Tamsui Oxford Journal of Mathematical Sciences **24(3)** (2008) 257-268
Aletheia University

Exact Solutions of Time-dependent Navier-Stokes Equations by Hodograph-Legendre Transformation Method *

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Received February 16, 2007, Accepted March 26, 2007.

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*2000 *Mathematics Subject Classification*. 76A05.
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Abstract

In this paper, we present analytic solutions of two-dimensional Navier-Stokes equations governing the unsteady incompressible flow The solutions have been obtained using hodograph-Legendre transform method. The obtained solutions are also compared with the existing results.

Keywords and Phrases: *Unsteady flow, Hodograph method, Hodograph-Legendre transform.*

1. Introduction

A Newtonian fluid is defined as one which satisfies Newton's law of viscosity. The equations which deals with the flow of Newtonian fluids are Navier-Stokes equations. There is no reason to believe that all fluids of low molecular weight should satisfy these equations. The Navier-Stokes equations are highly non-linear. The analytic solution of such equations have great value.

There are various methods and techniques which have been used to solve the Navier-Stokes equations. Some of these are based on intuition. One technique, which has become one of the powerful analytical tools and which has gained a considerable importance, is the method of transformations. In such methods, where applicable, either the system is linearized, the non-linear partial differential equations are reduced to a system of non-linear ordinary differential equations which can be solved, or some other type of reduction is performed to minimize the complexity. A comprehensive review of such transformations is given in the monograph of Ames [1].Amongst many,the hodograph transformation have gained a considerable success in various fields of research such as gas dynamics [2,3], linear viscous fluids [4], non-Newtonian fluids [5,6] and MHD Newtonian and non-Newtonian fluid flows [7,8]. The objective of the present communication is to obtain the analytic solutions of the Navier-Stokes equations. For this purpose, we consider two-dimensional time dependent equations. These equations are first transformed in the hodograph plane (U, V) from the plane (X, Y) and then solutions are constructed by introducing the four Legengdre transform functions. It is noted that the results of several previous studies appear as the limiting cases of the present investigation.

2. Governing Equations

For unsteady plane flow, the velocity field is defined as

$$V = [U(X,Y,t), V(X,Y,t), 0], \tag{1}$$

where U and V are the velocities in X and Y directions, respectively.

The continuity and momentum equations give

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{2}$$

$$\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}\right)U = -\frac{\partial P}{\partial X} + \mu\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)U, \tag{3}$$

$$\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}\right)V = -\frac{\partial P}{\partial Y} + \mu\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)V \tag{4}$$

in which ρ is the constant density, μ is dynamic viscosity of the fluid, and P is the scalar pressure. we see the above three partial differential differential equations have three unknowns $U, V$and P as functions of three independent variables X, Y,and t.

Using the transformation

$$\begin{aligned} x = X - ct, y = Y, \\ u = U - c, v = V, \end{aligned} \tag{5}$$

in equations $(2)-(4)$ and then introducing the vorticity

$$\omega = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{6}$$

into the resulting equations we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$\frac{\partial h}{\partial x} - \rho v\omega = -\mu\frac{\partial \omega}{\partial y} \tag{8}$$

$$\frac{\partial h}{\partial y} + \rho u\omega = \mu\frac{\partial \omega}{\partial x} \tag{9}$$

where c is a parameter and the modified pressure is

$$h = P + \frac{1}{2}\left(u^2 + v^2\right) \tag{10}$$

The equations $(7) - (9)$ constitute a system of four partial differential equations having four unknown functions $u(x, y)$, $w(x, y)$,and $h(x, y)$.

Note that the transformation (5) and Eq. (6) have been used to reduced the number of independent variables (from three to two) and to reduce the order (from two to one), respectively.

3. Equations in hodograph plane

Let us consider

$$u = u(x, y), \qquad v = v(x, y) \tag{11}$$

in such a way that their Jacobian

$$J = \frac{\partial\left(u, v\right)}{\partial(x, y)} \neq 0 \tag{12}$$

is not zero and $0 < |J| < \infty$.In such cases we may interchange the roles of dependent and independent variables. Thus on writing

$$x = x(u, v), \qquad y = (u, v) \tag{13}$$

the equations $(6) - (9)$ in the hodograph plane (u, v) take the following form

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0, \tag{14}$$

$$j\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right) = \omega, \tag{15}$$

$$-j\frac{\partial\left(h, y\right)}{\partial(u, v)} = \rho v\omega + \mu j W_1, \tag{16}$$

$$j\frac{\partial\left(h, y\right)}{\partial(u, v)} = \rho u\omega + \mu j W_2. \tag{17}$$

