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Some Theorems on Ramanujan's Cubic Continued Fraction and Related Identities *

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Abstract

On page 366 of his lost notebook [8], Ramanujan has recorded cubic continued fraction and several theorems analogous to Rogers-Ramanujan continued fractions. In this paper we establish several interesting results of cubic continued fraction which are analogous to Rogers-Ramanujan continued fractions.

Keywords and Phrases: *Cubic continued fraction, Modular equation, Theta-function.*

1. Introduction

In Chapter 16, of his second note book [1], [3, pp.257-262], Ramanujan develops the theory of theta-function and his theta-function is defined by

$$\begin{aligned} f(a,b) &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \ \ |ab|<1, \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \end{aligned}$$

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where, $(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$, $|q|<1$.

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty, \tag{1.3}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{1.4}$$

Let

$$R(q) := \frac{q^{\frac{1}{5}}}{1+} \frac{q}{1+} \frac{q^2}{1+\cdots}, \qquad |q|<1, \tag{1.5}$$

denote the Rogers-Ramanujan continued fraction. On page 365 of his lost notebook [8], Ramanujan wrote five identities which shows the relation between $R(q)$ and the five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$.

On page 366 of his lost notebook [8], Ramanujan has recorded cubic continued fraction

$$V(q) := \frac{q^{\frac{1}{3}}}{1+} \frac{q+q^2}{1+} \frac{q^2+q^4}{1+\cdots}, \qquad |q|<1, \tag{1.6}$$

and claimed that there are many results of $V(q)$ which are analogous to $R(q)$.

In Section 2, we establish several modular equations of degrees 3 and 9. In Section 3, we establish general formulas to find explicit evaluations of $V(q)$ and reciprocity theorems. In Section 4, we establish the relation between $\mu(q)$ and the other four identities $\mu(-q)$, $\mu(q^2)$ and $\mu(q^3)$, where $\mu(q) := 2V(q)V(q^2)$. We also establish reciprocity theorems, integral representations and several explicit evaluations of $\mu(q)$.

2. Modular Equations of Degrees 3 and 9

Theorem 2.1. *We have*

$$\frac{f^6(-q)}{f^6(-q^3)} = \frac{\psi^2(q)}{\psi^2(q^3)}\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)}, \tag{2.1}$$

$$\frac{f^6(-q^2)}{qf^6(-q^6)} = \frac{\varphi^2(-q)}{\varphi^2(-q^3)}\frac{\varphi^4(-q)-9\varphi^4(-q^3)}{\varphi^4(-q)-\varphi^4(-q^3)}, \tag{2.2}$$

$$\frac{f^{12}(-q)}{qf^{12}(-q^3)} = \frac{\varphi^8(-q)}{\varphi^8(-q^3)}\frac{\varphi^4(-q)-9\varphi^4(-q^3)}{\varphi^4(-q)-\varphi^4(-q^3)}, \tag{2.3}$$

$$\frac{f^{12}(-q^2)}{f^{12}(-q^6)} = \frac{\psi^8(q)}{\psi^8(q^3)}\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)}. \tag{2.4}$$

Proof of (2.1)**.** By using Theorem 9.9 and Theorem 9.10 of Chapter 33 of Ramanujan notebooks [5, p.148], in Theorem 10.5 of Chapter 33 of Ramanujan's notebooks [5, p.156], we find that

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = \frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)}. \tag{2.5}$$

Using Entry 24(ii) of Chapter 16 of Ramanujan's notebooks [3, p.39] in (2.5), we obtain (2.1).
As the proofs of the identities (2.2)-(2.4) are similar to the proof of the identity (2.1). So we omit the details.

