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On Starlike and Convex Functions with Respect to $2k$-Symmetric Conjugate Points *

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Abstract

In the present paper, two new classes $\mathcal{S}_{sc}^{(k)}(\phi)$ and $\mathcal{C}_{sc}^{(k)}(\phi)$ of functions starlike and convex with respect to $2k$-symmetric conjugate points are introduced. The integral representations for functions belonging to these classes are provided, the convolution conditions, growth, distortion and covering theorems for these classes are also provided.

Keywords and Phrases: *Starlike functions, Convex functions, Close-to-convex functions, Quasi-convex functions, Differential subordination, Hadamard product, $2k$-symmetric conjugate points.*

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbf{C} : z < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$ (see, for details, [3, 5, 6, 9]). Also let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ (z \in \mathcal{U}),$$

which satisfy the condition $\Re\{p(z)\} > 0$.

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $\omega(z) \leq z$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [7]).

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\phi)$ if $f(z)$ satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$. The class $\mathcal{S}^*(\phi)$ and a corresponding convex class $\mathcal{K}(\phi)$ were defined by Ma and Minda [4]. And the results about the convex class $\mathcal{K}(\phi)$ can be easily obtained from the corresponding results of functions in $\mathcal{S}^*(\phi)$.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*_{sc}(\phi)$ if $f(z)$ satisfies the condition

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$. And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_{sc}(\phi)$ if and only if $zf'(z) \in \mathcal{S}^*_{sc}(\phi)$. The classes $\mathcal{S}^*_{sc}(\phi)$ of functions starlike with respect to symmetric conjugate points and $\mathcal{C}_{sc}(\phi)$ of functions convex with respect to symmetric conjugate points were considered recently by Ravichandran [8]. Furthermore, Chen, Wu and Zou [2] discussed a class of functions α-starlike with respect to symmetric conjugate points.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^{(k)}_s(\phi)$ if $f(z)$ satisfies the condition

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$, $k \geq 2$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z) \quad (\varepsilon = \exp(2\pi i/k);\ z \in \mathcal{U}).$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_s^{(k)}(\phi)$ if and only if $zf'(z) \in \mathcal{S}_s^{(k)}(\phi)$. The classes $\mathcal{S}_s^{(k)}(\phi)$ of functions starlike with respect to k-symmetric points and $\mathcal{C}_s^{(k)}(\phi)$ of functions convex with respect to k-symmetric points were considered recently by Wang, Gao and Yuan [10].

Al-Amiri, Coman and Mocanu [1] once introduced and investigated a class of functions starlike with respect to $2k$-symmetric conjugate points, which satisfy the following inequality

$$\Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

where $k \geq 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality

$$f_{2k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[\varepsilon^{-\nu} f(\varepsilon^{\nu} z) + \varepsilon^{\nu} \overline{f(\varepsilon^{\nu} \overline{z})}\right] \quad (\varepsilon = \exp(2\pi i/k);\ z \in \mathcal{U}). \tag{1.2}$$

But until now, we can not give the definition of functions starlike with respect to k-conjugate points ($k \geq 3$), this is still an unsolved problem. Motivated by the above mentioned classes, we now introduce the following two classes of functions starlike and convex with respect to $2k$-symmetric conjugate points, and obtain some interesting results.

Definition 1. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}_{sc}^{(k)}(\phi)$ if $f(z)$ satisfies the condition*

$$\frac{zf'(z)}{f_{2k}(z)} \prec \phi(z) \quad (z \in \mathcal{U}), \tag{1.3}$$

where $\phi(z) \in \mathcal{P}$ and $f_{2k}(z)$ is defined by the equality (1.2). And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_{sc}^{(k)}(\phi)$ if and only if $zf'(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$.

In the present paper, we shall provide the integral representations for functions belonging to the classes $\mathcal{S}_{sc}^{(k)}(\phi)$ and $\mathcal{C}_{sc}^{(k)}(\phi)$, we shall also provide the convolution conditions, growth, distortion and covering theorems for these classes.