In above equations

$$W_1 = W_1(u,v) = \frac{\partial\,(x,\omega)}{\partial\,(u,v)},\ W_2 = W_2(u,v) = \frac{\partial\,(-y,\omega)}{\partial\,(u,v)},\ J = J(u,v) = \left[\frac{\partial\,(x,y)}{\partial\,(u,v)}\right]^{-1}. \tag{18}$$

Note that Eqs. (14) – (17) form a system of four partial differential equations in four unknown functions x, y, ω, h in the hodograph plane (u, v). now if $x = x(u, v)$, $y = y(u, v), \omega = \omega(u, v), h = h(u, v)$ are known then one can easily find out $u = u(x, y)$, $v = v(x, y)$, $\omega = \omega(x, y)$, $h = h(x, y)$, through Eqs.(6)-(9). The expressions for U, V and P can be calculated through Eqs.(5) and (10).

4. Equations for the Legendre-transform function

The Eq. (2) implies the existence of the following stream function $\Psi(x, y, t)$

$$U = \frac{\partial\Psi}{\partial Y}, \qquad V = -\frac{\partial\Psi}{\partial X}. \tag{19}$$

Also Eq. (7) implies the existence of stream functions $\psi(x, y)$ such that

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{20}$$

and the two steam functions are related through

$$\Psi = \Psi(X, Y, t) = \psi(X - ct) - CY + \text{constt.} \tag{21}$$

In a similar fashion, the Eq. (14) implies the existence of a function $L(u, v)$, called the Legendre-transform function of the stream function $\psi(x, y)$ such that

$$L_u = \frac{\partial L}{\partial u} = -y, \qquad L_v = \frac{\partial L}{\partial v} = x \tag{22}$$

and the two functions $\psi(x, y)$ and $L(u, v)$ are related by

$$L(u, v) = vx - u, y + \psi(x, y). \tag{23}$$

Introducing $L(u, v)$ the system (14)−(18), we find that Eq. (14) is satisfied identically and other equations after eliminating $h(u, v)$ between them take the form

$$\mu\left[\frac{\partial\left(L_{v},jw_{1}\right)}{\partial\left(u,v\right)}+\frac{\partial\left(L_{vv},jw_{2}\right)}{\partial\left(u,v\right)}\right]=\rho\left(uW_{2}+vW_{1}\right), \tag{24}$$

where

$$W_{1}=\frac{\partial\left(L_{vv},w\right)}{\partial\left(u,v\right)},\ W_{2}=\frac{\partial\left(L_{uv},w\right)}{\partial\left(u,v\right)},\omega=j\left(Luu+Lvv\right),j=\left(LuuLvv-L^{2}uv\right)^{-1}. \tag{25}$$

Given a solution $L = L(u, v)$, of equation (24), we can the velocity components as function of (x, y). Once a steady state solution is in hand, we can then use Eq. (5) to obtain a time dependent solution.

5. Solutions

Here, we determine the solutions of the flow problems by selecting the specific forms of the Legendre-transform function.

We seek a Legendre-transform function

$$L(u,v)=vF(u)+G(u) \tag{26}$$

such that $F(u) \neq 0$. Using Eq. (26) in Eqs. (24) and (25), we find that

$$\nu\left[\begin{array}{c}\nu\left(\frac{F^{iv}}{F'^{2}}-10\frac{F'''F''}{F'^{3}}+15\frac{F''^{3}}{F'^{4}}\right)\\+\left(\frac{G^{iv}}{F'^{2}}-6\frac{G'''F''}{F'^{3}}-4\frac{G''F'''}{F'^{3}}+15\frac{G''F''^{2}}{F'^{4}}\right)\end{array}\right]=-\nu\frac{F''}{F'}+uv\left[\begin{array}{c}\frac{F''}{F'}-3\frac{F''^{2}}{F'^{2}}\\+u\left(\frac{G'''}{F'}-3\frac{G''F''}{F'^{2}}\right)\end{array}\right], \tag{27}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. Equation (27) gives rise to the following differential equations

$$\nu\left[F^{iv}F'^{2}-10F'''F''F'+15F''^{3}\right]=u\left(F'''F'^{3}-3F''^{2}F'^{2}\right)-F''F'^{3}, \tag{28}$$

$$\nu\left[G^{iv}F'^{2}-6G'''F''F'-4G''F'''F'+15G''F''^{2}\right]=u(G'''F'^{3}-3G''F''F'^{2}). \tag{29}$$

It seems to be very difficult to obtain the general solution of these equations. We, therefore, examine some special cases.