Theorem 2.2. *We have*

$$\frac{f^3(-q)}{f^3(-q^9)} = \frac{\psi(q)}{\psi(q^9)}\left(\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)}\right)^2, \tag{2.6}$$

$$\frac{f^3(-q^2)}{f^3(-q^{18})} = \frac{\psi^2(q)}{\psi^2(q^9)}\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)}, \tag{2.7}$$

$$\frac{f^3(-q)}{qf^3(-q^9)} = \frac{\varphi^2(-q)}{\varphi^2(-q^9)}\frac{\varphi(-q)-3\varphi(-q^9)}{\varphi(-q)-\varphi(-q^9)}, \tag{2.8}$$

$$\frac{f^3(-q^2)}{q^2f^3(-q^{18})} = \frac{\varphi(-q)}{\varphi(-q^9)}\left(\frac{\varphi(-q)-3\varphi(-q^9)}{\varphi(-q)-\varphi(-q^9)}\right)^2. \tag{2.9}$$

Proofs of the identities (2.6)-(2.9) are similar to the proof of the identity (2.1). So we omit the details.

Theorem 2.3 *We have*

$$\begin{aligned}
\varphi(q)-\varphi(q^3) &= 2q\chi(q)f(-q,-q^{11}), &(2.10)\\
\varphi(q)+\varphi(q^3) &= 2\chi(q)f(-q^5,-q^7), &(2.11)\\
\varphi^2(q)-\varphi^2(q^3) &= 4q\chi^2(q)\psi(q^6)f(-q,-q^5), &(2.12)\\
\varphi^2(q)+\varphi^2(q^3) &= 2\chi^2(q)\varphi(-q^3)f(q^2,q^4). &(2.13)
\end{aligned}$$

Proof of (2.10)**.** Using (1.1), we find that

$$\begin{aligned}
\varphi(-q)-\varphi(-q^3) &= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1-\frac{(-q;q)_\infty(q^3;q^3)_\infty}{(q;q)_\infty(-q^3;q^3)_\infty}\right]\\
&= (q;q^2)_\infty[f(-q,-q^2)-f(q,q^2)]. \qquad (2.14)
\end{aligned}$$

Using Entry 30(iii) of Chapter 16 of Ramanujan's notebooks [3, p.46] and then changing q by $-q$, we obtain (2.10).
Proofs of the identities (2.11)-(2.13) are similar to the proof of the identity (2.10). So we omit the details.

Corollary 2.1. *We have*
(i)

$$\frac{\varphi(q)}{\varphi(q^3)}=\frac{1+P}{1-P}, \quad \text{where} \quad P:=P(q)=\frac{qf(-q,-q^{11})}{f(-q^5,-q^7)}. \qquad (2.15)$$

For more details, one can see [6].
(ii) If $V(q)$ *is defined as in* (1.6), *then*

$$\frac{\varphi^2(q)}{\varphi^2(q^3)}=\frac{1+\mu}{1-\mu}, \quad \text{where} \quad \mu:=\mu(q)=2V(q)V(q^2), \qquad (2.16)$$

(iii)

$$\frac{\psi^2(q^2)}{q\psi^2(q^6)}=\frac{2-\mu}{\mu}, \qquad (2.17)$$

(iv) If $t=\frac{q^{\frac{1}{12}}(-q^3;q^3)_\infty}{(-q;q)_\infty}$, *then*

$$\mu(-q)=\frac{1-\sqrt{x}}{1+\sqrt{x}}, \quad \text{where} \quad x=\sqrt[4]{\frac{1+t^{12}+\sqrt{t^{24}-34t^{12}+1}}{2}}, \qquad (2.18)$$

(v) If $\mu(q) < 1$, *then*

$$\frac{f^6(-q^2)}{qf^6(-q^6)} = \frac{(1+\mu)(1-2\mu)(2-\mu)}{\mu(1-\mu)}, \tag{2.19}$$

(vi)

$$\frac{\psi^4(-q)}{q\psi^4(-q^3)} = \frac{(2\mu-1)(\mu-2)}{\mu}. \tag{2.20}$$

Proof of (i). Using (2.6) and (2.7), we find that

$$\frac{\varphi(q)-\varphi(q^3)}{\varphi(q)+\varphi(q^3)} = \frac{qf(-q,-q^{11})}{f(-q^5,-q^7)}.$$

Hence, we complete the proof.

