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# On Volterra and Fredholm Type Integrodifferential Equations *

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**Abstract**

This paper deals with the existence, uniqueness and other properties of the solutions of certain Volterra and Fredholm type integrodifferential equations. The well known Banach fixed point theorem coupled with Bielcki type norm and the integral inequalities with explicit estimates are used to establish the results.

**Keywords and Phrases:** *Volterra and Fredholm type, Integrodifferential equations, Banach fixed point theorem, Bielcki type norm, Inequalities with explicit estimates, Existence and uniqueness, Continuous dependence.*

## 1. Introduction

Consider the nonlinear Volterra and Fredholm integrodifferential equations of the forms

$$x(t) = g(t) + \int_a^t f(t, s, x(s), x'(s))\, ds, \tag{1.1}$$

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and

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s))\, ds, \tag{1.2}$$

for $-\infty < a \le t \le b < \infty$, where $x, g, f$ are real vectors with $n$ components and $'$ denotes the derivative. Let $R^n$ denotes the n-dimensional Euclidean space with appropriate norm denoted by $|.|$ and $R$ the set of real numbers. Let $I = [a, \infty)$, $J = [a, b]$, $R_+ = [0, \infty)$ be the given subsets of $R$ and $C(S_1, S_2)$ denotes the class of continuous functions from the set $S_1$ to the set $S_2$ . For $-\infty < a \le s \le t \le b < \infty$, $u, v \in R^n$ , the functions $g(t)$ and $f(t, s, u, v)$ are continuous and are continuIusly differentiable with respect to $t$.

The literature concerning the Volterra and Fredholm integral equations of the forms (1.1) and (1.2) when the function $f$ is of the form $f(t, s, x)$ is particularly rich. A good deal of information on such equations may be found in [3,5,6,8,12] and some of the references cited therein. The purpose of this paper is to study the existence, uniqueness and other properties of solutions of equations (1.1) and (1.2) under various assumptions on the functions involved in equations (1.1) and (1.2). The main tools employed in the analysis are based on the applications of the well known Banach fixed point theorem (see [5, p. 37]) coupled with Bielcki type norm (see [2]) and the integral inequalities with explicit estimates given in [10, p. 20] and [11, p. 41].

## 2. Existence and uniqueness

By a solution of equation (1.1) or (1.2) we mean a continuous function $x(t)$ for $-\infty < a \le t \le b < \infty$ which is continuously differentiable with respect to $t$ and satisfy the corresponding equation (1.1) or (1.2). For every continuous function $x(t)$ in $R^n$ together with its continuous first derivative $x'(t)$ for $-\infty < a \le t \le b < \infty$ we denote by $|x(t)|_1 = |x(t)| + |x'(t)|$ . Let $E$ be a space of those continuous functions $x(t)$ in $R^n$ together with its continuous first derivative $x'(t)$ in $R^n$ for $-\infty < a \le t \le b < \infty$ which fulfil the condition

$$|x(t)|_1 = \mathrm{O}(\exp(\lambda t)) \tag{2.1}$$

where $\lambda$ is a positive constant. In the space $E$ we define the norm (see [2,9])

$$|x|_E = \sup_{-\infty < a \le t \le b < \infty} \{|x(t)|_1 \exp(-\lambda t)\}. \tag{2.2}$$

It is easy to see that $E$ with the norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there is a constant $N \geq 0$ such that

$$|x(t)|_1 \leq N \exp(\lambda t).$$

Using this fact in (2.2) we observe that

$$|x|_E \leq N. \tag{2.3}$$

We need the following special versions of the inequalities given in [10, p. 20] and [11, p. 41]. We shall state them in the following lemmas for completeness.

**Lemma 1.** *Let* $u(t), p(t) \in C(I, R_+)$, $k(t,\sigma), \frac{\partial}{\partial t}k(t,\sigma) \in C(D, R_+)$ *where* $D = \{(t,\sigma) \in I^2 : a \leq \sigma \leq t < \infty\}$. *If*

$$u(t) \leq p(t) + \int_a^t k(t,\sigma)\, u(\sigma) d\sigma,$$

*for* $t \in I$, *then*

$$u(t) \leq p(t) + \int_a^t B(\sigma) \exp\left(\int_\sigma^t A(\tau)\, d\tau\right) d\sigma,$$

*for* $t \in I$, *where*

$$A(t) = k(t,t) + \int_a^t \frac{\partial}{\partial t} k(t,s) ds, \tag{2.4}$$

$$B(t) = k(t,t)\, p(t) + \int_a^t \frac{\partial}{\partial t} k(t,s)\, p(s) ds, \tag{2.5}$$

*for* $t \in I$.

**Lemma 2.** *Let* $u(t), p(t), q(t), e(t) \in C(J, R_+)$ *and suppose that*

$$u(t) \leq p(t) + q(t) \int_a^b e(s)\, u(s) ds,$$

*for* $t \in J$. *If*

$$d = \int_a^b e(s)\, q(s)\, ds < 1, \tag{2.6}$$

*then*

$$u(t) \le p(t) + q(t) \left\{ \frac{1}{1-d} \int_a^b e(s)\, p(s)\, ds \right\},$$

*for* $t \in J$.