On assuming

$$F(u) = Au^m,\ m \neq 0 \tag{30}$$

in equation (28), we find that

$$\begin{aligned} & vu^{3m-6}\left[(m-2)(m-3) - 10(m-1)(m-2) + 15(m-1)^2\right] \\ = \; & mA\left[(m-2) - 3(m-1) - 1\right]u^{4m-5} \end{aligned} \tag{31}$$

we see that Eq. (32) is satisfied if $m = -1$ and $A = -6v$, and thus Eq. (30) becomes F(u)=-6vu^{-1}.(0)Substitution of Eq. (32) into Eq. (29) yields

$$u^4 G^{iv} + 6u^3 G^{'''} = 0 \tag{33}$$

whose general solution is

$$G = C_1 + C_2 u + C_3 u^2 + C_4 u^{-3}, \tag{34}$$

where $C_i (i = 1...4)$ are arbitrary constants. Using Eqs. (32) and (34) in Eq. (26) we obtain the Legendre-transformation function

$$L(u, v) = -6vu^{-1} + C_1 + C_2 u + C_3 u^2 + C_4 u^{-3}. \tag{35}$$

Now using Eq. (35) in Eq. (22) and solving for $u(x, y)$ and $v(x, y)$, we get

$$u(x, y) = -6v/x, \quad v(x, y) = -6vyx^{-2} - 6vC_2 x^{-2} + 72v^2 C_3 x^{-3} + \frac{C_4}{72v^3}x^2 \tag{36}$$

and thus the stream function $\psi(x, y)$ becomes

$$\psi(x, y = -6vyx^{-1} - 6vC_2 x^{-2} + 36C_3 v^2 x^{-2} + \frac{Cu}{216v^3}x^3 + a_0, \tag{37}$$

where a_0 is an arbitrary constant. The time dependent stream function $\Psi(X, Y, t)$ turns out to be

$$\Psi(X, Y, t) = -\frac{6v}{(x-ct)}Y - \frac{6vC_2}{(x-ct)} + \frac{36v^2 C_3}{(x-ct)^2} + \frac{C_4 (x-ct)^3}{216v^3} - cY + a_0. \tag{38}$$

Here we choose

$$F(u) = \frac{1}{a} \ln \frac{u - A}{KA}, \tag{39}$$

where a, K and A are constants. If we substitute Eq. (39) in Eq. (28) we find that the later is satisfied provided $A = va$ and Eq. (40) becomes

$$F(u) = \frac{1}{a} \ln \left(\frac{u - va}{Kva} \right). \tag{40}$$

With the help of Eq. (29) and (40) we can write

$$G^{iv} + \frac{6va - u}{av\,(u - va)} G''' + \frac{7va - 3u}{va\,(u - va)^2} G'' = 0. \tag{41}$$

The order of the Eq. (41) can be reduced from four to two by setting $G^{//} = H$ and accordingly, we have

$$H^{//} + \frac{6va - u}{av\,(u - va)} H^{/'''} + \frac{7va - 3u}{va\,(u - va)^2} H = 0. \tag{42}$$

Since this equation has a regular singular point at $u = va$, we may assume a Frobenious type solution

$$H =_{n=0}^{\infty} C_n\,(u - va)^{n+r} \tag{43}$$

and thus the general solution of Eq. (43) is

$$H = \overline{C}_1 H_0 + \overline{C_2} H_1 \phi, \tag{44}$$

where

$$\begin{aligned} H_1 &= \frac{1}{(u - va)^2} \left[e^{\frac{u - va}{va}} \right]^n, \; H_0 = C_0\,(u - va)_{n=0}^{\infty} \frac{1}{n!} \left(\frac{u - va}{va} \right)^n, \\ \phi &= \ln \left(\frac{u - va}{va} \right) +_{n=1}^{\infty} \frac{(-1)^n}{n!n} \left(\frac{u - va}{va} \right)^n. \end{aligned}$$

Accordingly we obtain the the Legendre transform function $L(u, v)$ and then steady state stream function are found to be

$$\psi(x,y) = va(1+Ke^{ax})y + K_3e^{an} + K_2^z d\alpha^\alpha \frac{C^{k\beta}}{\beta^2}d\beta + K_1^z d\alpha^\alpha \frac{C^{k\beta}}{\beta^2}d\beta^\beta \frac{C^{-k\infty}}{\theta}d\theta, \tag{45}$$

where $Z = C^{ax}$ and K_1, K_2, K_3, are constants. We consider special case of the solution where $K_1 = K_2 = K_3 = 0$, such that Eq. (45) becomes $\psi(x,y) = va(1+Ke^{an})y$, which is Riabouchinksy's solution [9]. This solution was derived by choosing the steady state stream function of the form $\psi(x,y) = yF(x) + G(x)$. Finally, we write the time dependent stream function

$$\begin{aligned}\Psi(X,Y,t) &= va\left[1+Ke^{a(X-ct)}\right]Y + K_3C^{a(X-ct)}CY \\ &\quad + K_2^z d\alpha^\alpha \frac{C^{k\beta}}{\beta^2}d\beta + K_1^z d\alpha^\alpha \frac{C^{k\beta}}{\beta^2}d\beta^\beta \frac{C^{-k\beta}}{\theta}d\theta.\end{aligned} \tag{46}$$