Proofs of (ii)-(vi) are similar to proof of (i). So we omit the details.

Theorem 2.4. *If* $\alpha = \frac{1-i\sqrt{3}}{2}$ *and* $\beta = \frac{1+i\sqrt{3}}{2}$, *then*

$$\varphi(-q) + i\sqrt{3}\varphi(-q^3) = \frac{(1+i\sqrt{3})\chi(-q)f(-q^4)}{\prod_{n\equiv 0,2,3(\mod ,4)}(1-\alpha q^n)\prod_{n\equiv 0,1,2(\mod ,4)}(1-\beta q^n)}, \tag{2.21}$$

$$\varphi(-q) - i\sqrt{3}\varphi(-q^3) = \frac{(1-i\sqrt{3})\chi(-q)f(-q^4)}{\prod_{n\equiv 0,1,2(\mod ,4)}(1-\alpha q^n)\prod_{n\equiv 0,2,3(\mod ,4)}(1-\beta q^n)}, \tag{2.22}$$

$$\varphi^2(-q) + 3\varphi^2(-q^3) = 4\chi^2(-q)f(-q^4)\prod_{n\equiv 0(\mod ,3)}(1+q^n)(1+q^{2n}). \tag{2.23}$$

Proof of (2.21)**.** Let $\omega = e^{\frac{2\pi i}{3}}$ then putting $\omega = -\alpha$ and $\omega^2 = -\beta$. Since $\beta - \alpha = i\sqrt{3}$. Using (1.1), we obtain

$$\begin{aligned}\varphi(-q) + i\sqrt{3}\varphi(-q^3) &= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \frac{(\beta-\alpha)(q^3;q^3)_\infty(-q;q)_\infty}{(-q^3;q^3)_\infty(q;q)_\infty}\right] \\ &= \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \omega(1-\omega)\prod_{i=1}^{2}\frac{(\omega^i q;q)_\infty}{(-\omega^i q;q)_\infty}\right] \\ &= \chi(-q)\left[\frac{f(\omega,\omega^2 q) - f(-\omega,-\omega^2 q)}{(-\omega;q)_\infty(-\omega^2 q;q)_\infty}\right].\end{aligned}$$

Using Entry 30(iii) of Chapter 16 of Ramanujan's notebooks [3, p.46], we find that

$$\varphi(-q) + i\sqrt{3}\varphi(-q^3) = \frac{2\omega\chi(-q)f(\omega q,\omega^2 q^3)}{(1+\omega)(-\omega q;q)_\infty(-\omega^2 q;q)_\infty}.$$

On simplification of the above identity, we obtain (2.21).
Proofs of the identities (2.22) and (2.23) are similar to the proof of the identity (2.21). So we omit the details.

Theorem 2.5. *We have*

$$\left[27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right]^{\frac{1}{3}} = \frac{1}{V}+4V^2, \tag{2.24}$$

$$\left[27+\frac{f^{12}(-q^2)}{q^2f^{12}(-q^6)}\right]^{\frac{1}{3}} = \frac{1}{V^2}-2V, \tag{2.25}$$

$$3+\frac{f^3(-q^{\frac{1}{3}})}{q^{\frac{1}{3}}f^3(-q^3)} = \frac{1}{V}+4V^2, \tag{2.26}$$

$$3+\frac{f^3(-q^{\frac{2}{3}})}{q^{\frac{2}{3}}f^3(-q^6)} = \frac{1}{V^2}-2V. \tag{2.27}$$

Proof of (2.24)**.** By Entry 1(i) of Chapter 20 of Ramanujan's notebooks [3, p.345], we have

$$1+\frac{1}{V^3}=\frac{\psi^4(q)}{q\psi^4(q^3)}. \tag{2.28}$$

Using (2.28) in (2.1), we find that

$$\frac{f^{12}(-q)}{qf^{12}(-q^3)}=(1+\frac{1}{V^3})(1-8V^3)^2. \tag{2.29}$$

On simplification of the above identity (2.29), we obtain (2.24).
As the proofs of the identities (2.25)-(2.27) are similar to the proof of the identity (2.24). So we omit the details.

