1}) - 2f(bu) - 2f(cv)| \\ &+\theta\,[\psi(\gamma(bu)) + \psi(\gamma(cv))]. \end{aligned}$$

Therefore

$$\begin{aligned} &|f(bcu^cv) + f(bc^{-1}(uv^{-1})^{c^{-1}}) - 2f(bu) - 2f(cv)| \\ \leq&\,\delta + \theta\,[\psi(\gamma(bu)) + \psi(\gamma(cv))] \end{aligned} \qquad (4.2)$$

and if $b = c$, then

$$\begin{aligned} &|f(c^2u^cv) + f((uv^{-1})^{c^{-1}}) - 2f(cu) - 2f(cv)| \\ \leq&\,\delta + \theta[\psi(\gamma(cu)) + \psi(\gamma(cv))]. \end{aligned} \qquad (4.3)$$

Since $f \in PQ_{\psi,\gamma}(G)$ and $c^2u^cu = (cu)^2$, we obtain

$$f(c^2u^cu) = 4f(cu). \qquad (4.4)$$

Letting $c^2 = 1$ and $u = 1$ in (4.3) implies

$$|f(v) + f((v^{-1})^{c^{-1}}) - 2f(cv) - 2f(c)| \leq \delta + \theta\,[\psi(\gamma(c)) + \psi(\gamma(cv))].$$

Since c is an element of finite order and $f \in PQ_{\psi,\gamma}(G)$, $f(c) = 0$ and from the last inequality, we have

$$|f(v) + f((v^{-1})^{c^{-1}}) - 2f(cv)| \leq \delta + \theta\,[\psi(\gamma(c)) + \psi(\gamma(cv))]. \qquad (4.5)$$

By Lemma 3.4, we have $f(v) = f(v^{-1}) = f((v^{-1})^{c^{-1}})$ and hence (4.5) yields

$$|2f(v) - 2f(cv)| \leq \delta + \theta\,[\psi(\gamma(c)) + \psi(\gamma(cv))]$$

which is

$$|f(v) - f(cv)| \leq \frac{\delta}{2} + \frac{\theta}{2}\,[\psi(\gamma(c)) + \psi(\gamma(cv))]. \tag{4.6}$$

From (4.4) and (4.6) we have

$$|f(u^c u) - 4f(u)| \leq 2\delta + 2\theta\,[\psi(\gamma(c)) + \psi(\gamma(cu))] \tag{4.7}$$

Next, letting $c^2 = 1,\ v = 1$, into (4.3), we get

$$|f(u^c) + f(u^c) - 2f(cu)| \leq \delta + \theta\,[\psi(\gamma(cu)) + \psi(\gamma(c))]$$

and by Lemma 3.4 the latter reduces to

$$|f(u) - f(cu)| \leq \frac{\delta}{2} + \frac{\theta}{2}\,[\psi(\gamma(cu)) + \psi(\gamma(c))]. \tag{4.8}$$

From (4.8), it follows

$$|4f(u) - 4f(cu)| \leq 2\delta + 2\theta\,[\psi(\gamma(cu)) + \psi(\gamma(c))].$$

Now taking into account (4.4) and relation $c^2 = 1$ we get

$$|f(u^c u) - 4f(u)| \leq 2\delta + 2\theta\,[\psi(\gamma(cu)) + \psi(\gamma(c))]. \tag{4.9}$$

Lemma 4.1. *Let G be an arbitrary group and $f \in PQ_{\psi,\gamma}(G)$. For $u, c \in G$, let $c^2 = 1$ and $u^c u = u u^c$. Then*

$$f(u^c u) = 4f(u). \tag{4.10}$$

Proof. For any $n \in \mathrm{N}$, we have

$$\begin{aligned} n^2|f(u^c u) - 4f(u)| &= |f((u^c u)^n) - 4f(u^n)| \\ &= |f((u^n)^c u^n) - 4f(u^n)| \\ &\leq \delta + \theta\,[\psi(\gamma(cu^n)) + \psi(\gamma(u^n))] \\ &\leq \delta + \theta\,[\psi(\gamma(c)) + \psi(\gamma(u^n)) + \psi(d) + \psi(\gamma(u^n))] \\ &\leq \delta + \theta\,[\psi(\gamma(c)) + \psi(d) + 2\,\psi(n(\,\gamma(u) + d))] \\ &\leq \delta + \theta\,[\psi(\gamma(c)) + \psi(d) + 2\,n\,\psi(\gamma(u) + d)]. \end{aligned}$$