The following theorem concerning the existence of a unique solution of equation (1.1) holds.

**Theorem 1.** *Assume that* $(i)$ *the function* $f$ *in equation (1.1) and its derivative with respect to* $t$ *satisfy the conditions*

$$|f(t,s,u,v) - f(t,s,\bar{u},\bar{v})| \le h_1(t,s)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{2.7}$$

$$\left|\frac{\partial}{\partial t} f(t,s,u,v) - \frac{\partial}{\partial t} f(t,s,\bar{u},\bar{v})\right| \le h_2(t,s)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{2.8}$$

*where for* $i = 1, 2$ *and* $a \le s \le t < \infty$, $h_i(t,s) \in C(I^2, R_+)$,

$(ii)$ *there exist nonnegative constants* $\alpha_1, \alpha_2$ *such that* $\alpha_1 + \alpha_2 < 1$ *and*

$$\int_a^t h_1(t,s) \exp(\lambda s)\, ds \le \alpha_1 \exp(\lambda t), \tag{2.9}$$

$$h_1(t,t) \exp(\lambda t) + \int_a^t h_2(t,s) \exp(\lambda s)\, ds \le \alpha_2 \exp(\lambda t), \tag{2.10}$$

*for* $t \in I$, *where* $\lambda$ *is as given in (2.1),* *(iii) there exist nonnegative constants $P_1, P_2$ such that*

$$|g(t)| + \int_a^t |f(t,s,0,0)|\,ds \le P_1 \exp(\lambda t), \tag{2.11}$$

$$|g'(t)| + |f(t,s,0,0)| + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,0,0)\right| ds \le P_2 \exp(\lambda t), \tag{2.12}$$

*where $g, f$ are defined in equation (1.1) and $\lambda$ is as given in (2.1).*

Then equation (1.1) has a unique solution $x(t)$ in $E$ on $I$.

**Proof.** Let $x(t) \in E$ and define the operator

$$(Tx)(t) = g(t) + \int_a^t f(t,s,x(s),x'(s))\,ds. \tag{2.13}$$

Differentiating both sides of (2.13) with respect to $t$ we get

$$(Tx)'(t) = g'(t) + f(t,t,x(t),x'(t)) + \int_a^t \frac{\partial}{\partial t} f(t,s,x(s),x'(s))\,ds. \tag{2.14}$$

Now, we show that $Tx$ maps $E$ into itself. Evidently, $(Tx)$ ,$(Tx)'$ are continuous on $I$ and $(Tx), (Tx)' \in R^n$. We verify that (2.1) is fulfilled. From (2.13), (2.14) and using the hypotheses and (2.3) we have

$$
\begin{aligned}
|(Tx)(t)| &\le |g(t)| + \int_a^t |f(t,s,x(s),x'(s)) - f(t,s,0,0) + f(t,s,0,0)|\,ds \\
&\le |g(t)| + \int_a^t |f(t,s,0,0)|ds + \int_a^t h_1(t,s)\,|x(s)|_1\,ds \\
&\le P_1 \exp(\lambda t) + |x|_E \int_a^t h_1(t,s) \exp(\lambda s)\,ds \\
&\le [P_1 + N\alpha_1] \exp(\lambda t),
\end{aligned} \tag{2.15}
$$

and

$$\left|(Tx)'(t)\right| \leq |g'(t)| + |f(t,t,x(t),x'(t)) - f(t,t,0,0) + f(t,t,0,0)|$$
$$+\int_a^t \left|\frac{\partial}{\partial t} f(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} f(t,s,0,0) + \frac{\partial}{\partial t} f(t,s,0,0)\right| ds$$
$$\leq \ |g'(t)| + |f(t,t,0,0)| + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,0,0)\right| ds + h_1(t,t)|x(t)|_1$$
$$+\int_a^t h_2(t,s)|x(s)|_1 \, ds$$
$$\leq \ P_2 \exp(\lambda t) + |x|_E \, h_1(t,t)\exp(\lambda t) + |x|_E \int_a^t h_2(t,s)\exp(\lambda s)\, ds$$
$$\leq \ [P_2 + N\alpha_2]\exp(\lambda t). \tag{2.16}$$

From (2.15) and (2.16) we get

$$|(Tx)(t)|_1 \leq [P_1 + P_2 + N(\alpha_1 + \alpha_2)]\exp(\lambda t). \tag{2.17}$$

From (2.17) it follows that $Tx \in E$ . This proves that $T$ maps $E$ into itself.