2. Integral Representations

We first give some inclusion relationships for the classes $\mathcal{S}_{sc}^{(k)}(\phi)$ and $\mathcal{C}_{sc}^{(k)}(\phi)$, which tell us that $\mathcal{S}_{sc}^{(k)}(\phi)$ is a subclass of close-to-convex functions, and $\mathcal{C}_{sc}^{(k)}(\phi)$ is a subclass of quasi-convex functions.

Theorem 1. *Let $\phi(z) \in \mathcal{P}$, then we have $\mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}$.*

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, it suffices to show that $f_{2k}(z) \in \mathcal{S}^* \subset \mathcal{S}$. From the condition (1.3), we have

$$\Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}) \tag{2.1}$$

since $\Re\{\phi(z)\} > 0$. Substituting z by $\varepsilon^\mu z$ $(\mu = 0, 1, 2, \ldots, k-1)$ in (2.1), then (2.1) is also true, that is,

$$\Re\left\{\frac{\varepsilon^\mu z f'(\varepsilon^\mu z)}{f_{2k}(\varepsilon^\mu z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.2}$$

From inequality (2.2), we have

$$\Re\left\{\frac{\overline{\varepsilon^\mu \overline{z}}\ \overline{f'(\varepsilon^\mu \overline{z})}}{\overline{f_{2k}(\varepsilon^\mu \overline{z})}}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.3}$$

Note that $f_{2k}(\varepsilon^\mu z) = \varepsilon^\mu f_{2k}(z)$ and $\overline{f_{2k}(\varepsilon^\mu \overline{z})} = \varepsilon^{-\mu} f_{2k}(z)$, then inequalities (2.2) and (2.3) can be written as

$$\Re\left\{\frac{zf'(\varepsilon^\mu z)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}), \tag{2.4}$$

and

$$\Re\left\{\frac{z\overline{f'(\varepsilon^\mu \overline{z})}}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.5}$$

Summing inequalities (2.4) and (2.5), we can get

$$\Re\left\{\frac{z\left[f'(\varepsilon^\mu z) + \overline{f'(\varepsilon^\mu \overline{z})}\right]}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.6}$$

Let $\mu = 0, 1, 2, \ldots, k-1$ in (2.6), respectively, and summing them we can get

$$\Re\left\{\frac{z\left[\frac{1}{2k}\sum_{\mu=0}^{k-1}\left(f'(\varepsilon^{\mu}z)+\overline{f'(\varepsilon^{\mu}\overline{z})}\right)\right]}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

or equivalently,

$$\Re\left\{\frac{zf'_{2k}(z)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

that is $f_{2k}(z) \in \mathcal{S}^* \subset \mathcal{S}$. This means that $\mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}$, and hence the proof of Theorem 1 is complete.

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\phi)$, we have

Corollary 1. *Let $\phi(z) \in \mathcal{P}$, then we have $\mathcal{C}_{sc}^{(k)}(\phi) \subset \mathcal{C}^* \subset \mathcal{C}$.*

We now provide the integral representations for functions belonging to the classes $\mathcal{S}_{sc}^{(k)}(\phi)$ and $\mathcal{C}_{sc}^{(k)}(\phi)$.

Theorem 2. *Let $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, then we have*

$$f_{2k}(z) = z \cdot \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi(\omega(\varepsilon^{\mu}\zeta)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))} - 2\right] d\zeta\right\}, \tag{2.7}$$

where $f_{2k}(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $\omega(z) < 1$.

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, from the definition of $\mathcal{S}_{sc}^{(k)}(\phi)$, we have

$$\frac{zf'(z)}{f_{2k}(z)} = \phi(\omega(z)), \tag{2.8}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $\omega(z) < 1$. Substituting z by $\varepsilon^{\mu}z$ ($\mu = 0, 1, 2, \ldots, k-1$) in (2.8), we have

$$\frac{\varepsilon^{\mu}zf'(\varepsilon^{\mu}z)}{f_{2k}(\varepsilon^{\mu}z)} = \phi(\omega(\varepsilon^{\mu}z)). \tag{2.9}$$

From equality (2.9), we have

$$\frac{\overline{\varepsilon^{\mu}\overline{z}}\;\overline{f'(\varepsilon^{\mu}\overline{z})}}{\overline{f_{2k}(\varepsilon^{\mu}\overline{z})}} = \overline{\phi(\omega(\varepsilon^{\mu}\overline{z}))}. \tag{2.10}$$