Theorem 2.6. *We have*

$$\frac{\varphi(q^{\frac{1}{3}})}{\varphi(q^3)}=1+\sqrt[3]{\frac{4\mu(q)}{(1-\mu(q))^2}}, \tag{2.30}$$

$$\frac{\psi(q^{\frac{2}{3}})}{q^{\frac{2}{3}}\psi(q^6)}=1+\sqrt[3]{\frac{4(1-\mu(q))}{\mu(q)}}. \tag{2.31}$$

Proof of (2.30)**.** Using (2.16) in (24.29) of Chapter 18 of Ramanujan notebooks [3, p.218], we obtain (2.30).

Proof of (2.31)**.** Using (2.17) in (24.28) of Chapter 18 of Ramanujan notebooks [3, p.218], we obtain (2.31).

3. General Formulas For Explicit Evaluations of $V(q)$

In this section, we prove some general formulas for explicit evaluations of $V(q)$.

Theorem 3.1. *We have*

(i) For $q = e^{-2n\sqrt{\frac{n}{3}}}$,

$$3(1+A_n^2)^{\frac{1}{3}} = \frac{1}{V} + 4V^2, \text{ where } A_n := \frac{1}{3\sqrt{3}} \frac{f^6(-q)}{q^{\frac{1}{2}} f^6(-q^3)}, \tag{3.1}$$

where $V := V(q)$ *is defined as in* (1.6),

(ii) For $q = e^{-\pi\sqrt{\frac{n}{3}}}$,

$$3(1+B_n^2)^{\frac{1}{3}} = \frac{1}{V^2} - 2V, \text{ where } B_n := \frac{1}{3\sqrt{3}} \frac{f^6(-q^2)}{q f^6(-q^6)}, \tag{3.2}$$

(iii) For $q = e^{-\pi\sqrt{\frac{n}{3}}}$,

$$3(1-\lambda_n^2)^{\frac{1}{3}} = \frac{1}{V(-q)} + 4V(-q)^2, \text{ where } \lambda_n := \frac{1}{3\sqrt{3}} \frac{f^6(q)}{q^{\frac{1}{2}} f^6(q^3)}, \tag{3.3}$$

(iv) For $q = e^{-2\pi\sqrt{n}}$,

$$3(1+\sqrt{3}D_n^3) = \frac{1}{V} + 4V^2, \text{ where } D_n := \frac{1}{\sqrt{3}} \frac{f(-q^{\frac{1}{3}})}{q^{\frac{1}{9}} f(-q^3)}, \tag{3.4}$$

(v) For $q = e^{-\pi\sqrt{n}}$,

$$3(1+\sqrt{3}C_n^3) = \frac{1}{V^2} - 2V, \text{ where } C_n := \frac{1}{\sqrt{3}} \frac{f(-q^{2/3})}{q^{2/9} f(-q^6)}, \tag{3.5}$$

(vi) For $q = e^{-\pi\sqrt{n}}$,

$$3(1-\sqrt{3}F_n^3) = \frac{1}{V(-q)} + 4V^2(-q), \text{ where } F_n := \frac{1}{\sqrt{3}} \frac{f(q^{\frac{1}{3}})}{q^{\frac{1}{9}} f(q^3)}. \tag{3.6}$$

Theorem(3.1) can be easily proved by using the results in Theorem(2.5).