Now, we verify that the operator $T$ is a contraction map. Let $x(t), y(t) \in E$. From (2.13) and (2.14) and using the hypotheses we have

$$|(Tx)(t) - (Ty)(t)| \leq \int_a^t |f(t,s,x(s),x'(s)) - f(t,s,y(s),y'(s))| ds$$
$$\leq \ \int_a^t h_1(t,s)|x(s) - y(s)|_1 \, ds$$
$$\leq \ |x-y|_E \int_a^t h_1(t,s)\exp(\lambda s)\, ds$$
$$\leq \ |x-y|_E \, \alpha_1 \exp(\lambda t), \tag{2.18}$$

and

$$\left|(Tx)'(t) - (Ty)'(t)\right| \leq |f(t,t,x(t),x'(t)) - f(t,t,y(t),y'(t))|$$
$$+ \int_a^t \left|\frac{\partial}{\partial t} f(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} f(t,s,y(s),y'(s))\right| ds$$
$$\leq h_1(t,t)\,|x(t) - y(t)|_1 + \int_a^t h_2(t,s)\,|x(s) - y(s)|_1\, ds$$
$$\leq |x-y|_E\, h_1(t,t)\exp(\lambda t) + |x-y|_E \int_a^t h_2(t,s)\exp(\lambda s)\, ds$$
$$\leq |x-y|_E\, \alpha_2 \exp(\lambda t). \tag{2.19}$$

From (2.18) and (2.19) we get

$$|(Tx)(t) - (Ty)(t)|_1 \leq |x-y|_E\,(\alpha_1 + \alpha_2)\exp(\lambda t). \tag{2.20}$$

From (2.20) we obtain

$$|Tx - Ty|_E \leq (\alpha_1 + \alpha_2)\,|x-y|_E\,.$$

Since $\alpha_1 + \alpha_2 < 1$ , it follows from Banach fixed point theorem (see [5,p. 37]) that $T$ has a unique fixed point in $E$ . The fixed point of $T$ is however a solution of equation (1.1). The proof is complete.

Our result on the existence of a unique solution of equation (1.2) is embodied in the following theorem.

**Theorem 2.** *Assume that* $(i)$ *the function* $f$ *in equation (1.2) and its derivative with respect to* $t$ *satisfy the conditions*

$$|f(t,s,u,v) - f(t,s,\bar{u},\bar{v})| \leq r_1(t,s)\,[|u-\bar{u}| + |v-\bar{v}|]\,, \tag{2.21}$$

$$\left|\frac{\partial}{\partial t} f(t,s,u,v) - \frac{\partial}{\partial t} f(t,s,\bar{u},\bar{v})\right| \leq r_2(t,s)\,[|u-\bar{u}| + |v-\bar{v}|]\,, \tag{2.22}$$

*where for* $i = 1,2$ *and* $a \leq s \leq t \leq b, r_i(t,s) \in C(J^2, R_+)\,,$

*(ii) there exist nonnegative constants $\beta_1, \beta_2$ such that $\beta_1 + \beta_2 < 1$ and*

$$\int_a^b r_1(t,s)\exp(\lambda s)\,ds \le \beta_1 \exp(\lambda t), \tag{2.23}$$

$$\int_a^b r_2(t,s)\exp(\lambda s)\,ds \le \beta_2 \exp(\lambda t), \tag{2.24}$$

*for $t \in J$, where $\lambda$ is as given in (2.1).*

*(iii) there exist nonnegative constants $Q_1, Q_2$ such that*

$$|g(t)| + \int_a^b |f(t,s,0,0)|\,ds \le Q_1 \exp(\lambda t), \tag{2.25}$$

$$|g'(t)| + \int_a^b \left|\frac{\partial}{\partial t} f(t,s,0,0)\right| ds \le Q_2 \exp(\lambda t), \tag{2.26}$$

*where $g, f$ are defined in equation (1.2) and $\lambda$ is as given in (2.1).*

Then equation (1.2) has a unique solution $x(t)$ in $E$ on $J$.

**Proof.** Let $x(t) \in E$ and define the operator

$$(Sx)(t) = g(t) + \int_a^b f(t,s,x(s),x'(s))\,ds. \tag{2.27}$$

Differentiating both sides of (2.27) with respect to $t$ we get

$$(Sx)'(t) = g'(t) + \int_a^b \frac{\partial}{\partial t} f(t,s,x(s),x'(s))\,ds. \tag{2.28}$$

The rest of the proof follows by the similar arguments as in the proof of Theorem 1 with suitable modifications. We omit the details.

**Remark 1.** *We note that the norm $|.|_E$ defined by (2.2) was first used by Bielecki [2] in 1956 for proving global existence and uniqueness of solutions of ordinary differential equations. For developments related to this topic, see [4,7] and the references given therein.*

Indeed, the following theorems are true concerning the uniqueness of solutions of equations (1.1) and (1.2) without existence parts.

**Theorem 3.** *Assume that the function $f$ in equation (1.1) and its derivative with respect to $t$ satisfy the conditions (2.7) and (2.8). Further assume that $h_1(t,s), h_2(t,s)$ are continuously differentiable with respect to $t$ and are non-negative and $h_1(t,t) \leq c$, where $c < 1$ is a constant. Then the equation (1.1) has at most one solution on $I$.*