Summing equalities (2.9) and (2.10), and making use of the same method as in Theorem 1, we have

$$\frac{zf'_{2k}(z)}{f_{2k}(z)} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\left[\phi(\omega(\varepsilon^{\mu}z)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{z}))}\right], \tag{2.11}$$

from equality (2.11), we can get

$$\frac{f'_{2k}(z)}{f_{2k}(z)} - \frac{1}{z} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{1}{z}\left[\phi(\omega(\varepsilon^{\mu}z)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{z}))} - 2\right]. \tag{2.12}$$

Integrating equality (2.12), we have

$$\log\left\{\frac{f_{2k}(z)}{z}\right\} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi(\omega(\varepsilon^{\mu}\zeta)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\ - 2\right]d\zeta. \tag{2.13}$$

From equality (2.13), we can get equality (2.7) easily. Hence the proof is complete.

Theorem 3. *Let $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, then we have*

$$f(z) = \int_0^z \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^{\xi}\frac{1}{\zeta}\left[\phi(\omega(\varepsilon^{\mu}\zeta)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\ - 2\right]d\zeta\right\}\cdot\phi(\omega(\xi))d\xi, \tag{2.14}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $\omega(z) < 1$.

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, from equalities (2.7) and (2.8), we can get

$$f'(z) = \frac{f_{2k}(z)}{z}\cdot\phi(\omega(z)) = \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{1}{\zeta}\left[\phi(\omega(\varepsilon^{\mu}\zeta)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\ - 2\right]d\zeta\right\}\cdot\phi(\omega(z)). \tag{2.15}$$

Integrating equality (2.15), we can get equality (2.14) easily. Hence the proof is complete.

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\phi)$, we have

Corollary 2. *Let $f(z) \in \mathcal{C}_{sc}^{(k)}(\phi)$, then we have*

$$f_{2k}(z) = \int_0^z \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^{\xi}\frac{1}{\zeta}\left[\phi(\omega(\varepsilon^{\mu}\zeta)) + \overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\ - 2\right]d\zeta\right\}d\xi,$$

where $f_{2k}(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $\omega(z) < 1$.

Corollary 3. *Let $f(z) \in \mathcal{C}_{sc}^{(k)}(\phi)$, then we have*

$$f(z) = \int_0^z \frac{1}{t} \int_0^t \exp\left\{ \frac{1}{2k} \sum_{\mu=0}^{k-1} \int_0^\xi \frac{1}{\zeta} \left[\phi(\omega(\varepsilon^\mu \zeta)) + \overline{\phi(\omega(\varepsilon^\mu \overline{\zeta}))} \ -2 \right] d\zeta \right\} \cdot \phi(\omega(\xi)) d\xi dt,$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $\omega(z) < 1$.

3. Convolution Conditions

In this section, we give the convolution conditions for the classes $\mathcal{S}_{sc}^{(k)}(\phi)$ and $\mathcal{C}_{sc}^{(k)}(\phi)$. Let $f,\ g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Theorem 4. *Let $f(z) \in \mathcal{A}$ and $\phi(z) \in \mathcal{P}$, then $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$ if and only if*

$$\frac{1}{z} \left[f * \left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2} h \right)(z) - \frac{\phi(e^{i\theta})}{2} \cdot \ \overline{(f * h)(\overline{z})} \right] \neq 0 \tag{3.1}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (3.6).

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, since

$$\frac{zf'(z)}{f_{2k}(z)} \prec \phi(z)$$

if and only if

$$\frac{zf'(z)}{f_{2k}(z)} \neq \phi(e^{i\theta}) \tag{3.2}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$. And the condition (3.2) can be written as

$$\frac{1}{z}\left[zf'(z) - f_{2k}(z)\phi(e^{i\theta})\right] \neq 0. \tag{3.3}$$

On the other hand, it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{3.4}$$

And from the definition of $f_{2k}(z)$, we know

$$f_{2k}(z) = \frac{1}{2}\left[(f*h)(z) + \overline{(f*h)(\overline{z})}\right], \tag{3.5}$$

where

$$h(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{z}{1-\varepsilon^{\upsilon}z}. \tag{3.6}$$

Substituting (3.4) and (3.5) into (3.3), we can get (3.1) easily. This completes the proof of Theorem 4.