Corollary 3.1. *We have*

$$A_1 = B_1 = C_1 = D_1 = F_1 = \lambda_1 = 1, \tag{3.7}$$

$$V(e^{-\frac{\pi}{\sqrt{3}}}) = 2^{-\frac{4}{3}}\left[\sqrt[3]{3+2\sqrt{2}} + \sqrt[3]{3-2\sqrt{2}} - 1\right]. \tag{3.8}$$

Proof of (3.7)**.** It follows from the definitions and transformation formulas in Entry 27 (iii), (iv) of Chapter 16 of Ramanujan's notebooks [3, p.43].
Proof of (3.8)**.** Putting n=1 in (3.2), we find that

$$B_1 = 1. \tag{3.9}$$

Using (3.9) in (3.2), we deduce that

$$3\sqrt[3]{2} = \frac{1}{x^2} - 2x, \quad where \quad x := V(e^{\frac{\pi}{\sqrt{3}}}). \tag{3.10}$$

Solving the above equation, we obtain the required result.

Theorem 3.2. *If* $V = V(q)$ *is defined as in* (1.6)*, then*

$$V(-e^{-\pi\sqrt{\frac{n}{3}}}) = -\frac{1}{2}\left[\sqrt[3]{\lambda_n + 1} - \sqrt[3]{\lambda_n - 1}\right], \ \lambda_n \geq 1, \tag{3.11}$$

where λ_n *is defined as in* (3.2)*.*
Proof. The equation (3.3) can be written as

$$4V^3(-q) - 3(1-\lambda_n^2)^{\frac{1}{3}}V(-q) + 1 = 0.$$

Solving the above equation, we obtain the required result.

Corollary 3.2. *We have*

$$V(-e^{\frac{-\pi}{\sqrt{3}}}) = \frac{-1}{\sqrt[3]{4}}. \tag{3.12}$$

Proof. Putting n=1 in (3.3), we find that

$$\lambda_1 = 1.$$

Using $\lambda_1 = 1$ in (3.11), we obtain (3.12).

Theorem 3.3. *If* $V(q)$ *is defined as in* (1.6)*, then*

$$V(-e^{-\pi\sqrt{n}}) = \frac{-1}{2}\left[\sqrt[3]{a+1} - \sqrt[3]{a-1}\right], \ \sqrt{3}F_n^3 - 1 \geq 0, \tag{3.13}$$

where

$$a = \sqrt{3\sqrt{3}F_n^3 - 9F_n^6 + 3\sqrt{3}F_n^9}$$

and F_n *is defined as in* (3.6).

Proof. The equation (3.6) can be written as

$$4V^3(-q) + 3(\sqrt{3}F_n^3 - 1)V(-q) + 1 = 0, \quad \sqrt{3}F_n^3 - 1 \geq 0.$$

Putting $x = -4V(-q)$ in the above equation, we find that

$$x^3 + 12(\sqrt{3}F_n^3 - 1)x - 16 = 0, \quad \sqrt{3}F_n^3 - 1 \geq 0.$$

Solving the above equation we obtain the required result.

Corollary 3.3. *We have*

$$V(-e^{-\pi}) = \frac{-1}{2}\left[\sqrt[3]{\sqrt{6\sqrt{3}-9}+1} - \sqrt[3]{\sqrt{6\sqrt{3}-9}-1}\,\right].$$

Proof. Putting $n = 1$ in (3.6), we find that

$$F_1 = 1.$$

Using $F_1 = 1$ in (3.13), we obtain the required result.

Remark. One can evaluate $V(q)$ by finding the explicit evaluations of λ_n and F_n, using Ramanujan's modular equations [4, pp.204-236] and transformation formulas [3, p.43].