**Proof.** Let $x(t)$ and $y(t)$ be two solutions of equation (1.1). Then we have

$$\begin{aligned}
& |x(t) - y(t)| + |x'(t) - y'(t)| \\
\leq & \int_a^t |f(t,s,x(s),x'(s)) - f(t,s,y(s),y'(s))| ds \\
& + |f(t,t,x(t),x'(t)) - f(t,t,y(t),y'(t))| \\
& + \int_a^t \left| \frac{\partial}{\partial t} f(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} f(t,s,y(s),y'(s)) \right| ds \\
\leq & \int_a^t h_1(t,s) [|x(s) - y(s)| + |x'(s) - y'(s)|] ds \\
& + h_1(t,t) [|x(t) - y(t)| + |x'(t) - y'(t)|] \\
& + \int_a^t h_2(t,s) [|x(s) - y(s)| + |x'(s) - y'(s)|] ds. \qquad (2.29)
\end{aligned}$$

From (2.29) and using the assumption $h_1(t,t) \le c$ we observe that

$$|x(t) - y(t)| + |x'(t) - y'(t)| \le \frac{1}{1-c}\int_a^t [h_1(t,s) + h_2(t,s)]\,[|x(s) - y(s)| + |x'(s) - y'(s)|]ds. \quad (2.30)$$

Now an application of Lemma 1 (when $p(t) = 0$ ) to (2.30) yields

$$|x(t) - y(t)| + |x'(t) - y'(t)| \le 0,$$

and hence $x(t) = y(t)$ . Thus there is at most one solution to equation (1.1) on $I$ .

**Theorem 4.** *Assume that the function $f$ in equation (1.2) and its derivative with respect to $t$ satisfy the conditions (2.21) and (2.22) with $r_i(t,s) = q(t)\,e_i(s)$ for $i = 1,2$ , where $q, e_i \in C(J, R_+)$ and let*

$$d_1 = \int_a^b [e_1(s) + e_2(s)]q(s)\,ds < 1.$$

*Then the equation (1.2) has at most one solution on $J$ .*

**Proof.** Let $x(t)$ and $y(t)$ be two solutions of equation (1.2). Then we have

$$\begin{aligned} &|x(t) - y(t)| + |x'(t) - y'(t)| \\ \le\ & \int_a^b |f(t,s,x(s),x'(s)) - f(t,s,y(s),y'(s))|ds \\ &+ \int_a^b \left|\frac{\partial}{\partial t}f(t,s,x(s),x'(s)) - \frac{\partial}{\partial t}f(t,s,y(s),y'(s))\right|ds \\ \le\ & q(t)\int_a^b [e_1(s) + e_2(s)]\,[|x(s) - y(s)| + |x'(s) - y'(s)|]ds. \quad (2.31) \end{aligned}$$

Now an application of Lemma 2 (when $p(t) = 0$ ) to (2.31) yields

$$|x(t) - y(t)| + |x'(t) - y'(t)| \le 0,$$

and hence $x(t) = y(t)$ , which proves the uniqueness of solutions of equation (1.2) on $J$ .

## 3. Estimates on solutions

In this section we obtain estimates on the solutions of equations (1.1) and (1.2) under some suitable conditions on the functions involved therein.

First, we shall give the following theorems concerning the estimates on the solutions of equations (1.1) and (1.2).

**Theorem 5.** *Assume that the functions* $g, f$ *in equation (1.1) and their derivatives with respect to* $t$ *satisfy the conditions*

$$|g(t)| + |g'(t)| \leq r(t), \tag{3.1}$$

$$|f(t,s,u,v)| \leq h_1(t,s)\left[|u| + |v|\right], \tag{3.2}$$

$$\left|\frac{\partial}{\partial t} f(t,s,u,v)\right| \leq h_2(t,s)\left[|u| + |v|\right], \tag{3.3}$$

*where* $r(t) \in C(I, R_+)$ *and for* $i = 1,2,\ a \leq s \leq t < \infty,\ h_i(t,s), \frac{\partial}{\partial t} h_i(t,s) \in C(I^2, R_+)$ . *Let* $\bar{k}(t,s) = h_1(t,s) + h_2(t,s)$ *and assume that* $h_1(t,t) \leq c$, *where* $c < 1$ *is a constant. If* $x(t)$ , $t \in I$ *is any solution of equation (1.1), then*

$$|x(t)| + |x'(t)| \leq \frac{r(t)}{1-c} + \int_a^t B_1(\sigma) \exp\left(\int_\sigma^t A_1(\tau)\, d\tau\right) d\sigma, \tag{3.4}$$

*for* $t \in I$ , *where* $A_1(t)$ *and* $B_1(t)$ *are defined respectively by the right hand sides of (2.4) and (2.5) by replacing* $k(t,s)$ *by* $\frac{\bar{k}(t,s)}{1-c}$ *and* $p(t)$ *by* $\frac{r(t)}{1-c}$ .

**Proof.** By using the fact that $x(t), t \in I$ is a solution of equation (1.1) and hypotheses we have

$$\begin{aligned}
|x(t)| + |x'(t)| &\leq |g(t)| + \int_a^t |f(t,s,x(s),x'(s))|\,ds \\
&\quad + |g'(t)| + |f(t,t,x(t),x'(t))| + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,x(s),x'(s))\right| ds \\
&\leq r(t) + \int_a^t h_1(t,s)\,[|x(s)| + |x'(s)|]\,ds \\
&\quad + h_1(t,t)\,[|x(t)| + |x'(t)|] + \int_a^t h_2(t,s)\,[|x(s)| + |x'(s)|]\,ds. \qquad (3.5)
\end{aligned}$$

From (3.5) and using the assumption $h_1(t,t) \leq c$ we observe that

$$|x(t)| + |x'(t)| \leq \frac{r(t)}{1-c} + \frac{1}{1-c}\int_a^t \bar{k}(t,s)\,[|x(s)| + |x'(s)|]\,ds. \qquad (3.6)$$

Now an application of Lemma 1 to (3.6) yields (3.4).