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\phi)$, we have

Corollary 4. *Let $f(z) \in \mathcal{A}$ and $\phi(z) \in \mathcal{P}$, then $f(z) \in \mathcal{C}_{sc}^{(k)}(\phi)$ if and only if*

$$\frac{1}{z}\left\{f * \left[z\left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2}h\right)'\right](z) - \frac{\phi(e^{i\theta})}{2} \cdot \overline{[f*(zh')](\overline{z})}\right\} \neq 0$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (3.6).

4. Growth, Distortion and Covering Theorems

Finally, we provide the growth, distortion and covering theorems for the classes $\mathcal{S}_{sc}^{(k)}(\phi)$ and $\mathcal{C}_{sc}^{(k)}(\phi)$. For the purpose of this section, assume that the function $\phi(z)$ is an analytic function with positive real part in the unit disk $\mathcal{U}$, $\phi(\mathcal{U})$ is convex and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. The functions $k_{\phi n}(z)$ $(n = 2, 3, \ldots)$ defined by $k_{\phi n}(0) = k'_{\phi n}(0) - 1 = 0$ and

$$1 + \frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)} = \phi(z^{n-1})$$

are important examples of functions in $\mathcal{K}(\phi)$. The functions $h_{\phi n}(z)$ satisfying $h_{\phi n}(z) = zk'_{\phi n}(z)$ are examples of functions in $\mathcal{S}^*(\phi)$. Write $k_{\phi 2}(z)$ simply as $k_\phi(z)$ and $h_{\phi 2}(z)$ simply as $h_\phi(z)$.

Note that if $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$, then $f_{2k}(z) \in \mathcal{S}^*(\phi)$. Therefore, by making use of the similar method as in Theorem 7 obtained by Ravichandran [8], we can get the following theorem, here we omit the details.

Theorem 5. *Let* $\min_{z=r} \phi(z) = \phi(-r)$, $\max_{z=r} \phi(z) = \phi(r)$, $z = r < 1$. *If* $f(z) \in \mathcal{S}_{sc}^{(k)}(\phi)$*, then we have*

$$h'_\phi(-r) \leq f'(z) \leq h'_\phi(r), \ -h_\phi(-r) \leq f(z) \leq h_\phi(r),$$

and

$$f(\mathcal{U}) \supset \{\omega : \omega \leq -h(-1)\}.$$

These results are sharp.

Similarly, note that if $f(z) \in \mathcal{C}_{sc}^{(k)}(\phi)$, then

$$f_{2k}(z) = z + \sum_{l=1}^{\infty} \frac{a_{lk+1} + \overline{a_{lk+1}}}{2} z^{lk+1} = z + \sum_{l=1}^{\infty} \Re(a_{lk+1}) z^{lk+1} \in \mathcal{K}(\phi).$$

Therefore, by making use of the similar method as in Theorem 9 obtained by Wang, Gao and Yuan [10], we can get the following theorem, here we also omit the details.

Theorem 6. *Let* $\min_{z=r} \phi(z) = \phi(-r)$, $\max_{z=r} \phi(z) = \phi(r)$, $z = r < 1$. *If* $f(z) \in \mathcal{C}_{sc}^{(k)}(\phi)$*, then we have*

$$\frac{1}{r}\int_0^r \phi(-t)[k'_\phi(-t^k)]^{1/k} dt \leq f'(z) \leq \frac{1}{r}\int_0^r \phi(t)[k'_\phi(t^k)]^{1/k} dt,$$

$$\int_0^r \frac{1}{s}\int_0^s \phi(-t)[k'_\phi(-t^k)]^{1/k} dtds \leq f(z) \leq \int_0^r \frac{1}{s}\int_0^s \phi(t)[k'_\phi(t^k)]^{1/k} dtds,$$

and

$$f(\mathcal{U}) \supset \left\{\omega : \omega \leq \int_0^1 \frac{1}{s}\int_0^s \phi(-t)[k'_\phi(-t^k)]^{1/k} dtds\right\}.$$

These results are sharp.

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