4. Relation Between Parameter $\mu(q)$ and $\mu(q^n)$

Theorem 4.1. *If* $u := \mu(q)$, $v := \mu(-q)$, $w := \mu(-q^2)$, $x := \mu(q^2)$, $y := \mu(q^3)$ *and* $z := \mu(q^5)$, *then*

$$v^2 + (u + \frac{1}{u} - 5)v + 1 = 0, \tag{4.1}$$

$$(1 - u + u^2)w^2 + (4 - u - 2u^2)w + u^2 = 0, \tag{4.2}$$

$$u^2 + 2ux - 2x - 2ux^2 + 2x^2 = 0, \tag{4.3}$$

$$u^3 + 3u^2y - 9u^2y^2 - 4u^3y + 4u^3y^2 - 4y + 6uy + 4y^2 - 3uy^2 + u^2y^3 - y^3 = 0. \tag{4.4}$$

Proof of (4.1)**.** Using (2.20) and (2.16) in (2.5), we find that

$$\left(\frac{1+\mu(-q)}{1-\mu(-q)}\right)^2 = \frac{(2\mu(q)-1)(\mu(q)-2)-9\mu(q)}{(2\mu(q)-1)(\mu(q)-2)-\mu(q)}. \tag{4.5}$$

On simplification of the above identity, we obtain (4.1).
Proof of (4.2)**.** Using (2.16) and (2.17) in (2.5), we deduce that

$$\left(\frac{1+\mu(-q^2)}{1-\mu(-q^2)}\right)^2 = \frac{9\mu^2(q)-(2-\mu(q))^2}{\mu^2(q)-(2-\mu(q))^2}. \tag{4.6}$$

On simplification of the above identity, we obtain (4.2).
Proof of (4.3)**.** Using (2.1), (2.4) and (2.16), we find that

$$x^2(2-u)^4(1-x)^2(1-u-2u^2) = u^4(1-u)(2-x)^2(1-x-2x^2).$$

We find that

$$-2(ux^2-2x+2+2ux-2u)(u^2x^2+2x-2ux-2u^2x+u^2)$$
$$(2x^2-2ux^2-2x+2ux+u^2) = 0.$$

The first two factors does not vanish in the neighbourhood of $q = e^{-\pi}$. But the third factor vanish in the neighbourhood of $q = e^{-\pi}$. So by the identity theorem it vanish identically. Hence, we complete the proof.
Proof of (4.4)**.** Using (2.16) in Entry 1(ii) of Chapter 20 of Ramanujan's notebooks [3, p.345], we find that

$$\left(u^3(2-y)^3+6u^2y(1-u)(1-y)\right)^2+3uy^2(2-u)^2(2-y)$$
$$= y(1-u)^3(1-y)^2+9u^2y^3(1-u).$$

On simplification of the above identity, we obtain (4.4).
Theorem 4.2. *If* $3\alpha\beta = 1$*, then*

$$\mu(e^{-\pi\beta}) = \frac{1-2\mu(e^{-\pi\alpha})}{2-\mu(e^{-\pi\alpha})}. \tag{4.7}$$

where $\mu(q)$ *is defined as in* (2.16)*.*
Proof. Putting $q = e^{-\pi\alpha}$ in (2.16), we find that

$$\frac{\varphi^2(e^{-\pi\alpha})}{\varphi^2(e^{-3\pi\alpha})} = \frac{1+\mu(e^{-\pi\alpha})}{1-\mu(e^{-\pi\alpha})}. \tag{4.8}$$

Replacing α by β in the above identity (4.8), we obtain

$$\frac{\varphi^2(e^{-\pi\beta})}{\varphi^2(e^{-3\pi\beta})} = \frac{1+\mu(e^{-\pi\beta})}{1-\mu(e^{-\pi\beta})}. \tag{4.9}$$

Using Entry 27(i) of Chapter 16 of Ramanujan's notebooks [3, p.43], (4.8) and (4.9), we find that

$$\frac{(1+\mu(e^{-\pi\alpha}))(1+\mu(e^{-\pi\beta}))}{(1-\mu(e^{-\pi\alpha}))(1-\mu(e^{-\pi\beta}))} = 3.$$

After some simplification, we obtain the required result.