The solution was also obtained by Dryden, Munaghan and Bateman [10] who searched the solutions for time dependent stream function of the form $\Psi(X,Y,t) = YF(X,t) + G(X,t)$.

We now seek the solution for

$$L(u,v) = A^{-1}F(u) + G(u-B)v, \quad A \neq 0. \tag{47}$$

On substituting Eq. (55) into Eq. (28) we find that $vAF^{iv} - UF^{///} = 0$, whose general solution leads to the following expression

$$F(u) = D_1 dudu \exp\left(\frac{u^2}{2vA}\right)du + D_2\frac{U^2}{2} + D_3u + D_4 \tag{48}$$

and thus the Legengre-transform function and the steady state stream functions turn out to be

$$L(u,v) = A^{-1}(u-B)v + A^{-1}F(u) \tag{49}$$

$$\begin{aligned}\psi(x,y) &= (Ax+B)y + \hat{C}_1 dxdx \exp\left[\frac{(Ax+B)^2}{2vA}\right]dx + \hat{C}_2\left(Ax+B\right)^2 \\ &\quad + \hat{C}_3\left(Ax+B\right) + \hat{C}_4\end{aligned} \tag{50}$$

where $D_i(i = 1,2,3,4)$, $\hat{C}_j(j = 1-4)$, and B are constants. The steady state solution (50) is similar to that of Jeffrey's solution [11] that searched for solution that satisfies $\Psi_{xx} + \Psi_{yy} = F(x)$. The only difference between this solution and Jeffrey's solution is that the kinematic viscosity ν is associated with quadratic term in our case while in the case of Jeffrey, viscosity appears with the term which is linear in y,and time time dependent stream function $\Psi(X, Y, t)$ is reduced to

$$\begin{aligned} \Psi(X,Y,t) &= Ax + B)y + \hat{C}_1 dxdx \exp\left[\frac{(Ax+B)^2}{2vA}\right] dx - CY + \hat{C}_2 (Ax+B)^2 \\ &\quad + \hat{C}_3 (Ax+B) + \hat{C}_4. \end{aligned} \tag{51}$$

In this case the Legendre-transform function is sought to be

$$L(u,v) = F(u) + G(v), \quad F^{/}(u) \neq 0, F^{/}(u) \neq 0, F^{//}(u) \neq 0, G^{/}(u) \neq 0, G^{//}(u) = 0. \tag{52}$$

Using Eq .(52) in Eq. (28) and solving the resulting equation, Chandna et al. [4] showed that

$$\psi(x,y) = v\overline{K}(y-x) + \widetilde{C}_1 e^{\overline{K}x} + \widetilde{C}_2 e^{\overline{K}y}, \tag{53}$$

where $\overline{K}^i, N_i, (i = 1-4)$ are arbitrary constants and $\widetilde{C}_1 = \frac{e^{-\overline{K}N_L}}{v\overline{K}^4}, \widetilde{C}_2 = \frac{e^{-\overline{K}N_1}}{v\overline{K}^4}$. The steady state solution is similar to that of Berker [12]. Berker obtained this type of solution by assuming $\psi(x,y) = F(x) + G(y)$. The time dependent stream function , in this case becomes

$$\Psi(X,Y,t) = v\overline{K}Y + v\overline{K}(X-ct) + \widetilde{C}_1 e^{\overline{K}(X-ct)} + \widetilde{C}_2 e^{\overline{K}Y} - CY + \widetilde{D}, \tag{54}$$

where $\overline{K}$, C and $\widetilde{D}$ are constants.

6. Concluding remarks

Solutions for the Navier-Stokes equations are obtained by employing transformation methods. First, the time dependent Navier-Stokes equations are transformed into steady state form and then hodograph transformation is used to interchange the dependent and independent variables. Then Navier-Stokes equations are reduced in the form of the Legendre-transform function. Several illustrations are considered to point out the usefulness of the method. It is found that, for a particular Legendre-transform, the Navier-Stokes equations in the hodograph plane become relatively easy to solve.

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