Tamsui Oxford Journal of Mathematical Sciences **24(3)** (2008) 269-276
Aletheia University

On a Double Inequality for the Dirichlet Beta Function *

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Received February 22, 2007, Accepted March 26, 2007.

Abstract

Let $\beta(x)$ be the Dirichlet beta function. Then for all $x > 0$

$$c < 3^{x+1}\left[\beta(x+1) - \beta(x)\right] < d$$

with the best possible constant factors

$$c = 3\left(\frac{\pi}{4} - \frac{1}{2}\right) \approx 0.85619449 \quad \text{and} \quad d = 2.$$

The above result, and some variants, are used to approximate β at even integers in terms of known β at odd integers.

Keywords and Phrases: *Dirichlet Beta function, Inequalities, Approximation, Dirichlet L−function.*

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*2000 *Mathematics Subject Classification.* 26D99, 11M06, 26D07.

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1. Introduction

The Dirichlet beta function or Dirichlet $L-$function is given by [5]

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}, \quad x > 0, \tag{1.1}$$

where $\beta(2) = G$, Catalan's constant.

The beta function may be evaluated explicitly at positive odd integer values of x, namely,

$$\beta(2n+1) = (-1)^n \frac{E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}, \tag{1.2}$$

where E_n are the Euler numbers.

The Dirichlet beta function may be analytically continued over the whole complex plane by the functional equation

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^z \sin\left(\frac{\pi z}{2}\right) \Gamma(z) \beta(z).$$

The function $\beta(z)$ is defined everywhere in the complex plane and has no singularities, unlike the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which has a simple pole at $s = 1$.

The Dirichlet beta function and the zeta function have important applications in a number of branches of mathematics, and in particular in Analytic number theory. See for example [2], [4] – [7].

Further, $\beta(x)$ has an alternative integral representation [5, p. 56]

$$\beta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + e^{-t}} dt, \quad x > 0, \tag{1.3}$$

where

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

The function $\beta(x)$ is also connected to prime number theory [5] which may perhaps be best summarised by

$$\begin{aligned}\beta(x) &= \prod_{p \text{ prime}, p \equiv 1 (\text{mod} 4)} \left(1 - p^{-x}\right)^{-1} \cdot \prod_{p \text{ prime}, p \equiv 3 (\text{mod} 4)} \left(1 + p^{-x}\right)^{-1} \\ &= \prod_{p \text{ odd}, p \text{ prime}} \left(1 - (-1)^{\frac{p-1}{2}} p^{-x}\right)^{-1},\end{aligned}$$

where the rearrangement of factors is permitted because of absolute convergence.

It is the intention of the current article to present sharp bounds for the secant slope of the $\beta(x)$ over a distance of 1 apart. This will enable an approximation of the beta function at even integers given that beta at odd integers is as in (1.2). Cerone et al. [3] utlised a similar philosophy in investigating the zeta function, however, one of the bounds was not sharp. Alzer [1] proved a tighter lower bound and proved the sharpness of both the upper and lower bounds.

2. An Identity and Bounds Involving the Beta Function

The following lemma will play a significant role in obtaining bounds for the Dirichlet beta function, $\beta(x)$.

Lemma 1. *The following identity for the Dirichlet beta function holds. Namely,*

$$P(x) := \frac{2}{\Gamma(x+1)} \int_0^\infty \frac{e^{-t}}{(e^t + e^{-t})^2} \cdot t^x dt = \beta(x+1) - \beta(x). \tag{2.1}$$

Proof. From (1.3)

$$x\Gamma(x)\beta(x) = \int_0^\infty \frac{xt^{x-1}}{e^t + e^{-t}} dt, \quad x > 0$$

which upon integration by parts gives

$$\Gamma(x+1)\beta(x) = \int_0^\infty t^x \cdot \frac{e^t - e^{-t}}{(e^t + e^{-t})^2} dt. \tag{2.2}$$

Thus, from (1.3) and (2.2) upon simplification produces (2.1). □

The following theorem produces sharp bounds for the secant slope of $\beta(x)$.