**Theorem 6.** *Assume that the functions $g, f$ involved in equation (1.2) and their derivatives with respect to $t$ satisfy the conditions*

$$|g(t)| + |g'(t)| \leq h(t), \qquad (3.7)$$

$$|f(t,s,u,v)| \leq q(t)\,e_1(s)\,[|u| + |v|], \qquad (3.8)$$

$$\left|\frac{\partial}{\partial t} f(t,s,u,v)\right| \leq q(t)\,e_2(s)\,[|u| + |v|], \qquad (3.9)$$

*where $h(t), q(t), e_1(t), e_2(t) \in C(J, R_+)$ and let $d_1$ be as in Theorem 4. If $x(t), t \in J$ is any solution of equation (1.2), then*

$$|x(t)| + |x'(t)| \leq h(t) + q(t)\left\{\frac{1}{1-d_1}\int_a^b [e_1(s) + e_2(s)]h(s)\,ds\right\}, \qquad (3.10)$$

*for $t \in J$.*

The proof follows by the similar arguments as in the proof of Theorem 5 and applying Lemma 2. We leave the details to the readers.

**Remark 2.** *We note that the estimates obtained in (3.4) and (3.10) yield, not only the bounds on the solutions of equations (1.1) and (1.2) , but also the bounds on their derivatives. If the bounds on the right hand sides in (3.4) and (3.10) are bounded, then the solutions of equations (1.1) and (1.2) and their derivatives are bounded.*

Next, we shall obtain the estimates on the solutions of equations (1.1) and (1.2) assuming that the function $f$ and its derivative with respect to $t$ satisfy Lipschitz type conditions.

**Theorem 7.** *Assume that the function $f$ and its derivative with respect to $t$ satisfy the conditions (2.7) and (2.8). Let for $i = 1, 2$ , $h_i(t,s)$ , $\frac{\partial}{\partial t} h_i(t,s)$ , $\bar{k}(t,s)$ , $h_1(t,t)$ and $c$ be as in Theorem 5 and*

$$\alpha(t) = |f(t,t,g(t),g'(t))| + \int_a^t |f(t,s,g(s),g'(s))| + \int_a^t \left| \frac{\partial}{\partial t} f(t,s,g(s),g'(s)) \right|,$$

*where $g$ is defined as in equation (1.1). If $x(t), t \in I$ is any solution of equation (1.1), then*

$$\begin{aligned} & |x(t) - g(t)| + |x'(t) - g'(t)| \\ \leq \; & \frac{\alpha(t)}{1-c} + \int_a^t B_2(\sigma) \exp\left( \int_\sigma^t A_1(\tau)\, d\tau \right) d\sigma, \end{aligned} \tag{3.11}$$

*for $t \in I$ , where $A_1(t)$ and $B_2(t)$ are defined respectively by the right hand sides of (2.4) and (2.5) , replacing $k(t,s)$ by $\frac{\bar{k}(t,s)}{1-c}$ and $p(t)$ by $\frac{\alpha(t)}{1-c}$.*

**Proof.** Since $x(t)$ is a solution of equation (1.1), by using the hypotheses we have

$$|x(t)-g(t)| \leq \int_a^t |f(t,s,x(s),x'(s)) - f(t,s,g(s),g'(s))|ds$$
$$+\int_a^t |f(t,s,g(s),g'(s))|ds$$
$$\leq \int_a^t |f(t,s,g(s),g'(s))|ds + \int_a^t h_1(t,s)\left[|x(s)-g(s)| + |x'(s)-g'(s)|\right]ds. \tag{3.12}$$

and

$$|x'(t)-g'(t)| \leq |f(t,t,x(t),x'(t)) - f(t,t,g(t),g'(t))| + |f(t,t,g(t),g'(t))|$$
$$+\int_a^t \left|\frac{\partial}{\partial t}f(t,s,x(s),x'(s)) - \frac{\partial}{\partial t}f(t,s,g(s),g'(s))\right|ds$$
$$+\int_a^t \left|\frac{\partial}{\partial t}f(t,s,g(s),g'(s))\right|ds$$
$$\leq |f(t,t,g(t),g'(t))| + \int_a^t \left|\frac{\partial}{\partial t}f(t,s,g(s),g'(s))\right|ds$$
$$+h_1(t,t)\left[|x(t)-g(t)| + |x'(t)-g'(t)|\right]$$
$$+\int_a^t h_2(t,s)\left[|x(s)-g(s)| + |x'(s)-g'(s)|\right]ds. \tag{3.13}$$

From (3.12), (3.13) and using the assumption $h_1(t,t) \leq c$ we observe that

$$|x(t)-g(t)| + |x'(t)-g'(t)|$$
$$\leq \frac{\alpha(t)}{1-c} + \frac{1}{1-c}\int_a^t \bar{k}(t,s)\left[|x(s)-g(s)| + |x'(s)-g'(s)|\right]ds. \tag{3.14}$$

Now an application of Lemma 1 to (3.14) yields (3.11).