Hence

$$|f(u^c u) - 4f(u)| \leq \frac{\delta}{n^2} + \frac{\theta}{n^2}\,[\psi(\gamma(c)) + \psi(d)] + 2\,\theta\,\frac{n}{n^2}\,\psi(\gamma(u) + d)].$$

Therefore, by letting $n \to \infty$, we see that

$$f(u^c u) = 4f(u)$$

and the proof is now complete. □

Suppose that A and B arbitrary groups. For any $b \in B$ let $A(b)$ be the group isomorphic to A under isomorphism $a \to a(b)$. We denote by $H = A^{(B)} = \prod_{b\in B} A(b)$ the direct product of the group $A(b)$. Cearly, if $a(b_1)a(b_2)\cdots a(b_k)$ is some element of H, then for $b \in B$ the mapping

$$b^* : a(b_1)\, a(b_2) \cdots a(b_k) \to a(b_1 b)\, a(b_2 b) \cdots a(b_k b)$$

is an automorphism of H, and the mapping $b \to b^*$ is an embedding of B in $Aut\, H$. Hence, we can form a semidirect product $G = B \cdot H$. This group is the *wreath product* of the groups A and B and will be denoted by $G = A \wr B$. We shell identify the group A with subgroup $A(1)$ of H, where 1 is unit element of B. Thus, we may assume that A is a subgroup of H.

Let $\gamma_A : A \to \mathrm{R}_0^+$ and $\gamma_A(xy) \le \gamma_A(x) + \gamma_A(y)$ for any $x, y \in A$. Let $\gamma_B : B \to \mathrm{R}_0^+$ such that $\gamma_B(xy) \le \gamma_B(x) + \gamma_B(y)$ for any $x, y \in B$. Let γ be an extension of the function γ_A from A to H defined by

$$\gamma(a_1(b_1)\, a_2(b_2) \cdots a_m(b_m)) = \sum_{i=1}^{m} \gamma_A(a_i), \tag{4.11}$$

$$\gamma(b \cdot a_1(b_1)\, a_2(b_2) \cdots a_m(b_m)) = \gamma_B(b) + \gamma(a_1(b_1)\, a_2(b_2) \cdots a_m(b_m)). \tag{4.12}$$

Let C be the group of order 2 with generator c. Consider the group $A \wr C$.

Lemma 4.2. *If for some $a_1, b_1 \in A$ we have equality*

$$|f(a_1 b_1) + f(a_1 b_1^{-1}) - 2f(a_1) - 2f(b_1)| = \delta > 0,$$

then there exist $x, y \in H$ such that

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| = 4\delta.$$

Proof. Let $u = a_1 b_1$. Then $u^c u = u u^c$. Using relation (4.10) we get

$$\begin{aligned}
&f(a_1 a_1^c b_1 b_1^c) + f(a_1 a_1^c (b_1^{-1})^c b_1^{-1}) - 2f(a_1 a_1^c) - 2f(b_1 b_1^c)\\
=&f(a_1 b_1 a_1^c b_1^c) + f(a_1 b_1^{-1} a_1^c (b_1^{-1})^c) - 2f(a_1 a_1^c) - 2f(b_1 b_1^c)\\
=&4f(a_1 b_1) + 4f(a_1 b_1^{-1}) - 8f(a_1) - 8f(b_1)\\
=&4\delta.
\end{aligned}$$