Theorem 1. *For real numbers $x > 0$, we have*

$$\frac{c}{3^{x+1}} < \beta(x+1) - \beta(x) < \frac{d}{3^{x+1}}, \tag{2.3}$$

with the best possible constants

$$c = 3\left(\frac{\pi}{4} - \frac{1}{2}\right) = 0.85619449\ldots \quad and \quad d = 2. \tag{2.4}$$

Proof. Let $x > 0$. We first establish the left inequality in (2.3). From (2.1) we note that $0 < P(x)$ and consider

$$I = 2\int_0^\infty \frac{e^{-t}}{(e^t + e^{-t})^2}dt. \tag{2.5}$$

That is, making the change of variable $u = e^{-t}$ gives

$$I = 2\int_0^1 \frac{u^2}{(1+u^2)^2}du$$

from which a further change of variable $u = \tan\theta$ produces

$$I = 2\int_0^{\frac{\pi}{4}} \sin^2\theta d\theta = \frac{\pi}{4} - \frac{1}{2}. \tag{2.6}$$

From (2.1) and (2.5) we have

$$\begin{aligned}
&\Gamma(x+1)\left[3^{x+1}P(x) - 3\cdot I\right] \\
=&6\left[3^x\int_0^\infty \frac{e^{-t}\cdot t^x}{(e^t+e^{-t})^2}dt - \Gamma(x+1)\int_0^\infty \frac{e^{-t}}{(e^t+e^{-t})^2}dt\right] \\
=&6\left[\int_0^\infty \frac{e^{-3t}}{(1+e^{-2t})^2}\left[(3t)^x - \Gamma(x+1)\right]dt\right] \\
=&2\int_0^\infty \frac{e^{-t}}{\left(1+e^{-\frac{2}{3}t}\right)^2}\left[t^x - \Gamma(x+1)\right]dt \\
=&2\int_0^\infty u(t,x)\,v(t)\,dt,
\end{aligned}$$

where

$$u(t,x) = e^{-t}\left[t^x - \Gamma(x+1)\right] \quad \text{and} \quad v(t) = \left(1+e^{-\frac{2}{3}t}\right)^{-2}. \tag{2.7}$$

The function $v(t)$ is strictly increasing for $t \in (0,\infty)$. Now, let $t_0 = (\Gamma(x+1))^{\frac{1}{x}}$, then for $0 < t < t_0$, $u(t,x) < 0$ and $v(t) < v(t_0)$. Further, for $t > t_0$, then $u(t,x) > 0$ and $v(t) > v(t_0)$. Hence we have that $u(t,x)v(t) > u(t,x)v(t_0)$ for $t > 0$, $t \neq t_0$. This implies that

$$\int_0^\infty u(t,x)v(t)\,dt > v(t_0)\int_0^\infty \left(e^{-t}t^x - e^{-t}\Gamma(x+1)\right)dt = 0.$$

Hence

$$P(x) > \frac{3\left(\frac{\pi}{4}-\frac{1}{2}\right)}{3^{x+1}}, \quad x > 0. \tag{2.8}$$

Now for the right inequality.

We have from (2.1)

$$\begin{aligned}\Gamma(x+1)\left[2-3^{x+1}P(x)\right] &= 2\left[\Gamma(x+1) - 3^{x+1}\int_0^\infty \frac{e^{-t}\cdot t^x}{\left(e^t+e^{-t}\right)^2}dt\right]\\ &= 2\left\{\int_0^\infty e^{-t}t^x dt - 3\int_0^\infty \frac{e^{-3t}\cdot(3t)^x}{\left(1+e^{-2t}\right)^2}dt\right\}\\ &= 2\int_0^\infty e^{-t}t^x\left[1-v(t)\right]dt,\end{aligned}$$

where $v(t)$ is as given by (2.7). We note that $e^{-t}t^x$ is positive and $1-v(t)$ is strictly decreasing and positive so that

$$P(x) < \frac{2}{3^{x+1}}, \quad x > 0. \tag{2.9}$$

The results in (2.8) and (2.9) demonstrate lower and upper bounds respectively for $P(x)$. That the constants in (2.4) are best possible will now be shown. If (2.3) holds for all positive x then we have, on noting the identity (2.1),

$$c < 3^{x+1}P(x) < d. \tag{2.10}$$

Now, from (2.1), we have

$$3^{x+1}P(x) = \frac{3^{x+1}}{\Gamma(x+1)}\cdot 2\int_0^\infty \frac{e^{-t}\cdot t^x}{\left(e^t+e^{-t}\right)^2}dt \tag{2.11}$$

and so

$$\lim_{x\to 0} 3^{x+1}P(x) = 3\cdot I = 3\left(\frac{\pi}{4}-\frac{1}{2}\right), \tag{2.12}$$

where we have undertaken the permissible interchange of limit and integration and, we have used (2.5) – (2.6).