**Theorem 8.** *Suppose that the hypotheses of Theorem 4 hold and let*

$$\bar{\alpha}(t) = \int_a^b |f(t,s,g(s),g'(s))|ds + \int_a^b \left|\frac{\partial}{\partial t} f(t,s,g(s),g'(s))\right| ds,$$

*for $t \in J$. If $x(t)$ , $t \in J$. is any solution of equation (1.2), then*

$$\begin{aligned} & |x(t) - g(t)| + |x'(t) - g'(t)| \\ \leq \; & \bar{\alpha}(t) + q(t) \left\{ \frac{1}{1-d_1} \int_a^b [e_1(s) + e_2(s)] \, \bar{\alpha}(s) ds \right\}, \end{aligned} \tag{3.15}$$

*for $t \in J$.*

The proof follows by the similar arguments as in the proof of Theorem 7 with suitable modifications. We omit the details.

# 4. Continuous dependence

In this section we shall deal with the continuous dependence of solutions of equations (1.1) and (1.2) on the functions involved therein and also the continuous dependence of solutions of equations of the forms (1.1) and (1.2) on parameters.

Consider the equations (1.1) and (1.2) and the corresponding Volterra and Fredholm integral equations

$$y(t) = G(t) + \int_a^b F(t,s,y(s),y'(s))\, ds, \tag{4.1}$$

and

$$y(t) = G(t) + \int_a^b F(t,s,y(s),y'(s))\, ds, \tag{4.2}$$

for $-\infty < a \leq t \leq b < \infty$, where $y, G, F$ are real vectors with $n$ components. The functions $G(t)$ and $F(t,s,u,v)$ for $-\infty < a \leq s \leq t \leq b < \infty, u, v \in R^n$ are continuous and are continuously differentiable with respect to $t$ .

The following theorems deals with the continuous dependence of solutions of equations (1.1) and (1.2) on the functions involved therein.

**Theorem 9.** *Assume that the function $f$ in equation (1.1) and its derivative with respect to $t$ satisfy the conditions (2.7) and (2.8). Let for $i = 1, 2$ , $h_i(t,s), \frac{\partial}{\partial t}h_i(t,s)$ , $\bar{k}(t,s), h_1(t,t)$ and $c$ be as in Theorem 5. Suppose that*

$$|g(t) - G(t)| + \int_a^t |f(t,s,y(s),y'(s)) - F(t,s,y(s),y'(s))|\, ds \le r_1(t), \tag{4.3}$$

$$|g'(t) - G'(t)| + |f(t,t,y(t),y'(t)) - F(t,t,y(t),y'(t))| + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,y(s),y'(s)) - \frac{\partial}{\partial t} F(t,s,y(s),y'(s))\right| ds \le r_2(t), \tag{4.4}$$

*where $g, f$ and $G, F$ are the functions involved in equations (1.1) and (4.1) and $r_1(t), r_2(t) \in C(I, R_+)$ . Let $x(t)$ and $y(t)$ , $t \in I$ be the solutions of equations (1.1) and (4.1) respectively. Then the solution $x(t)$ , $t \in I$ of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).*

**Proof.** Using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.1) and (4.1) and the hypotheses we have

$$|x(t) - y(t)| \le |g(t) - G(t)| + \int_a^t |f(t,s,x(s),x'(s)) - f(t,s,y(s),y'(s))| ds + \int_a^t |f(t,s,y(s),y'(s)) - F(t,s,y(s),y'(s))| ds$$
$$\le r_1(t) + \int_a^t h_1(t,s)\left[|x(s) - y(s)| + |x'(s) - y'(s)|\right] ds, \tag{4.5}$$

and

$$\begin{aligned}
|x'(t)-y'(t)| &\le |g'(t)-G'(t)| + |f(t,t,x(t),x'(t)) - f(t,t,y(t),y'(t))| \\
&+ |f(t,t,y(t),y'(t)) - F(t,t,y(t),y'(t))| \\
&+ \int_a^t \left| \frac{\partial}{\partial t} f(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} f(t,s,y(s),y'(s)) \right| ds \\
&+ \int_a^t \left| \frac{\partial}{\partial t} f(t,s,y(s),y'(s)) - \frac{\partial}{\partial t} F(t,s,y(s),y'(s)) \right| ds \\
\le\ & r_2(t) + h_1(t,t)\left[|x(t)-y(t)| + |x'(t)-y'(t)|\right] \\
&+ \int_a^t h_2(t,s)\left[|x(s)-y(s)| + |x'(s)-y'(s)|\right] ds. \qquad (4.6)
\end{aligned}$$

From (4.5) and (4.6) and using the assumption that $h_1(t,t) \le c$, it is easy to observe that

$$|x(t)-y(t)| + |x'(t)-y'(t)|$$

$$\le \frac{r_1(t)+r_2(t)}{1-c} + \frac{1}{1-c}\int_a^t \bar{k}(t,s)\left[|x(s)-y(s)| + |x'(s)-y'(s)|\right] ds. \quad (4.7)$$