Theorem 4.3. *We have*

$$\mu(q) = \frac{-1+\exp(4\int_0^q \psi^2(-t)\psi^2(-t^3)dt)}{1+\exp(4\int_0^q \psi^2(-t)\psi^2(-t^3)dt)} \tag{4.10}$$

$$= \frac{-1+3\ \ \exp(-4\int_0^{\exp\left(\frac{\pi^2}{\log q}\right)} \psi^2(-t)\psi^2(-t^3)dt)}{1+3\ \ \exp(-4\int_0^{\exp\left(\frac{\pi^2}{\log q}\right)} \psi^2(-t)\psi^2(-t^3)dt)} \tag{4.11}$$

$$= \frac{2}{1+9\ \ \exp(4\int_q^1 \varphi^2(-t)\varphi^2(-t^3)\frac{dt}{t})} \tag{4.12}$$

Proof of (4.10)**.** Putting $x = y = 2$, $m = 1$, $k = 3$ in [2, Theorem 2.5(a)], we obtain

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} = \exp\left(4\int_0^q \psi^2(-t)\psi^2(-t^3)dt\right). \tag{4.13}$$

Using (2.18) and (4.13), we obtain (4.10).

Proof of (4.11)**.** In Entry 27(i) of Chapter 16 of Ramanujan's notebooks [3, p.43], let $\alpha^2 = \log\left(\frac{1}{q}\right)$ and $\beta^2 = \log\left(\frac{1}{Q^3}\right)$, then

$$\log^{\frac{1}{4}}\left(\frac{1}{q}\right)\varphi(q) = \log^{\frac{1}{4}}\left(\frac{1}{Q^3}\right)\varphi(Q^3), \tag{4.14}$$

where

$$3\log\left(\frac{1}{q}\right)\log\left(\frac{1}{Q}\right) = \pi^2.$$

Replacing q and Q by q^3 and $Q^{\frac{1}{3}}$ respectively, we obtain

$$\log^{\frac{1}{4}}\left(\frac{1}{q^3}\right)\varphi(q^3)=\log^{\frac{1}{4}}\left(\frac{1}{Q}\right)\varphi(Q). \tag{4.15}$$

Using (4.14) and (4.15), we find that

$$\frac{\varphi(q)}{\varphi(q^3)}=\sqrt{3}\frac{\varphi(Q^3)}{\varphi(Q)}. \tag{4.16}$$

Using (4.13), (4.16) and (2.16), we obtain (4.11).
Proof of (4.12)**.** Putting $x=y=m=2$ and $k=6$ in [2, Theorem 2.5(c)], we obtain

$$\frac{\psi^2(q^2)}{q\psi^2(q^6)}=9\ \exp\left(4\int_q^1\varphi^2(-t)\varphi^2(-t^3)\frac{dt}{t}\right). \tag{4.17}$$

Using (4.17) and (2.17), we obtain (4.12).
Theorem 4.4. *We have*

$$\mu(e^{-\pi}) = \frac{\sqrt{6\sqrt{3}-9}-1}{\sqrt{6\sqrt{3}-9}+1}, \tag{4.18}$$

$$\mu(e^{-\sqrt{3}\pi}) = \frac{3\sqrt{\sqrt[3]{2}-1}-\sqrt{\sqrt[3]{2}+1}}{3\sqrt{\sqrt[3]{2}-1}+\sqrt{\sqrt[3]{2}+1}}, \tag{4.19}$$

$$\mu(e^{-\sqrt{5}\pi}) = \frac{3-\sqrt{1+2\sqrt{3}+2\sqrt{5}}}{3+\sqrt{1+2\sqrt{3}+2\sqrt{5}}}, \tag{4.20}$$

$$\mu(e^{-\sqrt{7}\pi}) = \frac{12\sqrt{2}-\sqrt{32+\sqrt{5+\sqrt{21}}(\sqrt{5+\sqrt{21}}+\sqrt{\sqrt{21}-3})^3}}{12\sqrt{2}+\sqrt{32+\sqrt{5+\sqrt{21}}(\sqrt{5+\sqrt{21}}+\sqrt{\sqrt{21}-3})^3}}, \tag{4.21}$$