Alternatively, from (2.11), we have

$$3^{x+1}P(x) = \frac{3^{x+1}}{\Gamma(x+1)} \cdot 2\int_0^\infty \frac{e^{-3t} \cdot t^x}{(1+e^{-2t})^2}dt. \tag{2.13}$$

Now, since for $0 < w < 1$, $1 - 2w < (1+w)^{-2} < 1$, we have

$$1 - 2e^{-2t} < \frac{1}{(1+e^{-2t})^2} < 1$$

and so from (2.13)

$$2\left(1 - \left(\frac{3}{5}\right)^{x+1}\right) < 3^{x+1}P(x) < 2. \tag{2.14}$$

Thus, from (2.14), we have

$$\lim_{x\to\infty} 3^{x+1}P(x) = 2. \tag{2.15}$$

From (2.10), (2.12) and (2.15) we have $c \le 3\left(\frac{\pi}{4} - \frac{1}{2}\right)$ and $d \ge 2$ which means that in (2.3) the best possible constants are given by $c = 3\left(\frac{\pi}{4} - \frac{1}{2}\right)$ and $d = 2$. □

Corollary 1. *The bound*

$$\left|\beta(x+1) - \beta(x) - \frac{d+c}{2 \cdot 3^{x+1}}\right| < \frac{d-c}{2 \cdot 3^{x+1}} \tag{2.16}$$

holds where $c = 3\left(\frac{\pi}{4} - \frac{1}{2}\right)$ *and* $d = 2$.

Proof. From (2.3), let

$$L(x) = \beta(x) + \frac{c}{3^{x+1}} \quad \text{and} \quad U(x) = \beta(x) + \frac{d}{3^{x+1}}, \tag{2.17}$$

then

$$L(x) < \beta(x+1) < U(x) \tag{2.18}$$

and so

$$-\frac{U(x) - L(x)}{2} < \beta(x+1) - \frac{U(x) + L(x)}{2} < \frac{U(x) - L(x)}{2}.$$

□

Remark 1. *The form (2.16) is useful since we may write*

$$\beta(x+1) = \beta(x) + \frac{d+c}{2 \cdot 3^{x+1}} + E(x),$$

where $|E(x)| < \varepsilon$ *for*

$$x > x^* := \frac{\ln\left(\frac{d-c}{2 \cdot \varepsilon}\right)}{\ln(3)} - 1.$$

Corollary 2. *The Dirichlet beta function satisfies the bounds*

$$L_2(x) < \beta(x+1) < U_2(x), \tag{2.19}$$

where

$$L_2(x) = \beta(x+2) - \frac{d}{3^{x+2}} \quad \text{and} \quad U_2(x) = \beta(x+2) - \frac{c}{3^{x+2}}. \tag{2.20}$$

Proof. From (2.3)

$$-\frac{d}{3^{x+1}} < \beta(x) - \beta(x+1) < -\frac{c}{3^{x+1}}.$$

Replace x by $x+1$ and rearrange to produce (2.19) – (2.20). □

Remark 2. *Some experimentation with the Maple computer algebra package indicates that the lower bound* $L_2(x)$ *is better than* $L(x)$ *for* $x > x_* \approx 0.65827$ *and vice versa for* $x < x_*$. *Similarly,* $U(x)$ *is better than* $U_2(x)$ *for* $x > x^* \approx 3.45142$ *and vice versa for* $x < x^*$.

Corollary 3. *The Dirichlet beta function satisfies the bounds*

$$\max\{L(x), L_2(x)\} < \beta(x+1) < \min\{U(x), U_2(x)\},$$

where $L(x)$, $U(x)$ *are given by (2.17) and* $L_2(x)$, $U_2(x)$ *by (2.20).*

Remark 3. *Table 1 provides lower and upper bounds for* $\beta(2n)$ *for* $n = 1, \ldots, 5$ *utilising Theorem 2 and Corollary 2 with* $x = 2n-1$. *That is, the bounds are in terms of* $\beta(2n-1)$ *and* $\beta(2n+1)$ *where these may be obtained explicitly using the result (1.2). There is no known explicit expression for* $\beta(2n)$.

n	$L(2n-1)$	$L_2(2n-1)$	$\beta(2n)$	$U(2n-1)$	$U_2(2n-1)$
1	.8805308843	.8948720722	.9159655942	1.007620386	.9372352393
2	.9795164487	.9879273754	.9889445517	.9936375043	.9926343940
3	.9973323061	.9986400132	.9986852222	.9989013123	.9991630153
4	.9996850054	.9998480737	.9998499902	.9998593395	.9999061850
5	.9999641840	.9999830849	.9999831640	.9999835544	.9999895417

Table 1: Table of $L(2n-1)$, $L_2(2n-1)$, $\beta(2n)$, $U(2n-1)$ and $U_2(2n-1)$ as given by (2.17) and (2.20) for $n = 1, \ldots, 5$.

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