Now an application of Lemma 1 to (4.7) yields

$$|x(t)-y(t)| + |x'(t)-y'(t)| \le \frac{r_1(t)+r_2(t)}{1-c} + \int_a^t B_3(\sigma)\exp\left(\int_\sigma^t A_1(\tau)\,d\tau\right) d\sigma, \quad (4.8)$$

for $t \in I$, where $A_1(t)$ and $B_3(t)$ are defined respectively by the right hand sides of (2.4) and (2.5), replacing $k(t,s)$ by $\frac{\bar{k}(t,s)}{1-c}$ and $p(t)$ by $\frac{r_1(t)+r_2(t)}{1-c}$ . From (4.8) it follows that the solutions of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

**Theorem 10.** *Assume that the hypotheses of Theorem 4 hold. Suppose that*

$$|g(t)-G(t)| + \int_a^b |f(t,s,y(s),y'(s)) - F(t,s,y(s),y'(s))| ds \le \bar{r}_1(t), \quad (4.9)$$

$$|g'(t) - G'(t)| + \int_a^b \left| \frac{\partial}{\partial t} f(t, s, y(s), y'(s)) - \frac{\partial}{\partial t} F(t, s, y(s), y'(s)) \right| ds \le \bar{r}_2(t), \tag{4.10}$$

*where $g, f$ and $G, F$ are the functions involved in equations (1.2) and (4.2) and $\bar{r}_1(t), \bar{r}_2(t) \in C(J, R_+)$. Let $x(t)$ and $y(t)$ , $t \in J$ be the solutions of equations (1.2) and (4.2) respectively. Then the solution $x(t)$ , $t \in J$ of equation (1.2) depends continuously on the functions involved on the right hand side of equation (1.2).*

The proof follows by closely looking at the proof of Theorem 9 and by making use of Lemma 2. Here, we omit the details.

Next, we consider the following Volterra and Fredholm type integral equations

$$z(t) = g(t) + \int_a^t f(t, s, z(s), z'(s), \mu) ds, \tag{4.11}$$

$$z(t) = g(t) + \int_a^t f(t, s, z(s), z'(s), \mu_0) ds, \tag{4.12}$$

and

$$z(t) = g(t) + \int_a^b f(t, s, z(s), z'(s), \mu) ds, \tag{4.13}$$

$$z(t) = g(t) + \int_a^b f(t, s, z(s), z'(s), \mu_0) ds, \tag{4.14}$$

for $-\infty < a \le t \le b < \infty$, where $z, g, f$ are vectors with $n$ components and $\mu, \mu_0$ are real parameters. For $-\infty < a \le s \le t \le b < \infty, u, v \in R^n, \lambda \in R$, the functions $g(t)$ and $f(t, s, u, v, \lambda)$ are continuous and are continuously differentiable with respect to $t$ .

Finally, we present the following theorems which deals with the continuous dependency of solutions of equations (4.11), (4.12) and (4.13), (4.14) on parameters.

**Theorem 11.** *Assume that the function $f$ in equations (4.11),(4.12) and its derivative with respect to $t$ satisfy the conditions*

$$|f(t,s,u,v,\mu)-f(t,s,\bar{u},\bar{v},\mu)|\le h_1(t,s)\left[|u-\bar{u}|+|v-\bar{v}|\right], \tag{4.15}$$

$$|f(t,s,u,v,\mu)-f(t,s,u,v,\mu_0)|\le e_1(t,s)|\mu-\mu_0|, \tag{4.16}$$

$$\left|\frac{\partial}{\partial t}f(t,s,u,v,\mu)-\frac{\partial}{\partial t}f(t,s,\bar{u},\bar{v},\mu)\right|\le h_2(t,s)\left[|u-\bar{u}|+|v-\bar{v}|\right], \tag{4.17}$$

$$\left|\frac{\partial}{\partial t}f(t,s,u,v,\mu)-\frac{\partial}{\partial t}f(t,s,u,v,\mu_0)\right|\le e_2(t,s)|\mu-\mu_0|, \tag{4.18}$$

*where $h_1,h_2,e_1,e_2\in C(I^2,R_+)$. Let, for $i=1,2$, $h_i(t,s)$, $\frac{\partial}{\partial t}h_i(t,s)$, $\bar{k}(t,s)$, $h_1(t,t)$ and $c$ be as in Theorem 5 and*

$$\beta(t)=e_1(t,t)+\int_a^t[e_1(t,s)+e_2(t,s)]ds.$$

*Let $z_1(t)$ and $z_2(t)$ be the solutions of equations (4.11) and (4.12) respectively. Then*

$$|z_1(t)-z_2(t)|+|z_1'(t)-z_2'(t)|\le\frac{|\mu-\mu_0|}{1-c}\beta(t)+\int_a^t B_4(\sigma)\exp\left(\int_\sigma^t A_1(\tau)\,d\tau\right)d\sigma, \tag{2.19}$$