The proof is completed. □

Theorem 4.3. *Let A be a group and $\gamma : A \to R_0^+$ be a function satisfying relation $\gamma(xy) \le \gamma(x) + \gamma(y)$ for all $x, y \in A$, then A can be embedded into a group G such that the equation* (3.1) *is (ψ, γ)-stable on G.*

Proof. Let C_i denotes a group of order two for any $i \in \mathrm{N}$. Define function γ on C_i as zero function. Consider a chain of groups:

$$A_1 = A, \quad A_2 = A_1 \wr C_1, \quad A_3 = A_2 \wr C_2, \ \ldots, \quad A_{k+1} = A_k \wr C_k, \ldots$$

Now define the following chain of embeddings:

$$A_1 \to A_2 \to A_3 \to \cdots \to A_{k+1} \to \ldots \tag{4.13}$$

by identifying A_k with $A_k(1)$ – a subgroup of A_{k+1}. Let G be the direct limit of (4.13). Then $G = \cup_{k \in \mathrm{N}} A_k$ and

$$A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} \subset \ldots\ldots \subset G.$$

Let us extend function γ from group A_i onto A_{i+1} by the rule mentioned in (4.11) and (4.12)). Let $f \in PQ_{\psi,\gamma}(G)$. Suppose that there are a_1, b_1 in A_1, such that

$$|f(a_1 b_1) + f(a_1 b_1^{-1}) - 2f(a_1) - 2f(b_1)| = \delta > 0.$$

Put $a_2 = a_1 a_1^{c_1}$, $b_2 = b_1 b_1^{c_1}$. Then by Lemma 4.2 we get

$$|f(a_2 b_2) + f(a_2 b_2^{-1}) - 2f(a_2) - 2f(b_2)| = 4\delta.$$

Furthermore, for any $k \in \mathrm{N}$, we set $a_{k+1} = a_k a_k^{c_k}$, $b_{k+1} = b_k b_k^{c_k}$. Using Lemma 4.2 k times, we obtain

$$|f(a_{k+1} b_{k+1}) + f(a_{k+1} b_{k+1}^{-1}) - 2f(a_{k+1}) - 2f(b_{k+1})| = 4^k \delta.$$

From the way of extending γ (see (4.11) and (4.12)), it follows that

$$\gamma(a_2) = \gamma(a_1 a_1^{c_1}) = \gamma(a_1) + \gamma(a_1^{c_1}) = 2\gamma(a_1).$$

Similarly, $\gamma(b_2) = 2\gamma(b_1)$. Using induction on k, we get $\gamma(a_{k+1}) = 2^k \gamma(a_1)$ and $\gamma(b_{k+1}) = 2^k \gamma(b_1)$.

Now if r and θ nonnegative numbers, such that for any $x, y \in G$, we have relation

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \le r + \theta[\psi(\gamma(x)) + \psi(\gamma(y))],$$

then

$$\begin{aligned}4^k\delta &= |f(a_{k+1}b_{k+1}) + f(a_{k+1}b_{k+1}^{-1}) - 2f(a_{k+1}) - 2f(b_{k+1})| \\ &\le r + \theta\,[\psi(\gamma(a_{k+1})) + \psi(\gamma(b_{k+1}))] \\ &= r + \theta\,[\psi(2^k\,\gamma(a_1)) + \psi(2^k\,\gamma(b_1))] \\ &\le r + 2^k\,\theta\,[\psi(\gamma(a_1)) + \psi(\gamma(b_1))].\end{aligned}$$

Therefore

$$\delta \le \frac{r}{4^k} + \theta\,\frac{2^k}{4^k}\,[\psi(\gamma(a_1)) + \psi(\gamma(b_1))].$$

Because of the last relation is true for any $k \in \mathbb{N}$ we obtain $\delta = 0$. So $f\big|_{A_1} \in Q(A_1)$. Similarly, we verify that $f\big|_{A_k} \in Q(A_k)$ for any $k \in \mathbb{N}$. Therefore $f \in Q(G)$. The proof is now complete. □

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