$$\mu(e^{-3\pi}) = \tag{4.22}$$

$$\frac{3\sqrt{\sqrt[3]{2(\sqrt{3}-1)}-1}-\sqrt{\sqrt[3]{2(\sqrt{3}-1)}-1+2(3\sqrt{3}-5)(\sqrt[3]{2(\sqrt{3}+1)}+1)}}{3\sqrt{\sqrt[3]{2(\sqrt{3}-1)}-1}+\sqrt{\sqrt[3]{2(\sqrt{3}-1)}-1+2(3\sqrt{3}-5)(\sqrt[3]{2(\sqrt{3}+1)}+1)}},$$

$$\mu(e^{-\sqrt{13}\pi}) = \tag{4.23}$$

$$\frac{3\sqrt{2\sqrt{2}}-\sqrt{2\sqrt{2}+\sqrt{(2\sqrt{3}+\sqrt{13})(5\sqrt{13}-18)}(\sqrt[4]{3}+\sqrt{4+\sqrt{3}})^3}}{3\sqrt{2\sqrt{2}}+\sqrt{2\sqrt{2}+\sqrt{(2\sqrt{3}+\sqrt{13})(5\sqrt{13}-18)}(\sqrt[4]{3}+\sqrt{4+\sqrt{3}})^3}},$$

$$\mu(e^{-\frac{\pi}{\sqrt{2}}}) = \frac{2}{3\sqrt{3}+3\sqrt{2}+1}, \tag{4.24}$$

$$\mu(e^{-\frac{\pi}{\sqrt{3}}}) = 2-\sqrt{3}, \tag{4.25}$$

$$\mu(e^{-\frac{\pi}{\sqrt{6}}}) = \frac{2}{\sqrt{6}+\sqrt{3}+1}, \tag{4.26}$$

$$\mu(e^{-\frac{\pi}{3}}) = \frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}, \tag{4.27}$$

$$\mu(e^{-\pi\sqrt{\frac{5}{3}}}) = \frac{\sqrt{6}-\sqrt{3+\sqrt{5}}}{\sqrt{6}+\sqrt{3+\sqrt{5}}}, \tag{4.28}$$

$$\mu(e^{-\pi\sqrt{\frac{7}{3}}}) = \frac{3-\sqrt{1+2(2\sqrt{7}+3\sqrt{3})(3-\sqrt{7})}}{3+\sqrt{1+2(2\sqrt{7}+3\sqrt{3})(3-\sqrt{7})}}, \tag{4.29}$$

$$\mu(e^{-\pi\sqrt{\frac{11}{3}}}) = \frac{\sqrt{15+9\sqrt{3}}-\sqrt{15+4\sqrt{11}+\sqrt{3}}}{\sqrt{15+9\sqrt{3}}+\sqrt{15+4\sqrt{11}+\sqrt{3}}}, \tag{4.30}$$

$$\mu(e^{-\pi\sqrt{\frac{19}{3}}}) = \frac{3-\sqrt{1+2(2+\sqrt{3})^3(3\sqrt{19}-13)}}{3+\sqrt{1+2(2+\sqrt{3})^3(3\sqrt{19}-13)}}. \tag{4.31}$$

Proof of (4.18)**.** From [5, p.330] we have

$$\frac{\varphi^2(e^{-\pi})}{\varphi^2(e^{-3\pi})} = \sqrt{6\sqrt{3}-9}. \tag{4.32}$$

Using (4.32) in (2.16) and simplifying the resultant equation, we obtain the required result.

The identities (4.19)-(4.31) can be obtained by using Ramanujan's Class-Invariants[5, pp.189-199] and equation (4.5) of [5, eqn.(4.5), p.330] in (2.16) and (2.17). So we omit the details.

Remark. *We can also obtain several other evaluations of* $\mu(q)$*, using* (2.16), (2.17), (2.30) *and* (2.31) *and Ramanujan's class invariants.*

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