*where $A_1(t)$ and $B_4(t)$ are defined respectively by the right hand sides of (2.4) and (2.5), replacing $k(t,s)$ by $\frac{\bar{k}(t,s)}{1-c}$ and $p(t)$ by $\frac{|\mu-\mu_0|}{1-c}\beta(t)$.*

**Proof.** Let $w(t)=z_1(t)-z_2(t)$. Using the facts that $z_1(t)$ and $z_2(t)$ are the solutions of equations (4.11) and (4.12) we have

$$\begin{aligned}|w(t)| &\le \int_a^t|f(t,s,z_1(s),z_1'(s),\mu)-f(t,s,z_2(s),z_2'(s),\mu)|ds\\ &\quad+\int_a^t|f(t,s,z_2(s),z_2'(s),\mu)-f(t,s,z_2(s),z_2'(s),\mu_0)|ds\\ &\le \int_a^t h_1(t,s)\left[|w(s)|+|w'(s)|\right]ds+\int_a^t e_1(t,s)|\mu-\mu_0|\,ds \qquad (4.20)\end{aligned}$$

and

$$\begin{aligned} |w'(t)| &\leq |f(t,t,z_1(t),z_1'(t),\mu) - f(t,t,z_2(t),z_2'(t),\mu)| \\ &\quad + |f(t,t,z_2(t),z_2'(t),\mu) - f(t,t,z_2(t),z_2'(t),\mu_0)| \\ &\quad + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,z_1(s),z_1'(s),\mu) - \frac{\partial}{\partial t} f(t,s,z_2(s),z_2'(s),\mu)\right| ds \\ &\quad + \int_a^t \left|\frac{\partial}{\partial t} f(t,s,z_2(s),z_2'(s),\mu) - \frac{\partial}{\partial t} f(t,s,z_2(s),z_2'(s),\mu_0)\right| ds \\ &\leq h_1(t,t)\left[|w(t)| + |w'(t)|\right] + e_1(t,t)|\mu-\mu_0| \\ &\quad + \int_a^t h_2(t,s)\left[|w(s)| + |w'(s)|\right] ds + \int_a^t e_2(t,s)|\mu-\mu_0|\, ds. \end{aligned} \tag{4.21}$$

From (4.20) and (4.21) and using the assumption $h_1(t,t) \leq c$, it is easy to observe that

$$|w(t)| + |w'(t)| \leq \frac{|\mu-\mu_0|}{1-c}\beta(t) + \frac{1}{1-c}\int_a^t k(t,s)\left[|w(s)| + |w'(s)|\right] ds. \tag{4.22}$$

Now an application of Lemma 1 to (4.22) yields (4.19), which shows the dependency of solutions of equations (4.11) and (4.12) on parameters.

**Theorem 12.** *Assume that the function $f$ in equations (4.13), (4.14) and its derivative with respect to $t$ satisfy the conditions*

$$|f(t,s,u,v,\mu) - f(t,s,\bar{u},\bar{v},\mu)| \leq \bar{q}(t)\bar{e}_1(s)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{4.23}$$

$$|f(t,s,u,v,\mu) - f(t,s,u,v,\mu_0)| \leq \gamma_1(t,s)|\mu-\mu_0|, \tag{4.24}$$

$$\left|\frac{\partial}{\partial t} f(t,s,u,v,\mu) - \frac{\partial}{\partial t} f(t,s,\bar{u},\bar{v},\mu)\right| \leq \bar{q}(t)\bar{e}_2(s)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{4.25}$$

$$\left|\frac{\partial}{\partial t} f(t,s,u,v,\mu) - \frac{\partial}{\partial t} f(t,s,u,v,\mu_0)\right| \leq \gamma_2(t,s)|\mu-\mu_0|, \tag{4.26}$$

*where* $\bar{q}, \bar{e}_1, \bar{e}_2 \in C(J, R_+), \gamma_1, \gamma_2 \in C(J^2, R_+)$. *Let*

$$\gamma(t) = \int_a^b [\gamma_1(t,s) + \gamma_2(t,s)]ds,$$

*and suppose that*

$$d_2 = \int_a^b [\bar{e}_1(s) + e_2(s)]\bar{q}(s)\,ds < 1.$$

*Let* $z_1(t)$ *and* $z_2(t)$ *be the solutions of equations (4.13) and (4.14) respectively. Then*

$$\begin{aligned} & |z_1(t) - z_2(t)| + |z_1(t) - z_2(t)| \\ \leq\ & |\mu - \mu_0| \left[ \gamma(t) + \bar{q}(t) \left\{ \frac{1}{1-d_2} \int_a^b [\bar{e}_1(s) + e_2(s)]\,\gamma(s)\,ds \right\} \right], \end{aligned} \tag{4.27}$$

*for* $t \in J$.

The proof is similar to that of Theorem 11 with suitable modifications and by making use of Lemma 2. We omit the details.

**Remark 3.** *In a recent paper [1] , the authors have studied the existence, uniqueness and approximation of solutions of equation (1.2) in a Banach space by using Perov's fixed point theorem, the method of successive approximations and a trapezoidal quadrature rule. We note that our approach to the study of equations (1.1) and (1.2) is different and we believe that the results given here are of independent interest.*

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